On the adjacent vertex-distinguishing total colorings of some cubic graphs*

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Abstract

Suppose $G = (V, E)$ is a simple graph and $f : (V \cup E) \to \{1, 2, \ldots, k\}$ is a proper total k-coloring of G. Let $C(u) = \{f(u)\} \cup \{f(uv) : uv \in E(G)\}$ for each vertex u of G. The coloring f is said to be an adjacent vertex-distinguishing total coloring of G if $C(u) \neq C(v)$ for every $uv \in E(G)$. The minimum k for which such a coloring of G exists is called the adjacent vertex-distinguishing total chromatic number of G, and is denoted by $\chi_{at}(G)$. This paper considers three types of cubic graphs: a specific family of cubic hamiltonian graphs, snares and Generalized Petersen graphs. We prove that these cubic graphs have the same adjacent vertex-distinguishing total chromatic number 5. This is a step towards a problem that whether the bound $\chi_{at}(G) \leq 6$ is sharp for a graph G with maximum degree three.

Keywords: Adjacent vertex-distinguishing total coloring; Adjacent vertex-distinguishing total chromatic number; Cubic graphs; Snares; Generalized Petersen graphs

1 Introduction

Let $G = (V, E)$ be a simple graph and $T(G) = V(G) \cup E(G)$ be the set of vertices and edges of G. A proper *total k-coloring* of G is a mapping $f : T(G) \to \{1, 2, \ldots, k\}$ such that no two adjacent or incident elements of

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$T(G)$ receive the same color. Consider such a coloring f, denote by $C(u)$ the color set $\{f(u)\} \cup \{f(uv) : uv \in E(G)\}$ for each vertex u of G. The coloring f is said to be an *adjacent vertex-distinguishing total coloring* (AVDTC for short) if $C(u) \neq C(v)$ whenever $uv \in E(G)$. The minimum k for which such a coloring of G exists is called the *adjacent vertex-distinguishing total chromatic number* of G, and is denoted by $\chi_{at}(G)$. It was Zhang et al [13] who first introduced this kind of coloring.

It is worth to mention another related total coloring—neighbor sum distinguishing total coloring, which is defined as follows. In a total k-coloring f of G, let $S(u)$ denote the total sum of colors of the edges incident to u and the color of u. The coloring f is said to be a *neighbor sum distinguishing total coloring* if for each edge uv, then $S(u) \neq S(v)$. The minimum k for which such a coloring of G exists is called the *neighbor sum distinguishing total chromatic number* of G, and is denoted by $\chi''_{nsd}(G)$. If f is a neighbor sum distinguishing total coloring, then clearly it is also an adjacent vertex-distinguishing total coloring. Thus $\chi_{at}(G) \leq \chi''_{nsd}(G)$ for any graph G. For results about neighbor sum distinguishing total chromatic number, we refer readers to [5–7].

Let $\Delta(G)$ and $\delta(G)$ be the maximum degree and minimum degree of a graph G respectively. By definition, it is obvious that $\chi_{at}(G) \geq \Delta(G) + 1$. The following simple observation was also made by Zhang et al [13].

Proposition 1 *If G is a graph with two adjacent vertices of maximum degree, then* $\chi_{at}(G) \geq \Delta(G) + 2$.

They also proposed the following conjecture.

Conjecture 2 *If G is a simple graph, then* $\chi_{at}(G) \leq \Delta(G) + 3$.

It is easy to prove that $\chi_{at}(G) \le \Delta(G)+2$ for bipartite graphs G [1]. Thus the conjecture is true for bipartite graphs. It was also confirmed for outerplanar graphs [10] and K_4-minor free graphs [9]. Then the conjecture was proved for planar graphs with maximum degree at least 11 by Huang and Wang [2]. Recently, Wang and Huang [11] proved that $\chi_{at}(G) \le \Delta(G)+2$ for planar graphs with $\Delta(G) \ge 14$.

