Non-cubic, edge-critical Hamilton laceable bigraphs with 3m edges on 2m vertices

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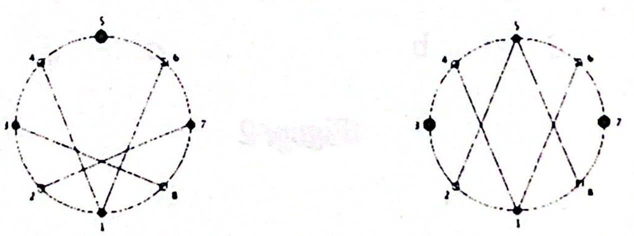
Abstract:

Constructions are given for non-cubic, edge-critical Hamilton laceable bigraphs with 3m edges on 2m vertices for all m ≥ 4. The significance of this result is that it shows the conjectured hard upper bound of 3m edges for edge-critical bigraphs on 2m vertices is populated by both cubic and non-cubic cases for all m. This is unlike the situation for the hard 3m-edge lower bound for edge-stable bigraphs where the bound is populated exclusively by cubics.

1. Introduction:

A bigraph is *equitable* if the two parts have the same cardinality and *Hamilton laceable* if there exists a Hamilton path between every pair of vertices in different parts. The bigraphs of interest here are all equitable on 2m vertices but may or may not be Hamilton laceable. Let G be a graph, not necessarily bipartite, which exhibits a graph property and ***G*** the E element collection of sub-graphs formed by deleting each edge in turn from G. G is *edge-stable* with respect to the property if every graph in ***G*** exhibits the property or *edge-critical* if none do. It is easy to see that 3m edges is a hard lower bound for edge-stability with respect to Hamilton laceability for equitable bipartite graphs on 2m vertices since no vertex can have degree 2 and cubic examples abound: the m-prisms and the Mobius ladders for example. In [1] it was conjectured 3m is a hard upper bound for the number of edges an edge- critical bigraph can have. The conjecture was based on the fact that Weisstein's m-crossed prisms [4] when V ≡ 0 mod 4 and Simmons' sausage graphs [3] when V ≡ 2 mod 4 provide cubic examples for all m ≥ 5, plus the known existence of a few isolated "four leaf clover' non-cubic cases with 3m edges. The conjecture is still open. What will be shown here is that non-cubic cases, far from being rare, exist for all m.≥4.

The smallest m for which such a graph could exist is m=4. Up to isomorphism there are only two Hamilton laceable candidate bigraphs on 8 vertices.



a b

Figure 1

Graph 1a is not edge-critical since the punctured graph resulting from deleting edge 1-2 (or 1-8) is Hamilton laceable. Graph 1b though is easily shown to be edge-critical. By symmetry the edges can be partitioned into two groupings of transitive edges; the four incident on a vertex of degree 2 and the eight incident on a vertex of degree 4. Deletion of an edge in the first group results in a vertex of degree 1 which cannot be an interior vertex in a path. Deletion of an edge in the second group results in two adjacent vertices of degree 2, either in a circular segment cut off by another edge from that group, or an arrangement easily converted to that form by simply interchanging vertices 1 and 5 while preserving edges. There cannot be a Hamilton path between the endpoints of that edge since they form a cut-set in the punctured graph. The graph in Figure 1b is therefore the smallest example of a non-cubic, edge-critical Hamilton laceable bigraph.

This is a convenient place to introduce the notation we will use to denote non-cubic, edge-critical Hamilton laceable bigraphs. The degrees, and multiplicities, of the vertices in each part which differ from 3 will be grouped in parentheses. Vertices of degree 3 will be lumped together and shown outside the parentheses. The bigraph in Figure 1a would be denoted by $(4,2), 3^6$ and the one in Figure 1b by $(4^2,2^2), 3^4$. As mentioned earlier, these are the only two possibilities when m=4. $(4,2)(4,2), 3^6$ would denote a bigraph with vertices of degree 4 and 2 in each part, for which an edge-critical example does exist as will be shown in the next section.

2. The case m = 5:

Ten vertices are too few to accommodate the iterative constructions used for larger values of m, so a case by case analysis is forced for $(4,2), 3^8$; $(4^2,2^2), 3^6$; $(4,2)(4,2), 3^6$; $(5,2^2), 3^7$ and $(5,2^2)(4,2), 3^5$.

