ON THE CONSTRUCTION OF COGRAPH COLOR CRITICAL GRAPHS

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ABSTRACT. A *cograph* is a simple graph that does not contain an induced path on 4 vertices. A graph G is k-*ℭcolorable* if the vertices of G can be colored in k colors such that, for each color, the subgraph induced by the vertices assigned the color is a cograph. A graph that is k-ℭcolorable and is not $(k-1)$-ℭcolorable, but becomes $(k-1)$-ℭcolorable whenever a vertex is removed, is called k-*ℭcritical* graph. Two general constructions are provided that produce ℭcritical graphs from color critical graphs and hypergraphs. A characterization is also given for when a general composition of graphs (path-joins) is ℭcritical. The characterization is used to provide an upper bound for the fewest number of vertices of a k-ℭcritical graph.

1. INTRODUCTION

For notational simplicity, we define for a positive integer t, $[t] = \{1, \ldots, t\}$. In the definitions that follow, let k be a positive integer. We reserve $C_k = \{c_1, \ldots, c_k\}$ to be a finite set of size k whose elements $c_1, \ldots, c_k$ are called *colors*. Let G be a finite, undirected, simple graph with vertices $V(G)$ and edges $E(G)$. A function $\gamma : V(G) \to C_k$ is called a *k-coloring* of G (or *coloring* if k is not specified). The function γ is said to *color* G and to *use* colors from C_k. If for some vertex v and color c_i, $\gamma(v) = c_i$, then v is said to be *colored* c_i by γ (or just colored c_i if γ is understood). A set of vertices is *monochromatic* under γ if γ colors all the vertices of the set the same color. The *color classes* of γ are the elements of the partition $\{V_{c_1}, \ldots, V_{c_k}\}$ of preimages of members of C_k where $V_{c_i} = \gamma^{-1}(\{c_i\})$ for $i \in [k]$.

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Let $\mathcal{B}$ be a (not necessarily finite) set of finite, undirected, simple graphs. A coloring γ of G using colors from C_k is a *k-$_{\mathcal{B}}$coloring* if the induced subgraph of each color class of γ is a member of $\mathcal{B}$ (i.e. if $G[V_{c_i}] \in \mathcal{B}$ for all $i \in [k]$). A graph G is *k-$_{\mathcal{B}}$colorable*, if there exists a k-$_{\mathcal{B}}$coloring of G. The *$_{\mathcal{B}}$chromatic* number of G, denoted $\chi_{\mathcal{B}}(G)$, is the minimum integer k for which G is k-$_{\mathcal{B}}$colorable. If $\chi_{\mathcal{B}}(G) = k$, then graph G is said to be *k-$_{\mathcal{B}}$chromatic.* We define a 1-$_{\mathcal{B}}$*critical* graph to be K_1, the single vertex graph. For $k \geq 2$, a k-$_{\mathcal{B}}$chromatic graph G is *k-$_{\mathcal{B}}$critical* if $G - v$ is $(k-1)$-$_{\mathcal{B}}$colorable for all vertices $v \in V(G)$.

Let $\mathcal{E}$ be the set of finite empty graphs (no edges). Connecting the classical study of the chromatic number of graphs to the definitions here, a proper k-coloring of G is a k-$_{\mathcal{E}}$coloring of G. Moreover, G is k-colorable (respectively k-chromatic, k-vertex-critical) is equivalent to G is k-$_{\mathcal{E}}$colorable (respectively k-$_{\mathcal{E}}$chromatic, k-$_{\mathcal{E}}$critical). These definitions extend to hypergraphs (edges need not have size two). A hypergraph whose edges all have the same size is called a *uniform* hypergraph. A proper k-coloring of G is a k-$_{\mathcal{E}}$coloring of G (i.e. no edge is monochromatic). A graph or hypergraph G is k-critical, however, if G is k-chromatic and $G-e$ is $(k-1)$-colorable for each edge e of G. Graphs and uniform hypergraphs that are k-critical have been extensively studied and exist for all k and edge sizes (see Toft [9]).

Two graphs are *disjoint* if they have no vertex in common. If H_1 and H_2 are two disjoint graphs, then their *disjoint union*, $H_1 \cup H_2$, is the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Moroever, their *join*, $H_1 \oplus H_2$, is the graph obtained from their disjoint union by adding every edge that has one of its vertices in $V(H_1)$ and the other in $V(H_2)$. The *complement* of a graph G, denoted $\overline{G}$, is the graph with vertices $V(\overline{G}) = V(G)$ and edges $E(\overline{G}) = \{uv : u, v \in V(\overline{G}), uv \notin E(G)\}$.

Let P_ℓ be a path on ℓ vertices. A *cograph* (*complement reducible* graph, see [4]) is a graph that contains no induced P_4 subgraph. Let $\mathcal{C}$ be the set of all cographs. This paper sets its focus on constructions of graphs that are k-$_{\mathcal{C}}$critical.