Wang in [8] showed that this conjecture is true for any graph G with $\Delta(G)=3$. Short and concise proofs were given by Chen in [1] and Hulgan in [3], independently. However, many graphs with maximum degree three, including K_4, $K_{3,3}$, and Petersen graphs, have an AVDTC with only 5 colors. Therefore, Hulgan in [3] proposed the following problem.

Problem 3 *For a graph G with $\Delta(G)=3$, is the bound $\chi_{at}(G) \le 6$ sharp?*

Actually, the following simple result tells us that we only need to focus on cubic graphs.

Proposition 4 *Let G be a cubic graph, if H is a subgraph of G, then $\chi_{at}(H) \le \chi_{at}(G)$.*

Proof. Let $\chi_{at}(G)=k$, then $5 \le k \le 6$. Suppose f is a k-AVDTC of G. If $f|_H$, the restriction of f on H, is a k-AVDTC of H, then we are done, otherwise, there exist two adjacent vertices of degree 2 having the same color set. Consider suspend trail $v_0v_1v_2\ldots v_tv_{t+1}$, i.e., a trail $v_0v_1v_2\ldots v_tv_{t+1}$ such that $d_H(v_0) \ne 2 \ne d_H(v_{t+1})$, $d_H(v_i)=2$, $i=1,2,\ldots,t$. Let $f|_H = f^{(1)}$, $C_{f|_H}(v_i)=C^{(1)}(v_i)$. If $C^{(1)}(v_1)=C^{(1)}(v_2)$, without loss of generality, assume $f(v_1)=1, f(v_2)=2, f(v_1v_2)=3, f(v_1v_0)=2, f(v_2v_3)=1$. Then recolor v_2 by 4 if $f(v_3)\ne 4$ or 5 if $f(v_3)=4$. The resulting coloring, denoted by $f^{(2)}$, satisfies $C^{(2)}(v_1) \neq C^{(2)}(v_2)$. If $C^{(1)}(v_1) \neq C^{(1)}(v_2)$, then let $f^{(2)} = f^{(1)}$. For each vertex x of H we denote by $C^{(2)}(x)$ the color set of x under coloring $f^{(2)}$. If $C^{(2)}(v_2) \neq C^{(2)}(v_3)$, then let $f^{(3)} = f^{(2)}$, otherwise, we recolor the vertex v_3 similarly and denote the resulting coloring by $f^{(3)}$. Continuing in this way until $f^{(t)}$. For each suspend trail we do the above modifications. Then a k-AVDTC of H is obtained. ∎

Due to the above Proposition, we only need to concentrate on cubic graphs. Now we consider three types of cubic graphs which are defined as follows.

Definition 5 *Consider a class of cubic hamiltonian graphs as follows: there exists a hamiltonian cycle $u_1u_2\ldots u_nv_nv_{n-1}\ldots v_1$ such that all the matching edges are of the form u_iv_j $(1 \leq i, j \leq n)$, we use $\mathcal{H}_{2n}$ to denote the set of this kind of cubic hamiltonian graphs.*

The following definition can be found in [4].

Definition 6 *Consider two disjoint n-cycles $i_1i_2\ldots i_ni_1$ and $o_1o_2\ldots o_no_1$. Let π denote a permutation on n elements. Add to these two cycles the set of edges $i_jo_{\pi(j)}$ $(1 \leq j \leq n)$. We call the family of such graphs n-snares. If π is the identity permutation, we call the graph a drum and denote it by D_n.*

The well-known Generalized Petersen graph is defined below.

Definition 7 ([12]). *Generalized Petersen graph $G(n,k)$ $(n \geq 3, 1 \leq k < \frac{n}{2})$, is a graph with vertex set*

$$\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\},$$

and edge set

$$\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n-1\},$$

where subscripts are taken modulo n.

In this paper, we prove that each of these cubic graphs has a 5-AVDTC.