There are many ways five chords can be assigned to C_{10} consistent with a $(4,2), 3^8$ bigraph, all of which reduce to one of four non-isomorphic candidate bigraphs;

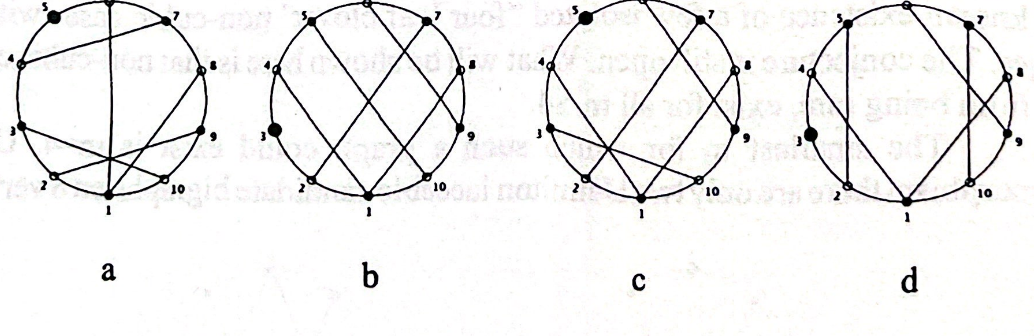
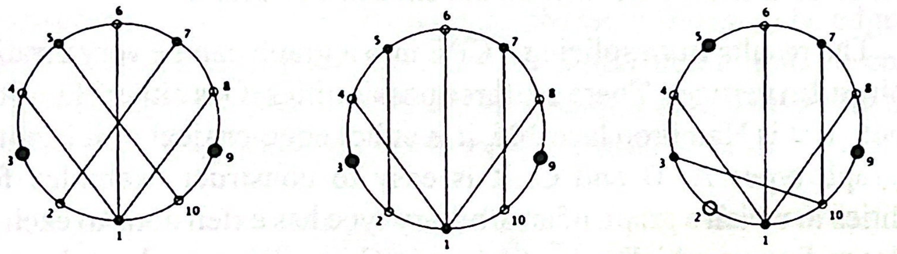


Figure 2

The graph in Figure 2d is not Hamilton laceable but the other three are, hosting 110, 112 and 126 Hamilton paths respectively. None of the Hamilton laceable cases are edge critical however, since the punctured graphs resulting from deleting edge 1-10 are all Hamilton laceable. Therefore, there are no edge-critical (4,2), 3^8 bigraphs, unlike the situation for larger m where it will be shown a (4,2), 3^{2m-2} bigraph always exists.

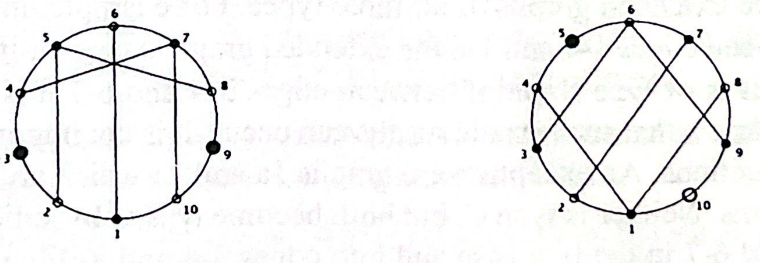
At the other extreme, $(5,2^2)$ requires a vertex in one part be connected to every vertex in the other part and forces two of the other vertices to be of degree 2. There are only two possibilities; $(5,2^2)(4,2)$, 3^5 and $(5,2^2)$, 3^7. $(5,2^2)(4,2)$, 3^5 has only a single isomorph realization, Figure 3c, which is not Hamilton laceable. $(5,2^2)$, 3^7 has two non-isomorphic realizations, Figures 3a and 3b: the first of which is Hamilton laceable, the second of which is not. The bigraph in Figure 3a is not edge-critical however, since the punctured graph resulting from deleting edge 1-6 is Hamilton laceable.



a b c

Figure 3

That leaves only the two cases, $(4^2,2^2)$, 3^6 and (4,2)(4,2), 3^6, both of which, fortunately for the objective of showing there are non-cubic edge-critical bigraphs for all m, have edge-critical realizations; Figure 4.