In 1968, Chartrand, Geller and Hedetniemi [2] introduced the ℓ-chromatic number of a graph, denoted χ_ℓ, which is equivalent to the $_{\mathcal{B}}$chromatic number when $\mathcal{B}$ is the set of graphs that do not contain a $P_{\ell+1}$ (regardless of whether the path is induced or not). Cockayne [3] defined the P-chromatic number where P is a property that defines the elements of $\mathcal{B}$. The number was denoted by χ_P and was defined in the more general context of uniform hypergraphs.

Mynhardt and Broere [8] were the first to show that for each k, there exists a graph G with $\chi_{\mathcal{C}}(G) > k$. Moreover, Brown and Corneil [1] first defined k-$_{\mathcal{C}}$critical graphs and gave a construction that showed they exist for all k. In both of these papers, the results were in terms of the more general k-$_{\mathcal{B}}$colorings. A k-$_{\mathcal{C}}$coloring is called a P_4-coloring in [8] and a $-P_4$ k-coloring in [1].

In 2010, Gimbel and Nešetřil [6] showed how $_{\mathcal{C}}$colorings fit in an axiomatization of well-studied coloring functions. They also provided bounds and complexity results for the $_{\mathcal{C}}$chromatic number which they called the c-chromatic number, denoted $c(G)$. Most recently, Dorbec, Montassier and Ochem [5] constructed 3-$_{\mathcal{C}}$chromatic triangle-free planar graphs as well as proving some associated complexity results.

Trivally, every k-coloring of a graph G is also a k-$_{\mathcal{C}}$coloring of G. Moreover, since the complement of a P_4 is also a P_4 subgraph, a coloring is a $_{\mathcal{C}}$coloring of G if and only if it is $_{\mathcal{C}}$coloring of the complement of G (as observed in [6]).

The main results of this paper are Constructions 6 and 11, and to a somewhat lesser degree, Lemmas 3 and 4. We now describe the significance of the results.

Although every k-$_{\mathcal{C}}$chromatic graph contains a k-$_{\mathcal{C}}$critical subgraph, until now only one general construction was known to produce k-$_{\mathcal{C}}$critical graphs explicitly. This construction was provided by Brown and Corneil [1] when they introduced the topic in the more general context of k-$_{\mathcal{B}}$critical graphs. The construction uses k-$_{\mathcal{B}}$critical graphs as input to build $(k+1)$-$_{\mathcal{B}}$critical graphs. In this paper, Lemma 3 extends the construction for $_{\mathcal{C}}$colorings, by allowing s-$_{\mathcal{C}}$critical graphs, $s \le k$, to be used as input. Lemma 3 gives a complete characterization of the inputs for which the construction produces $(k+1)$-$_{\mathcal{C}}$critical graphs. Consequently, new $(k+1)$-$_{\mathcal{C}}$critical graphs are described explicitly.

The natural quest to determine the fewest number of vertices in a k-$_{\mathcal{C}}$critical graph begins with a partial answer in Lemma 4. The construction of Lemma 3 is used in Lemma 4 to find an upper bound to this parameter that is polynomial in k.

Constructions 6 and 11 produce a $_{\mathcal{C}}$critical graph from a color critical hypergraph or graph. For $k \ge 3$, both constructions can be used to produce an infinite set of k-$_{\mathcal{C}}$critical graphs. In contrast, only two non-isomorphic 3-$_{\mathcal{C}}$critical graphs are produced from Lemma 3 or the construction in [1] (and only a finite number of k-$_{\mathcal{C}}$critical graphs if each of the input graphs are built from the construction as well). For $k \le 4$, Construction 11 produces k-$_{\mathcal{C}}$critical graphs that are outerplanar whereas all graphs produced by Construction 6 are nonplanar. We conclude the paper with some observations regarding the 4-$_{\mathcal{C}}$critical planar graphs produced by the constructions.

2. Coloring Path-Joins

We start this section with a definition of a graph that will be used throughout the paper.

Definition 1. *For $t \geq 2$, let $H_1, \ldots, H_t$ be a sequence of disjoint graphs. The* t-path-join *(or just* path-join*) of the sequence is defined to be*

$$P(H_1, \ldots, H_t) = \bigcup_{i=1}^{t-1} (H_i \oplus H_{i+1}).$$

Each of the H_i is called an element *of the path-join.*

Path-join graphs have a structure that is sufficiently general but at the same time controls where any induced P_4 subgraphs might be contained. The next lemma makes this last statement precise. The lemma is not new but is provided in this form for quick reference later.