2 Main results

Theorem 8 *Let $n \geq 2$ be an integer. Then, for any graph $H_{2n} \in \mathcal{H}_{2n}$, $\chi_{at}(H_{2n}) = 5$.*

Proof. Using the notations in definition 5, we suppose that $u_1 v_s$, $v_n u_t$, $v_1 u_r$ and $u_n v_w$ are edges in $E(H_{2n})$, where $2 \leq s, r \leq n$ and $1 \leq t, w \leq n-1$. By Proposition 1, $\chi_{at}(H_{2n}) \geq 5$. Thus it suffices to give a 5-AVDTC of H_{2n}. To this end, we prove the following two claims according to the parity of n.

Claim 1: If $n \geq 3$ and n is odd, then $\chi_{at}(H_{2n}) = 5$.

Initially, define a proper total 5-coloring f of H_{2n} as: alternately color the vertices $u_1, u_2, \ldots, u_n$ by 3 and 4, and alternately color the edges $u_1u_2, u_2u_3, \ldots, u_{n-1}u_n$ by 2 and 1; alternately color the vertices $v_1, v_2, \ldots, v_n$ by 1 and 2, and alternately color the edges $v_1v_2, \ldots, v_{n-1}v_n$ by 3 and 4; color u_1v_1 by 4 and u_nv_n by 2; color the remaining edges by 5.

Next, we construct a 5-AVDTC of H_{2n} by recoloring some vertices and edges (if necessary) according to the parity of s, t, r and w.

Case 1.1. If both s and t are odd, then f is a 5-AVDTC of H_{2n}.

Case 1.2. s is even and t is odd. If r is odd, then exchange the color of v_1 and v_1u_1; else, recolor u_r by 5, u_rv_1 by 4, v_1u_1 by 5, and u_1v_s by 1.

Case 1.3. s is odd and t is even. This case is similar to case 1.2. If w is odd, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 2, u_nv_n by 5, and v_nu_t by 3.

Case 1.4. Both s and t are even.

- If r is odd, then exchange the color of v_1 and v_1u_1. Further, if w is odd, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 2, u_nv_n by 5, and v_nu_t by 3.
- If r is even, then recolor u_r by 5, u_rv_1 by 4, v_1u_1 by 5, u_1v_s by 1. Further, if w is odd, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 2, u_nv_n by 5, and v_nu_t by 3.

Claim 2: If $n \geq 2$ and n is even, then $\chi_{at}(H_{2n}) = 5$.

The coloring strategy of claim 2 is similar to that of claim 1, therefore, we re-use the same coloring symbol f for this situation.

Properly total 5-coloring of H_{2n} in the same way as that of claim 1, except that $f(u_nv_n) = 1$. Then we construct a 5-AVDTC of H_{2n} by recoloring some vertices and edges (if necessary) according to the parity of s, t, r and w.

Case 2.1. If s is odd and t is even, then f is a 5-AVDTC of H_{2n}.

Case 2.2. Both s and t are odd. If w is even, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 1, u_nv_n by 5, and v_nu_t by 4.

Case 2.3. Both s and t are even. This case is similar to case 2.2. If r is odd, then exchange the color of v_1 and v_1u_1; else, recolor u_r by 5, u_rv_1 by 4, v_1u_1 by 5, and u_1v_s by 1.

Case 2.4. s is even and t is odd.

- If r is odd, then exchange the color of v_1 and v_1u_1. Further, if w is even, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 1, u_nv_n by 5, and v_nu_t by 4.

- If r is even, then recolor u_r by 5, u_rv_1 by 4, v_1u_1 by 5, u_1v_s by 1. Further, if w is even, then exchange the color of u_n and u_nv_n; else, recolor v_w by 5, v_wu_n by 1, u_nv_n by 5, and v_nu_t by 4.

∎

We now turn to n-*snares*. The following three propositions come from [4].

Proposition 9 $\chi_{at}(D_n) = 5$.

Proposition 10 *Let G be an even snare. Then* $\chi_{at}(G) = 5$.

Proposition 11 *Let G be an odd snare containing a C_4. Then* $\chi_{at}(G) = 5$.