a b

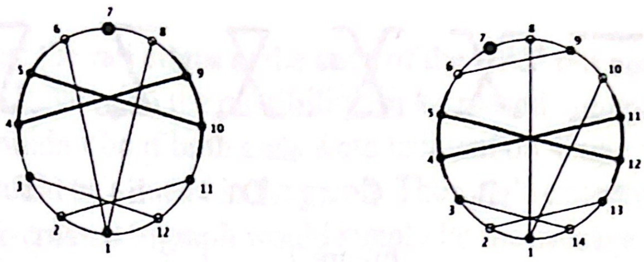
Figure 4

3. The cases m > 5:

In the preceeding sections it was shown that only a single $(4^2,2^2)$, 3^4 edge-critical Hamilton laceable bigraph exists when m = 4 and the pair, $(4^2,2^2)$, 3^6 and $(4,2)(4,2)$, 3^6, of such bigraphs when m = 5. The number of non-cubic edge-critical bigraphs undoubtably grows rapidly with m, but since only their existence is of concern here, it suffices to show there exists a $(4,2)$, 3^{2m-2} graph for all m > 5. This will be done by defining two starter edge-critical Hamilton laceable graphs, one for even m and one for odd m, and a sub-graph on four vertices which can be spliced into the starter graphs arbitrarily many times to increase m by increments of 2 while preserving Hamilton laceability and edge criticality. There are only two choices for the sub-graph on four vertices; a pair of parallel edges or a pair of crossed edges, i.e. a square or a twisted square. To accommodate vertex parity, parallel edges could only be spliced into host edges of opposite orientation and crossed edges into host edges of like orientation. For reasons that will become apparent, a crossed pair of edges, or *CPE* hereafter, will be the chosen sub-graph.

The results from splicing a CPE into a graph can be very erratic. Let G be a bigraph on 2m vertices. There are three possibilities: G is either Hamilton laceable or it is not. If it is Hamilton laceable, it is either edge-critical or it is not. Call these three graph types A, B and C. It is easy to construct examples for all nine possibilities in which a graph of an arbitrary type has extensions to each of the three types, depending on which pair of edges in G are chosen to host the spliced CPE. For the intended application to extending bigraphs by splicing in CPE that is about as bad as it could be; every type has extensions to every other type. For the purposes of this paper what is needed are a pair of type C bigraphs, one for even m and one for odd m, which can be extended indefinitely to type C bigraphs.

Three of the graphs in Figures 1 and 2 represent all three graph types: graph 2d is type A, 1a is type B and 1b is type C. As luck would have it there are pairs of host edges in each of these three graphs which, when a CPE is spliced into them, produce extended graphs of all three types. For example, in 1a if the splice is made between edges 1-4 and 1-6 the extended graph is type A, if between edges 2-3 and 7-8 it is of type B and if between edges 3-4 and 6-7 it is of type C. It is important to know that such erratic results can occur. It is not important to show all of the constructions. An exception are graphs 1a and 2a which are "seeds" for the starter bigraphs. Neither is type C, but both become type C by splicing a CPE into edges 3-4 and 6-7 in the first case and into edges 3-4 and 8-9 in the other. These will be the starter graphs used here: 5a for even m and 5b for odd m.



a (1a extended) b (2a extended)

Figure 5

In each case, computer testing has verified that splicing additional CPE parallel to the bold pairs in Figure 5 results in type C bigraphs for the first five extensions. Five is not an arbitrary choice, but is rather the number of extensions required to support an argument in the proof of Theorem 1.

Weisstein's m-crossed prisms replace the edges linking matching vertices in the base m-gons in an m-prism with crossed pairs of edges, i.e. with m copies of a CPE: obviously $V \equiv 0 \bmod 4$. In [2] these graphs were shown to be edge-critical with respect to Hamilton laceability for $m \geq 6$. A simple link to join the ends of an "opened" m-crossed prism, Figure 6, resulting in graphs resembling ring bologna, provides examples of edge-critical graphs for $V \equiv 2 \bmod 4$ for $m \geq 5$ [3].