Lemma 2. *Let $t \in \{2,3,4\}$ and $H_1, \ldots, H_t$ be disjoint graphs. If P is an induced P_4 of $P(H_1, \ldots, H_t)$, then either*

(1) *for some $i \in [t]$, P is a subgraph of H_i, or*
(2) *$t = 4$ and for all $i \in [4]$, P contains a vertex from H_i.*

Proof. Suppose (1) is not true. Using the language of Golumbic[7][1], since the path P is an induced subgraph of the composition graph $G = P_t(H_1, \ldots, H_t) = P(H_1, \ldots, H_t)$, $E(P)$ is contained in an implication class A of G and hence contained in its symmetric closure $\hat{A}$. Thus by Theorem 5.8 of [7], since (1) is not true, $\hat{A} \cap E(H_i) = \emptyset$ for all $i \in [t]$. Moreover, since $\overline{G} = \overline{P_t(H_1, \ldots, H_t)} = \overline{P_t}(\overline{H_1}, \ldots, \overline{H_t})$ and $\overline{P} \cong P_4$, $E(\overline{P})$ is contained in the symmetric closure $\hat{B}$ of some implication class B of $\overline{G}$ implying $\hat{B} \cap E(\overline{H_i}) = \emptyset$ for all $i \in [t]$. Thus, for $i \in [t]$, no pair of vertices of P are contained in H_i and (2) is true. □

We now focus our attention on 4-path-join graphs and provide a complete characterization of their $\mathfrak{e}$coloring properties.

Lemma 3. *Let H_1, H_2, H_3 and H_4 be disjoint graphs and let $k_i = \chi_{\mathfrak{e}}(H_i)$ for $i \in [4]$ with maximum $k^{\max} = \max\{k_1, k_2, k_3, k_4\}$ and sum $k^{\Sigma} = k_1 + k_2 + k_3 + k_4$. Let $G = P(H_1, H_2, H_3, H_4)$. For every positive integer ℓ,*

(1) *G is ℓ-$\mathfrak{e}$colorable if and only if $\ell \geq \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$,*
(2) *G is ℓ-$\mathfrak{e}$chromatic if and only if $\ell = \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$, and*
(3) *G is ℓ-$\mathfrak{e}$critical if and only if $\ell \geq k^{\max} + 1$, $k^{\Sigma} = 3(\ell - 1) + 1$, and H_i is k_i-$\mathfrak{e}$critical for all $i \in [4]$.*

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[1] In particular, the use of the terms *composition graph* and *implication class* in this proof.

Proof. In what follows, let ℓ be a some positive integer.

Suppose there exists an ℓ-$\mathfrak{e}$coloring, $\gamma : V(G) \to C_\ell$, of G. For $i \in [4]$, $j \in [\ell]$, let

$$I(H_i, c_j) = \begin{cases} 1 & \text{if } \gamma(v) = c_j \text{ for some vertex } v \text{ in } H_i \\ 0 & \text{otherwise.} \end{cases}$$

We shall count the number of these subgraph/color pairs in two ways.

For $j \in [\ell]$, let $I_j = \{i \in [4] : \text{some vertex } v \text{ of } H_i \text{ has } \gamma(v) = c_j\}$. Now, if $|I_j| = 4$ for some $j \in [\ell]$, then there is an induced P_4 of G all of whose vertices are colored c_j under $\gamma(v)$ formed by selecting a single vertex colored c_j from each of H_1, H_2, H_3, H_4 guaranteed by the size of I_j. This contradicts that γ is a ℓ-$\mathfrak{e}$coloring. Thus $|I_j| \leq 3$ for all $j \in [\ell]$ and hence

$$\sum_{j=1}^{\ell} \sum_{i=1}^{4} I(H_i, c_j) = \sum_{j=1}^{\ell} |I_j| \leq 3\ell.$$

On the other hand, considering the induced $\mathfrak{e}$colorings $\gamma_i = \gamma|_{V(H_i)}$ for $i \in [4]$, we have $\ell \geq k^{\max}$ and

$$\sum_{i=1}^{4} \sum_{j=1}^{\ell} I(H_i, c_j) = \sum_{i=1}^{4} |\gamma_i(V(H_i))| \geq \sum_{i=1}^{4} \chi_{\mathfrak{e}}(H_i) = k^{\Sigma}.$$

Thus we have $k^{\Sigma} \leq 3\ell$ and hence $\ell \geq \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$.

Suppose now that $\ell \geq \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$. We shall construct an ℓ-$\mathfrak{e}$coloring of G. For $i \in [4]$, let $k_i' = \ell - k_i$. Then $k_1' + k_2' + k_3' + k_4' = 4\ell - k^{\Sigma} \geq \ell$. Hence we can find a covering B_1', B_2', B_3', B_4' of the set of colors C_ℓ such that $C_\ell = B_1' \cup B_2' \cup B_3' \cup B_4'$ and $|B_i'| = k_i'$ (note that some of these sets may be empty). Now for each $i \in [4]$, $\chi_{\mathfrak{e}}(H_i) = k_i$, and so there exists a k_i-$\mathfrak{e}$coloring of H_i, $\gamma_i : V(H_i) \to C_\ell \setminus B_i'$. Let $\gamma : V(G) \to C_\ell$ be the functional extension of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. We claim that γ is a ℓ-$\mathfrak{e}$coloring of G. By Lemma 2, an induced P_4, P, of G is either contained in H_i for some $i \in [4]$, or P contains a vertex from each element of the path-join G. If P is contained in an H_i, then P is not monochromatic under $\gamma_i = \gamma|_{V(H_i)}$. Suppose therefore that P contains a vertex from each element of the path-join G. Consider an arbitrary color c_j of C_ℓ. By construction, $c_j \in B_i'$ for some $i \in [4]$ and hence γ_i does not use c_j to color the vertices of H_i. Thus the vertex of P in H_i is not colored c_j under γ. Since c_j is arbitrary, P is not monochromatic under γ. Hence G is ℓ-$\mathfrak{e}$colorable.