We will deal with the final case, i.e., odd *snares* without C_4. Since every 3-*snare* is a drum and the only 5-*snare* without C_4 is the Petersen graph, so we consider n-*snares* with $n \geq 7$ in the following context.

Lemma 12 *Let G be an odd snare without C_4. Then* $\chi_{at}(G) = 5$.

Proof. By Proposition 1, $\chi_{at}(G) \geq 5$. Thus it suffices to give a 5-AVDTC G. Let the two cycles of G be $i_1i_2\ldots i_ni_1$ and $o_1o_2\ldots o_no_1$ with $i_no_n \in E(G)$. Suppose i_1o_s, $i_{n-1}o_t$, o_1i_r, $o_{n-1}i_w \in E(G)$, where $2 \leq s,t,r,w \leq n-2$, since G contains no C_4.

Initially, define a proper total 5-coloring f of G as follows: alternately color the vertices $i_1, i_2, \ldots, i_{n-1}$ by 2 and 1, and alternately color the edges $i_1i_2, i_2i_3, \ldots, i_{n-2}i_{n-1}$ by 3 and 4, color i_n by 4, i_ni_1 by 1, i_ni_{n-1} by 2;

alternately color the vertices $o_1, o_2, \ldots, o_{n-1}$ by 4 and 3, and alternately color the edges of $o_1o_2, o_2o_3, \ldots, o_{n-2}o_{n-1}$ by 1 and 2, color o_n by 2, o_no_1 by 3, o_no_{n-1} by 4; color all the edges $i_to_{\pi(t)}$ by 5, where $1 \le t \le n$.

Next, we construct a 5-AVDTC of G by recoloring some vertices and edges (if necessary) according to the parity of s, t, r and w.

Case 1. If all of s, t, r and w are odd, then f is a 5-AVDTC of G.

Case 2. Only one of s, t, r and w is even, without loss of generality, suppose s is even. Recolor o_ni_n by 1, i_ni_1 by 5, i_1o_s by 4.

Case 3. Two of s, t, r and w are even. By symmetry, we only need to consider two subcases below.

Subcase 3.1. Both s and t are even. Recolor o_ni_n by 1, i_n by 3, i_ni_1 by 2, i_1 by 5, i_1o_s by 4, i_ni_{n-1} by 5, $i_{n-1}o_t$ by 4.

Subcase 3.2. Both s and r are even.

Subcase 3.2.1. $|s-t| = 1$.

Alternately recolor the edges $o_1o_2, o_2o_3, \ldots, o_{n-2}o_{n-1}$ by 2 and 1.

If $s = t+1$ then we do the following recoloring. Firstly, recolor i_1i_n by 5, i_n by 3, i_no_n by 4, o_no_{n-1} by 3, o_{n-1} by 5, $o_{n-1}i_w$ by 1, o_no_1 by 1. Secondly, exchange the color of o_t and o_to_{t+1}, ie., o_to_s, recolor o_s by 5, o_so_{s+1} by 3, o_si_1 by 1. Suppose $o_{s+1}i_p \in E(G)$. If p is even, then $f(o_{s+1}i_p) = 5$, else, exchange the color of o_s and o_so_{s+1}, i.e., $f(o_s) = 3$, $f(o_so_{s+1}) = 5$. Recolor $o_{s+1}i_p$ by 1.

Now suppose $t = s+1$. If $s = 2$, then recolor o_1 by 2, o_1o_2 by 4, o_2(i.e., o_s) by 5, o_so_t by 3, o_si_1 by 1, i_1i_n by 5, i_ni_{n-1} by 4, i_n by 3, i_no_n by 2, o_n by 4, o_no_{n-1} by 3, o_{n-1} by 5, $o_{n-1}i_w$ by 1, o_no_1 by 1. If $s > 2$, then recolor o_so_t by 4, o_t by 5, o_si_1 by 1, o_ti_{n-1} by 1, i_1i_n by 5, i_{n-1} by 4, i_n by 3, i_no_n by 4, o_no_1 by 1, o_no_{n-1} by 3, o_{n-1} by 1.