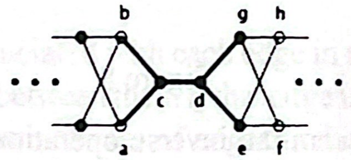


Figure 6

The inductive proofs that m-crossed prisms and the closely related sausage bigraphs are Hamilton laceable and edge-critical [2,3] depended critically on the fact that there are only a few easily described spanning paths possible in a chain of CPE's, closed or open, and argued a few special cases for the sausage bigraphs when one or both endpoints, or a deleted edge, were in the link. In a rare bit of serendipity almost the same proof technique can be used here.

Although the following observations barely warrant being dignified as a lemma, their importance to the proof of Theorem 1 does.

Lemma 1: Let G be a bigraph in which there exists a Hamilton path between endpoints x and y and which contains a pair of adjacent CPE, neither of which hosts x or y, then there exists a Hamilton path between x and y in a reduced G with one of the CPE deleted.

Proof:

A Hamilton path through a CPE connecting endpoints x and y not in the CPE simply spans the four vertices in the CPE and exits on the side opposite the one on which it entered. This is possible in six ways as shown in Figure 7; two ways when the path enters on both rails and four when it enters on only one.

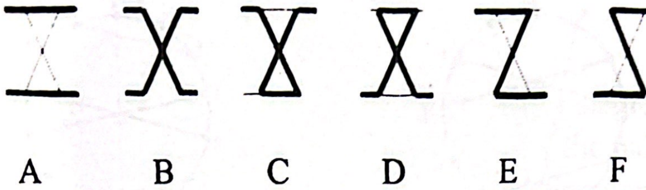


Figure 7

Of interest is the case in which there are two adjacent such CPE's, i.e neither hosts an endpoint of the Hamilton path. The emergent edges from the lefthand CPE must match the entrant edges of the righthand CPE, so there are only twelve possible paths through the two: E followed by E is not possible, E followed by F is and has the same effect on the path as C in a single CPE. In fact every pair of concatenated paths through the two CPE's is equivalent to some path through a single CPE: summarized in Figure 8. The paths through the lefthand CPE are indexed on the left of the arrays, the path through the righthand CPE on the top.

	A	B
A	A	B
B	B	A

	C	D	E	F
C	C	.	E	.
D	.	D	.	F
E	.	E	.	C
F	F	.	D	.

Figure 8

It might at first seem the inverse operation, replacing a path through a single CPE with an equivalent concatenation of paths through two adjacent CPE's, would be an easy way to show Hamilton laceability or edge criticality of bigraphs formed by splicing in a CPE. It is not, since both the contraction and the expansion presupposes neither an endpoint of the assumed Hamilton path nor the deleted edge in testing edge criticality are in the pair of adjacent CPE's; something that is not true in general but will be for the constructions considered in Theorem 1.

k-crossed bigraphs are a vertex join of a CPE^k , k>1, (the left hand subgraph in Figure 9) with a bigraph B (the subfigure in the right hand circle in Figure 9) where the two rail edges at one end of the CPE^k are incident on a single vertex in B and the two at the other end are not.

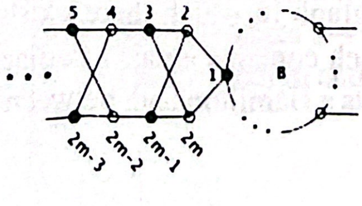


Figure 9

The restriction that the rail edges at the ends of the CPE^k not both lie on common vertices in B is to leave open the possibility for k-crossed bigraphs to be Hamilton laceable. They couldn't be if both ends were incident on single vertices since that pair of vertices would be a cutset in the graph. The single exception is when B is an edge so that the k-crossed bigraph would simply be the sausage bigraph shown in Figure 6.The condition is necessary, but not sufficient, as shown by the two k-crossed bigraphs in Figure 10: those in 10a are Hamilton laceable (and edge critical) while those in 10b are neither.