Therefore (1) is true.

Part (2) follows by applying (1) twice to G with the fact that two integers satisfy $\ell - 1 < \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\} \leq \ell$ if and only if $\ell = \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$.

We use the following in the next two paragraphs. Consider a vertex h of some H_t, $t \in [4]$. If H_t contains only vertex h, then $k^{\Sigma} \leq 3k^{\max} + 1$ since each $k_i \leq k^{\max}$, and $k_t = 1$. Moreover, we can $k^{\max}$-$\mathfrak{e}$color each of the other elements of the path-join G to get a $k^{\max}$-$\mathfrak{e}$coloring of $G-h$ (note that in this case, H_t is 1-$\mathfrak{e}$critical). Otherwise, if H_t has more than one vertex, let $H'_t = H_t - h$ and define $x(h) = k_t - \chi_{\mathfrak{e}}(H_t - h) \in \{0,1\}$. For $i \in [4]$, $i \neq t$, let $H'_i = H_i$. Then the path-join $G'_h = P(H'_1, H'_2, H'_3, H'_4) = G - h$ satisfies $\chi_{\mathfrak{e}}(H'_1) + \chi_{\mathfrak{e}}(H'_2) + \chi_{\mathfrak{e}}(H'_3) + \chi_{\mathfrak{e}}(H'_4) = k^{\Sigma} - x(h)$.

Suppose now, that G is ℓ-$\mathfrak{e}$critical. By (2), $\ell = \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$. If $\ell = k^{\max}$, then we have $\chi_{\mathfrak{e}}(H_i) = k^{\max}$ for some $i \in [4]$, and by choosing $h \in V(H_j)$ for some $j \in [4]$, $j \neq i$, we find that H_i is a subgraph of $G-h$ with $\chi_{\mathfrak{e}}(G-h) \geq \chi_{\mathfrak{e}}(H_i) = k^{\max} = \ell$, a contradiction. Thus $\ell \geq k^{\max}+1$. We therefore have $\ell = \lceil k^{\Sigma}/3 \rceil$ and hence $k^{\Sigma} \geq 3(\ell-1)+1$. Now consider $G-h$ for a vertex h of some H_t, $t \in [4]$. If $V(H_t) = \{h\}$, then, from the previous paragraph, H_t is 1-$\mathfrak{e}$critical, $k^{\Sigma} \leq 3k^{\max}+1$ and we can $k^{\max}$-$\mathfrak{e}$color $G-h$ and so $\ell - 1 \leq k^{\max}$. Thus $\ell-1 = k^{\max}$ and hence $k^{\Sigma} \leq 3(\ell-1)+1$. If $V(H_t) \neq \{h\}$, then, again from the previous paragraph, $G-h$ is $(\ell-1)$-$\mathfrak{e}$colorable implies by (1) that $\ell - 1 \geq \left\lceil \frac{k^{\Sigma}-x(h)}{3} \right\rceil \geq \frac{k^{\Sigma}-1}{3}$. But $\ell = \lceil k^{\Sigma}/3 \rceil$ and so $x(h) = 1$. Hence, H_t is k_t-$\mathfrak{e}$critical and $k^{\Sigma} \leq 3(\ell-1)+1$. In both cases, we have that $k^{\Sigma} = 3(\ell-1)+1$ and H_t is k_t-$\mathfrak{e}$critical.

Conversely, suppose $\ell \geq k^{\max}+1$, $k^{\Sigma} = 3(\ell-1)+1$, and each H_i is k_i-$\mathfrak{e}$critical, $i \in [4]$. Then $\lceil k^{\Sigma}/3 \rceil = \ell$ and so $\ell = \max\{k^{\max}, \lceil k^{\Sigma}/3 \rceil\}$. Thus by (2), G is ℓ-$\mathfrak{e}$chromatic. Consider a vertex h of some H_t, $t \in [4]$. If H_t contains only vertex h, then from above, $G-h$ is $k^{\max}$-$\mathfrak{e}$colorable and hence $(\ell-1)$-$\mathfrak{e}$colorable since $\ell - 1 \geq k^{\max}$. Otherwise, $x(h) = 1$ since H_t is k_t-$\mathfrak{e}$critical and $G-h$ is a path-join $G' = (H'_1, H'_2, H'_3, H'_4)$ with $\chi_{\mathfrak{e}}(H'_1)+\chi_{\mathfrak{e}}(H'_2)+\chi_{\mathfrak{e}}(H'_3)+\chi_{\mathfrak{e}}(H'_4) = k^{\Sigma}-1$, and $\lceil k^{\Sigma-1}/3 \rceil = \ell-1 \geq k^{\max}$. Thus by (1), $G-h$ is $(\ell-1)$-$\mathfrak{e}$colorable. Therefore, G is ℓ-$\mathfrak{e}$critical and (3) is true. □