Subcase 3.2.2. $|s-t|>1$.

Firstly, exchange the color of $i_j i_{j+1}$ and i_{j+1} $(1 \le j \le n-2)$. Recolor i_n by 5, $i_n i_1$ by 4. Secondly, exchange the color of $o_j o_{j+1}$ and o_{j+1} $(1 \le j \le n-2)$. Recolor o_n by 3, $o_n o_1$ by 4, o_1 by 2, $o_n o_{n-1}$ by 2. Recolor $i_n o_n$ by 1.

We continue to do recoloring according to the value of s. If $s=2$, then recolor $i_1 o_s$, i.e., $i_1 o_2$ by 3, $o_2 o_1$ by 5, $o_1 i_r$ by 3, i_r by 5. If $s>2$, then recolor $i_1 o_s$ by 3, $o_s o_{s-1}$ by 5. Suppose $o_{s-1} i_p \in E(G)$. If p is odd, then recolor $o_{s-1} i_p$ by 3, else, recolor i_p by 5, $i_p o_{s-1}$ by 3.

Case 4. Three of s, t, r and w are even. Without loss of generality, suppose s, t, and r are even.

Recolor $o_s i_1$ by 4, i_1 by 5, $i_1 i_n$ by 2, $i_n i_{n-1}$ by 1, i_{n-1} by 5, $i_{n-1} o_t$ by 4, $i_n o_n$ by 3, $o_n o_1$ by 5, $o_1 i_r$ by 2.

Case 5. All of s, t, r and w are even.

Recolor $o_s i_1$ by 4, i_1 by 5, $i_1 i_n$ by 2, $i_n i_{n-1}$ by 1, i_{n-1} by 5, $i_{n-1} o_t$ by 4, $i_n o_n$ by 3, $o_n o_{n-1}$ by 5, $o_{n-1} i_w$ by 2, $o_n o_1$ by 4, o_1 by 5, $o_1 i_r$ by 2.

∎

Theorem 13 *Let G be an n-snare with $n \ge 3$, then $\chi_{at}(G)=5$.*

Proof. Since Petersen graph has a 5-AVDTC, the conclusion follows by Proposition 9, Proposition 10, Proposition 11 and Lemma 12. ∎

We conclude this section by proving that Generalized Petersen graphs have an AVDTC with only 5 colors. The greatest common divisor (gcd) of two positive integers a and b is the largest divisor common to a and b. For example, gcd (2, 7)=1, gcd (12, 18)=6, and gcd (15, 90)=15. Please keep in mind that subscripts are taken modulo n in the following.

Theorem 14 *If $n \geq 3$ and $1 \leq k < \frac{n}{2}$, then $\chi_{at}(G(n,k)) = 5$.*

Proof. By Proposition 1, $\chi_{at}(G(n,k)) \geq 5$, so it suffices to give a 5-AVDTC of $G(n,k)$. Let g=gcd(n, k) and $p = \frac{n}{g}$. Observe that the subgraph induced by $v_0, v_1, \ldots, v_{n-1}$ is the disjoint union of g cycles with the same length p.

Case 1. n is even.

Subcase 1.1. p is even.

Alternately color the vertices $u_0, u_1, \ldots, u_{n-1}$ by 1 and 2, and alternately color the edges $u_0u_1, \ldots, u_{n-2}u_{n-1}, u_{n-1}u_0$ by 3 and 4. For $0 \leq i \leq g-1$, alternately color the vertices of each cycle $v_iv_{i+k}\cdots v_{i+(p-1)k}$ by 3 and 4, and alternately color the edges of each cycle $v_iv_{i+k}\cdots v_{i+(p-1)k}$ by 1 and 2. For $0 \leq i \leq n-1$, color u_iv_i by 5. It is clear that, for $0 \leq i \leq n-1$,

$$C(u_i) = \begin{cases} \{1,3,4,5\}, & i \text{ is even}, \\ \{2,3,4,5\}, & i \text{ is odd}. \end{cases} \tag{1}$$

And for $0 \leq i \leq g-1$, $0 \leq j \leq p-1$,

$$C(v_{i+jk}) = \begin{cases} \{1,2,3,5\}, & j \text{ is even}, \\ \{1,2,4,5\}, & j \text{ is odd}. \end{cases} \tag{2}$$

Obviously, f is a 5-AVDTC of $G(n,k)$.