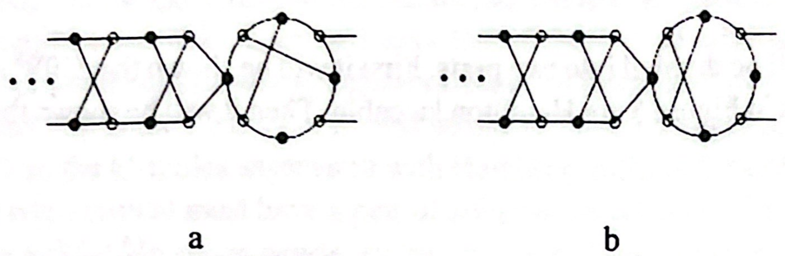


Figure 10

Lemma 2: Let G be a Hamilton laceable k-crossed bigraph. Deleting any edge in CPE^k causes G to no longer be Hamilton laceable.

Proof:

There is a pair of vertices associated with each edge in the CPE^k in Figure 9 which cannot have a Hamilton path between them if that edge is deleted. We first note that there are only two isomorphism classes of edges in the CPE^k: those internal to a CPE and those connecting adjacent CPE's. The convention will be that edges connecting CPE are on the side opposite the common vertex; 1 in Figure 9. A k-crossed bigraph is invariant under a half twist of the right hand portion of the CPE^k at the midpoint of any CPE, which simply converts the crossed edges in that CPE into edges on the rail and vice versa.

If edge d-e in Figure 11a is deleted, there cannot be Hamilton path between vertices 1 and b. Since vertices d and e are interior points in the assumed Hamilton path, the bold sub-paths are forced. If b connects to g the path is forced to close prematurely leaving at least vertices a, d and c isolated. If b connects to c either the path on 1 spanning the vertices to its left is isolated, or else the vertices themselves are isolated. In either case a Hamilton path cannot exist.

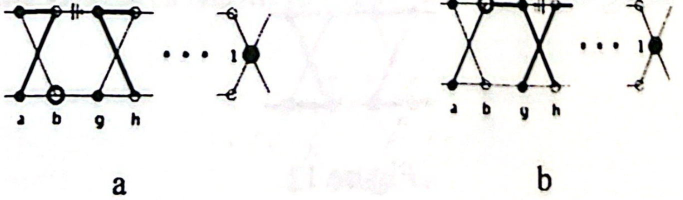


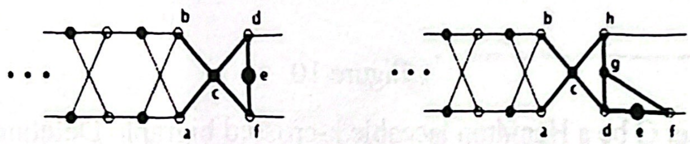
Figure 11

If edge e-f in Figure 11b is deleted, there cannot be a Hamilton path between vertices 1 and d. Since vertices e and f are interior points in the assumed Hamilton path, the bold sub-paths are forced. If g connects to h the path closes prematurely isolating vertices to the left of d. If g does not connect to h the path either returns through g-f leaving vertices to the left of 1 isolated or else closes to vertex 1 prematurely. In either case a Hamilton path cannot exist.

Theorem 1: There exist (4,2), 3^{2m-2} edge-critical Hamilton laceable bigraphs for all $m \geq 6$.

Proof:

The proof will be divided into two parts. First it will be shown that CPE^k extensions of the starters in Figure 5 are Hamilton laceable. Then it will be shown that they are edge-critical.



a b

Figure 12

Figure 12 shows the starter graphs redrawn as k-crossed bigraphs. Associate a k-bit binary number with CPE^k, where a 1 indicates that at least one endpoint of the assumed Hamilton path is in the associated CPE. By Lemma 1 a 0 in the k-tuple can be extended to a 00 in a (k+1)-tuple of a Hamilton path in the k-crossed bigraph with CPE^{k+1}. For example if it is known that a k-crossed bigraph is Hamilton laceable for CPE^4, Lemma 1 guarantees the k-crossed bigraph with CPE^5 has Hamilton paths for all endpoint pairs except for one associated with 4-tuple 0110. A path can reverse direction in a CPE if and only if an endpoint is in it which means it is possible to have a pair of adjacent CPE, represented by 1's in the associated k-tuple, which have no edge of the Hamilton path joining them; endpoints x and y in Figure 13. A CPE could not be interpolated between the host CPE for x and y, i.e. a 0 interpolated between the pair 11 in k-tuple 0110 to form 01010, since the four vertices in it would be isolated in the extension of the Hamilton path.