The path-join $P(G_1, G_2, G_3, G_4)$ for disjoint copies of a graph G is described as the composition graph $P_4[G]$ in Mynhardt and Broere [8].[2] Applying Lemma 1 of [8], they showed that $\chi_{\mathfrak{e}}(P_4[G]) > \chi_{\mathfrak{e}}(G)$ for all graphs G. Using Lemma 3, we have that $\chi_{\mathfrak{e}}(P_4[G]) = \left\lceil \frac{4\chi_{\mathfrak{e}}(G)}{3} \right\rceil$ as well as that $P_4[G]$ is s-$\mathfrak{e}$critical if and only if $s \equiv 2 \bmod 4$ and G is $(\frac{3}{4}(s-2)+1)$-$\mathfrak{e}$critical.

When P is chosen to be the cograph property, Theorem 2.8 of [1] is implied by Lemma 3 using the case when $H_1 \cong K_1$ and $k_2 = k_3 = k_4$. Note that G in Theorem 2.8 is forced to be P_4 for the cograph property since the only 2-$\mathfrak{e}$critical graph is P_4.

Since there is only one 1-$\mathfrak{e}$critical graph, namely K_1, only one 2-$\mathfrak{e}$critical 4-path-join graph can be constructed. We define $PJ_2 = P(H_1, H_2, H_3, H_4)$ where each element is isomorphic to K_1. For $k \geq 2$, define $PJ_{k+1} =$

[2] Described as the composition graph $P_4[G,G,G,G]$ in [7].

$P(H_1, H_2, H_3, H_4)$ with disjoint elements where for $i \in [3]$, H_i is a copy of PJ_k, and H_4 is a copy of K_1. Using Lemma 3 inductively, we have that PJ_{k+1} is $(k+1)$-ecritical. This was proven first in Theorem 2.8 of Brown and Corneil [1] where the graph PJ_3 and its complement were first described. Gimbel and Nešetřil [6] also describe PJ_3 but give it the notation $(P_4)'$.

It is natural to ask questions regarding the structure of those k-ecritical graphs with the fewest number vertices. For $k \geq 1$, we define $f(k)$ to be the fewest number vertices of a k-ecritical graph and in what follows, let A_k be a k-ecritical graph with $f(k)$ vertices. From the previous paragraphs, we have $f(1) = 1$ and $f(2) = 4$.

Question 1 in the Remarks of [1] may be rephrased, when restricted to the property of cographs and using the terminology of this paper, to ask whether the smallest $(k+1)$-ecritical graph is a $P(A_k, A_k, A_k, K_1)$ (or its complement). The authors answered Question 1 negatively at the end of the paper by mentioning that the circulant graph of order 11 with distances $\{1, 4\}$ is 3-ecritical, thus showing $f(3) \leq 11$. A 10 vertex 4-regular 3-ecritical graph named J (and implicitly a 5-regular one forming the complement of J) is given in [6]. A computational search reported in [10] found no 3-ecritical graph on less than 10 vertices. Thus $f(3) = 10$.

A variety of ecritical graphs can be constructed from Lemma 3. For example, for $k = 4$ (abusing the notation), both path-joins $P(J, J, J, K_1)$ and $P(J, J, PJ_2, PJ_2)$ are 4-ecritical graphs, the first with 31 vertices, the second with 28 vertices. Although the answer to Question 1 was already resolved, $P(J, J, PJ_2, PJ_2)$ illustrates that the answer is still negative when the question of whether the smallest 4-path-join k-ecritical graph is a $P(A_{k-1}, A_{k-1}, A_{k-1}, K_1)$ (or its complement). Moreover, $P(A_{k-1}, A_{k-1}, A_{k-1}, K_1)$ provides the upper bound $f(k) \leq 3^{k-1} + 3^{k-2}$. The following lemma improves this upper bound for $f(k)$ and disproves Conjecture 3.20 of [10] that $f(k) = 2^k + 2^{k-1} + 1$.

Lemma 4. *For $k \geq 4$, the fewest number of vertices of a k-ecritical graph satisfies*

$$f(k) < \left(\tfrac{10}{4^5}\right) k^{\frac{1}{\log_4(5)-1}} < \left(\tfrac{10}{4^5}\right) k^{6.22}.$$

Proof. For each positive integer k', $k' \leq k$, every k-ecritical graph contains a k'-ecritical graph (see Theorem 2.3 of [1]). Hence f is monotonically increasing.