Subcase 1.2. p is odd.

For $0 \leq i \leq n-1$, let

$$f(u_i) = \begin{cases} 1, & i \text{ is even}, \\ 2, & i \text{ is odd}. \end{cases} \tag{3}$$

and

$$f(u_iu_{i+1}) = \begin{cases} 5, & i \text{ is even and } 0 \leq i \leq g-2, \\ 3, & i \text{ is even and } g \leq i \leq n-2, \\ 4, & \text{Otherwise}. \end{cases} \tag{4}$$

For $0 \leq i \leq g-1$, alternately color the vertices of $v_iv_{i+k}\cdots v_{i+(p-2)k}$ by 4 and 3. Further, if i is even, then alternately color the edges of

$v_iv_{i+k}\cdots v_{i+(p-2)k}$ by 2 and 1, color $v_{i+(p-2)k}v_{i+(p-1)k}$ by 4, $v_{i+(p-1)k}$ by 2, $v_{i+(p-1)k}v_i$ by 1; if i is odd, then alternately color the edges of $v_iv_{i+k}\cdots v_{i+(p-2)k}$ by 1 and 2, color $v_{i+(p-2)k}v_{i+(p-1)k}$ by 4, $v_{i+(p-1)k}$ by 1, $v_{i+(p-1)k}v_i$ by 2. Let

$$f(u_iv_i) = \begin{cases} 3, & 0 \le i \le g-1, \\ 5, & g \le i \le n-1. \end{cases} \tag{5}$$

Obviously, f is a proper total 5-coloring of $G(n,k)$.

From above coloring, we can obtain the color set of each vertex of $G(n,k)$ as follows. For $0 \le i \le n-1$,

$$C(u_i) = \begin{cases} \{1,3,4,5\}, & i \text{ is even}, \\ \{2,3,4,5\}, & i \text{ is odd}. \end{cases} \tag{6}$$

And for $0 \le i \le g-1$, $0 \le j \le p-1$,

$$C(v_{i+jk}) = \begin{cases} \{1,2,3,4\}, & j = 0, \\ \{1,2,3,5\}, & j \text{ is odd and } 1 \le j \le p-3, \\ \{1,2,4,5\}, & j \text{ is even and } 1 \le j \le p-1, \\ \{2,3,4,5\}, & i \text{ is even}, \ j = p-2, \\ \{1,3,4,5\}, & i \text{ is odd}, \ j = p-2. \end{cases} \tag{7}$$

It can be verified that f is a 5-AVDTC of $G(n,k)$.

Case 2. n is odd.

If $g=1$ then $G(n,k)$ is an odd-*snare*. By Proposition 11 and Lemma 12, $G(n,k)$ has a 5-AVDTC. Thus we assume $g \ge 3$. For $0 \le i \le n-1$, let

$$f(u_i) = \begin{cases} 5, & i = 0, \\ 4, & i = 1, \\ 5, & i = 2, \\ 1, & i \text{ is odd and } i \ne 1, \\ 2, & \text{otherwise}. \end{cases} \tag{8}$$

$$f(u_iu_{i+1}) = \begin{cases} 1, & i = 0, \\ 2, & i = 1, \\ 4, & 2 \le i \le n-1 \text{ and } i \text{ is even}, \\ 5, & 3 \le i < g-1 \text{ and } i \text{ is odd}, \\ 3, & \text{otherwise}. \end{cases} \tag{9}$$

and

$$f(u_iv_i) = \begin{cases} 3, & 0 \le i \le g-1, \\ 5, & \text{otherwise.} \end{cases} \tag{10}$$