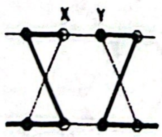


Figure 13

At the time starter bigraphs 5a and 5b were defined it was remarked that it had been computer verified that extensions for $k \le 5$ were all Hamilton laceable and edge-critical. The binary 5-tuples with at most two 1's can be extended to all binary 6-tuples with at most two 1's using Lemma 1. Therefore the CPE^k extensions of the starters in Figure 5 are all Hamilton laceable.

The proof that the extensions are also edge-critical will be by contradiction. Since the first five extensions have been shown to be edge-critical, either all extensions are edge-critical or there is a least k, say k', for which there exists an edge in the extension which can be deleted and the punctured graph will still be Hamilton laceable. Lemma 1 says that if there is an edge in a Hamilton laceable k-crossed bigraph whose deletetion leaves a Hamilton laceable punctured graph, it can't be in the $CPE^{k'}$, i.e. it must be one of the bold edges in either 12a or 12b. k' > 5 so the k'-tuples associated with Hamilton paths in the extension which fails to be edge-critical must have a pair of adjacent 0's which by Lemma 1 can be reduced to k=k'-1. In other words, every pair of endpoints x and y that have a Hamilton path between them in the extension by $CPE^{k'}$must also have had a Hamilton path between them in the extension by $CPE^{k'-1}$, which contradicts the assumption k' was the least value of k for which there existed an edge whose deletion left a Hamilton laceable punctured bigraph.

Therefore the $(4,2), 3^{2m-2}$ bigraphs introduced here are all edge-critical with respect to Hamilton laceability for $m \ge 6$.

4. Conclusion, etc.:

Non-cubic edge-critical bigraphs on 2m vertices having 3m edges have been exhibited for all $m \ge 4$. The incidental motivation was the conjecture that no such bigraph can have more edges. It was shown that there is a unique $(4^2,2^2), 3^4$ such bigraph when m = 4, a pair of $(4^2,2^2), 3^6$ and $(4,2)(4,2), 3^6$ bigraphs when m = 5 and at least a $(4,2),3^{2m-2}$ bigraph for all $m \ge 6$. The $(4,2),3^{2m-2}$ cases, however, are almost cubic, not just because they have 3m edges, but because there is a simple operation which transforms them into the archetypical cubic edge-critical bigraphs:

m-crossed prisms when m is even and sausage bigraphs when m is odd. In either case the operation consists of simply pivoting an edge in a starter bigraph about one of it's endpoints to a different endpoint for the other.

For m even, pivot edge a-c in Figure 12 to become edge a-e. The result is a pair of adjacent CPE's, as shown in Figure 14, i.e. a sub-graph of an m-crossed prism.

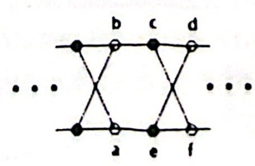


Figure 14

For m odd, pivot edge h-c in Figure 12 to become edge h-e. The result is the sausage bigraph link shown in Figure 6.

The **etc.** in the heading of this section refers to non-cubic edge-critical bigraphs in general, i.e. to cases not so simply related to cubics. There are 19 partitions of integers ≤ 12 into parts ≥ 4. There are 29 ways these parts can be assigned to two sets. Each such assignment is a potential degree sequence of supernumerary vertices for a family of non-cubic edge-critical bigraphs. 12 was chosen as the limiting case since it is the smallest integer which could have three parts, i.e. which could lead to $(4^3,-)$ or $(4^2,-)(4,-)$ bigraphs. A computer search found edge-critical bigraphs realizing all 29 possibilities, often with several non-isomorphic realizations. For example, unique $(4^3,2^3)$, 3^6 and $(4^2,2^2)(4,2),3^{10}$ cases were found on 12 and 16 vertices respectively and a, probably not minimal or unique, $(12, 2^9),3^{26}$ edge-critical case on 36 vertices, i.e. an edge-critical bigraph with a vertex of degree 12. This suggests the 3m-edge upper bound for edge criticality is probably densely populated by non-cubic bigraphs as m increases, making the bound all the more remarkable if true.

Acknowledgement:

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