Using (3) of Lemma 3, each row of Table 1 lists a k-ecritical graph, a (loose but useful) recursive upper bound on the number of vertices of the graph, and the statement that $\lfloor \frac{4}{5} k \rfloor$ is an upper bound to the input of f used for the recursive upper bound. The information in the table thus proves for $k \geq 4$ (i.e. $t \geq 1$) that

(1) $$f(k) \leq 4f\left(\left\lfloor \tfrac{4}{5}k \right\rfloor\right).$$

TABLE 1. k-ecritical graphs with bounds on the number of vertices.

k	k-ecritical graph	upper bound on the number of vertices	$\lfloor \frac{4}{5}k \rfloor \geq$
$4t$	$P(A_{3t-1}, A_{3t-1}, A_{3t}, A_{3t})$	$4f(3t)$	$3t$
$4t+1$	$P(A_{3t}, A_{3t}, A_{3t}, A_{3t+1})$	$4f(3t+1)$	$3t+1$
$4t+2$	$P(A_{3t+1}, A_{3t+1}, A_{3t+1}, A_{3t+1})$	$4f(3t+1)$	$3t+1$
$4t+3$	$P(A_{3t+1}, A_{3t+2}, A_{3t+2}, A_{3t+2})$	$4f(3t+2)$	$3t+2$

Let $k \geq 4$, $r+1 = \left\lceil \log_{\left(\frac{5}{4}\right)}(k) \right\rceil$, and note that $r+1 \geq 7$. Using the monotonicity of f, the recursive inequality from (1), and the fact $\lfloor x \lfloor y \rfloor \rfloor \leq \lfloor xy \rfloor$ for all nonnegative reals x, y, each once per recursive step, we have

$$\begin{aligned} f(k) &\leq f\left(\left\lfloor \left(\tfrac{5}{4}\right)^{r+1} \right\rfloor\right) \leq 4f\left(\left\lfloor \tfrac{4}{5}\left\lfloor \left(\tfrac{5}{4}\right)^{r+1} \right\rfloor \right\rfloor\right) \leq 4f\left(\left\lfloor \left(\tfrac{5}{4}\right)^{r} \right\rfloor\right) \\ &\leq 4^2 f\left(\left\lfloor \tfrac{4}{5}\left\lfloor \left(\tfrac{5}{4}\right)^{r} \right\rfloor \right\rfloor\right) \leq 4^2 f\left(\left\lfloor \left(\tfrac{5}{4}\right)^{r-1} \right\rfloor\right) \\ &\leq \cdots \leq 4^{r-5} f\left(\left\lfloor \left(\tfrac{5}{4}\right)^{6} \right\rfloor\right) \\ &= 4^{r-5} f(3) < \left(\tfrac{10}{4^5}\right) 4^{\log_{(5/4)}(k)} = \left(\tfrac{10}{4^5}\right) k^{\frac{1}{\log_4(5)-1}}. \end{aligned}$$

□

We finish this section by proving a strong coloring property of PJ_k that will be used in Lemma 9 for one of the main constructions.

Lemma 5. *Let $k \geq 2$. For every non-constant function $m : V(PJ_k) \to C_k$, there is a k-ecoloring $\gamma : V(PJ_k) \to C_k$ such that $\gamma(v) \neq m(v)$ for all vertices v of PJ_k.*

Proof. For $k \geq 2$, let $S(k)$ be the statement of lemma for the given k. We will prove $S(k)$ is true by induction.

For a given non-constant function $m : V(PJ_2) \to \{c_1, c_2\}$, we define γ such that $\gamma(v)$ is the only color of $C_2 \setminus \{m(v)\}$ for all $v \in V(PJ_2)$. Since m is non-constant, γ is as well, and hence γ is a 2-ecoloring of PJ_2, a P_4, with $\gamma(v) \neq m(v)$ for all vertices v of PJ_2. Thus $S(2)$ is true.

Suppose now that $S(k)$ is true for some $k \geq 2$ and consider PJ_{k+1} with a non-constant function $m : V(PJ_{k+1}) \to C_{k+1}$. Then $PJ_{k+1} = P(H_1, H_2, H_3, H_4)$ with disjoint elements where for $i \in [3]$, H_i is a copy of PJ_k, and H_4 is a copy of K_1. For all $i \in [3]$, define $m_i = m|_{V(H_i)}$ and let w be the vertex of H_4.

Case 1: For all $i \in [3]$, m_i is constant on $V(H_i)$.