For $0 \le i \le g-1$, let

$$f(v_i) = \begin{cases} 2, & i = 0, \\ 5, & i = 1, \\ 2, & i = 2, \\ 2, & 3 \le i \le g-1 \textit{ and } i \textit{ is odd}, \\ 1, & \text{otherwise.} \end{cases} \tag{11}$$

For $0 \le i \le g-1$, since $i + (p-1)k = n - k + i \ (mod\ n)$, while $(n - k + i) + (i + k) = n + 2i$, recall that n is odd, we conclude that $i+(p-1)k$ and $i+k$ have different parity. The remaining edges are colored in the following way.

For $i = 0, 1, 2$, or $3 \le i \le g-1$ and i is odd. If $i + k$ is even, then $f(v_{i+k}) = 1$ and $f(v_iv_{i+k}) = 4$, alternately color the vertices of $v_{i+2k} \cdots v_{i+(p-1)k}$ by 3 and 4, and alternately color the edges of $v_{i+k} \cdots v_{i+(p-1)k}v_i$ by 2 and 1. If $i+(p-1)k$ is even, then $f(v_{i+(p-1)k}) = 1$ and $f(v_iv_{i+(p-1)k}) = 4$, alternately color the vertices of $v_{i+(p-2)k} \cdots v_{i+k}$ by 3 and 4, and alternately color the edges of $v_{i+(p-1)k} \cdots v_{i+k}v_i$ by 2 and 1.

For $3 \le i \le g-1$ and i is even. If $i + k$ is odd, then $f(v_{i+k}) = 2$ and $f(v_iv_{i+k}) = 4$, alternately color the vertices of $v_{i+2k} \cdots v_{i+(p-1)k}$ by 3 and 4, and alternately color the edges of $v_{i+k} \cdots v_{i+(p-1)k}v_i$ by 1 and 2. If $i+(p-1)k$ is odd, then $f(v_{i+(p-1)k}) = 2$ and $f(v_iv_{i+(p-1)k}) = 4$, alternately color the vertices of $v_{i+(p-2)k} \cdots v_{i+k}$ by 3 and 4, and alternately color the edges of $v_{i+(p-1)k} \cdots v_{i+k}v_i$ by 1 and 2.

It is obvious that f is a proper total 5-coloring of $G(n, k)$. The color set of each vertex of $G(n,k)$ is presented as follows. For $0 \le i \le n-1$,

$$C(u_i) = \begin{cases} \{1,3,4,5\}, & i = 0, \\ \{1,2,3,4\}, & i = 1, \\ \{2,3,4,5\}, & i \text{ is even and } i \neq 0, \\ \{1,3,4,5\}, & \text{otherwise}. \end{cases} \quad (12)$$

For $0 \le i \le g-1$, $1 \le j \le p-1$, the color sets of vertices $v_{i+k}, \ldots, v_{i+(p-1)k}$ are $\{1,2,4,5\}$ and $\{1,2,3,5\}$ alternately or reverse. In addition, for $0 \le i \le g-1$,

$$C(v_i) = \begin{cases} \{1,3,4,5\}, & i = 1, \\ \{1,2,3,4\}, & \text{otherwise}. \end{cases} \quad (13)$$

It can be verified that f is a 5-AVDTC of $G(n,k)$.

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3 Concluding remarks

In this work, we prove that a class of cubic hamiltonian graphs have an AVDTC with only 5 colors. We also solve the tough problem (Lemma 12 in this paper) that Hulgan didn't solve in his Ph.D. Thesis [4]. In addition, we totally determine that Generalized Petersen graphs have an AVDTC with only 5 colors. To our best knowledge, it is still challenging to answer the question that whether 5 is an upper bound for any cubic graph G. Our work provides a basis for attacking this problem.

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