For all $i \in [3]$, let $j_i \in [k+1]$ be such that $m_i(u) = c_{j_i}$ for all $u \in V(H_i)$. Since m is non-constant on $V(PJ_{k+1})$, there exists $t \in [3]$ such that $c_{j_t} \neq m(w)$. Since each H_i is k-ecolorable, let γ_i be a k ecoloring $\gamma_i : V(H_i) \to (C_{k+1} \setminus \{c_{j_i}\})$. Then $\gamma_i(u) \neq c_{j_i} = m_i(u)$ for all $u \in V(H_i)$. Letting $\gamma : V(PJ_{k+1}) \to C_{k+1}$ be the functional extension of $\gamma_1, \gamma_2, \gamma_3$ with $\gamma(w) = c_{j_t} \neq m(w)$, we have $\gamma(u) \neq m(u)$ for all $u \in V(PJ_{k+1})$. Moreover any induced P_4 not contained in an element of the path-join PJ_{k+1} includes vertex w and a vertex h_t from H_t with $\gamma(w) = c_{j_t} \neq \gamma_t(h_t) = \gamma(h_t)$, and thus is not monochromatic under γ. Therefore, γ is a desired $(k+1)$-ecoloring and $S(k+1)$ is true.

Case 2: For some $t \in [3]$, m_t is non-constant on $V(H_t)$.

Consider the set of colors $R = m_t(V(H_t))$. We are interested in assigning an appropriate color to w by finding a color $c_j \neq m(w)$ that is not in R, or can be removed from R, so that we can apply the induction hypothesis $S(k)$.

If there exists $c_j \in C_{k+1}$ such that $c_j \neq m(w)$ and $c_j \notin R$, then since $R \subseteq C_{k+1} \setminus \{c_j\}$, by $S(k)$, there exists a k-ecoloring $\gamma_t : V(H_t) \to C_{k+1} \setminus \{c_j\}$ such that $\gamma_t(u) \neq m_t(u)$ for all $u \in V(H_t)$.

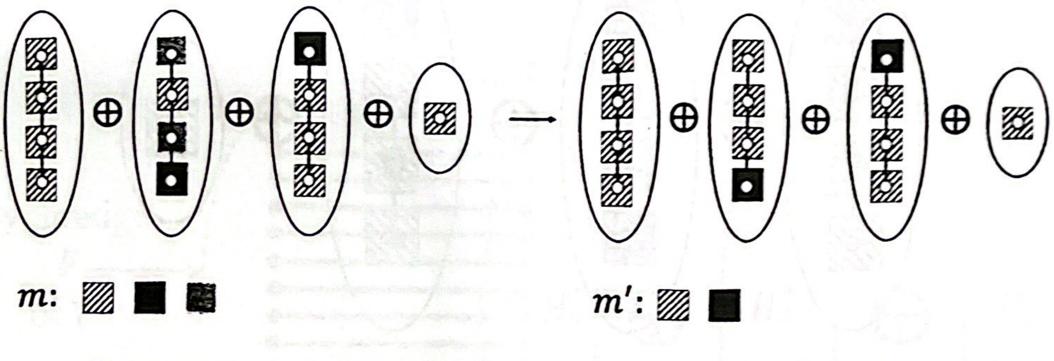

FIGURE 1. A reduction of an m to an m' for PJ_3 that is a Case 2 example with $t = 2$ and c_j the color grey.

Otherwise, $C_{k+1} \setminus \{m(w)\} \subseteq R$ and since $k+1 \geq 3$, $|R \setminus \{m(w)\}| \geq 2$ (see Figure 1). Thus, another non-constant function m'_t can be constructed from m_t by choosing an arbitrary color $c_j \in R \setminus \{m(w)\}$ and replacing it with $m(w)$. For all $i \in [3]$, define $m'_i : V(H_i) \to (C_{k+1} \setminus \{c_j\})$ such that for all $u \in V(H_i)$,

$$m'_i(u) = \begin{cases} m(w) & \text{if } m_i(u) = c_j \\ m_i(u) & \text{otherwise.} \end{cases}$$

By $S(k)$, there exists a k-ecoloring $\gamma_t : V(H_t) \to C_{k+1} \setminus \{c_j\}$ such that $\gamma_t(u) \neq m'_t(u)$ for all $u \in V(H_t)$. For $u \in V(H_t)$, $m_t(u) =$

c_j implies $\gamma_t(u) \neq c_j = m_t(u)$, and $m_t(u) \neq c_j$ implies $\gamma_t(u) \neq m'_t(u) = m_t(u)$. Thus $\gamma_t(u) \neq m_t(u)$ for all $u \in V(H_t)$.

In either of these two cases, we start the definition of $\gamma : V(PJ_{k+1}) \to C_{k+1}$ with $\gamma(w) = c_j$ and $\gamma|_{V(H_t)} = \gamma_t$.

Consider now H_ℓ where $\ell \in [3]$, $\ell \neq t$. Define $R_\ell = m_\ell(V(H_\ell))$. If $|R_\ell| = 1$, then let $C' = C_{k+1} \setminus R_\ell$ and let $\gamma_\ell : V(H_\ell) \to C'$ be a k-ȩcoloring of H_ℓ. Implicitly, $\gamma(u) \neq m_\ell(u)$ for all $u \in V(H_\ell)$. If $2 \leq |R_\ell| \leq k$, then let $C' \subset C_{k+1}$ such that $|C'| = k$ and $R \subseteq C'$. By $S(k)$, there exists a k-ȩcoloring $\gamma_\ell : V(H_\ell) \to C'$ with $\gamma(u) \neq m_\ell(u)$ for all $u \in V(H_\ell)$. Finally, if $|R_\ell| = k+1$, then since $k+1 \geq 3$, $m'_\ell : V(H_\ell) \to (C_{k+1} \setminus \{c_j\})$ as defined above, is non-constant, and so by $S(k)$, there exists $\gamma_\ell : V(H_\ell) \to C_{k+1} \setminus \{c_j\}$ such that $\gamma_\ell(u) \neq m'_\ell(u)$ for all $u \in V(H_\ell)$. An argument identical to the one used for γ_t can be used here to give $\gamma_\ell(u) \neq m_\ell(u)$ for all $u \in V(H_\ell)$. In each of these cases regarding the size of $|R_\ell|$, we find a k-ȩcoloring γ_ℓ of H_ℓ using colors from C_{k+1} such that $\gamma_\ell(u) \neq m_\ell(u)$ for all $u \in V(H_\ell)$. Finishing the definition of γ, we let $\gamma|_{V(H_\ell)} = \gamma_\ell$ (see Figure 2).

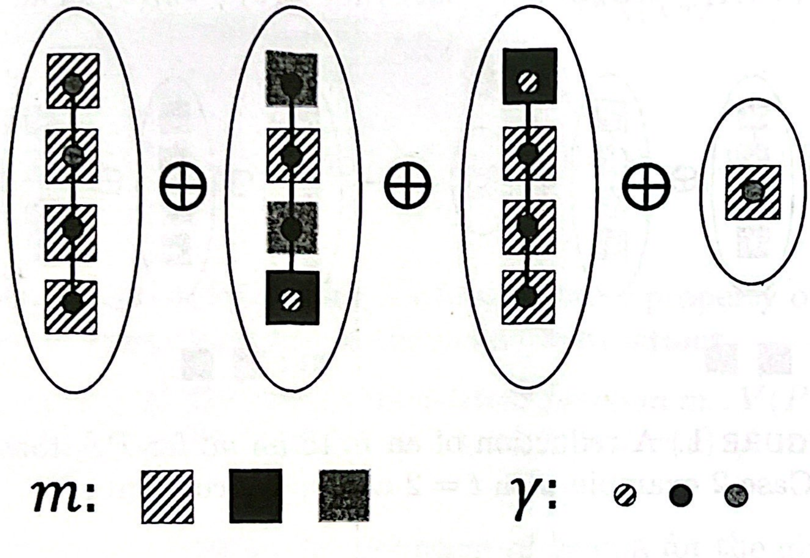

FIGURE 2. A 3-ȩcoloring γ of PJ_3 that is obtained from Case 2 with $t = 2$ and c_j the color grey.

Finally, γ is a $(k+1)$-ȩcoloring of $V(PJ_{k+1})$ since each element H_i, $i \in [3]$, has k-ȩcoloring using k colors from C_{k+1}, and any induced P_4 not contained in an element of the path-join PJ_{k+1} includes vertex w and a vertex h_t from H_t with $\gamma(w) = c_j \neq \gamma_t(h_t) = \gamma(h_t)$, and thus is not monochromatic under γ.

Therefore $S(k+1)$ is true. □

3. Main Constructions

This section provides two general constructions of $(k+1)$-$_e$critical graphs. The first construction, Construction 6, is built from a $(k+1)$-critical hypergraph while the second one, Construction 11, is built from a $(k+1)$-critical graph. Both constructions force any induced P_4 subgraph to either contain some part of the hypergraph/graph they are built from or to be contained in a k-$_e$critical subgraph (i.e. see Lemmas 8 and 12).

Construction 6. *With $k \geq 2$ and $n = |V(PJ_k)|$, let $\mathcal{F}$ be an n-uniform $(k+1)$-critical hypergraph. For each edge $F \in \mathcal{F}$, let H_F^1, H_F^2 and H_F^3 be three disjoint k-$_e$critical graphs with $H_F^1 \cong PJ_k$ and let $H_F = P(H_F^1, H_F^2, H_F^3)$. The graphs H_F, $F \in \mathcal{F}$, are required to be pairwise disjoint and to not share any vertex in common with $\mathcal{F}$. For each $F \in \mathcal{F}$, let M_F be a matching between vertices of F and H_F^1. With $H = \bigoplus_{F\in\mathcal{F}} H_F$ and $SU = \bigcup_{F\in\mathcal{F}} M_F$, define $G = H \cup SU$.*

Note that each vertex v of degree d in $\mathcal{F}$ is the center of a star $K_{1,d}$ in SU. Thus SU is a union of stars. Moreover, note G contains no edges between vertices of $\mathcal{F}$. See Figure 3.

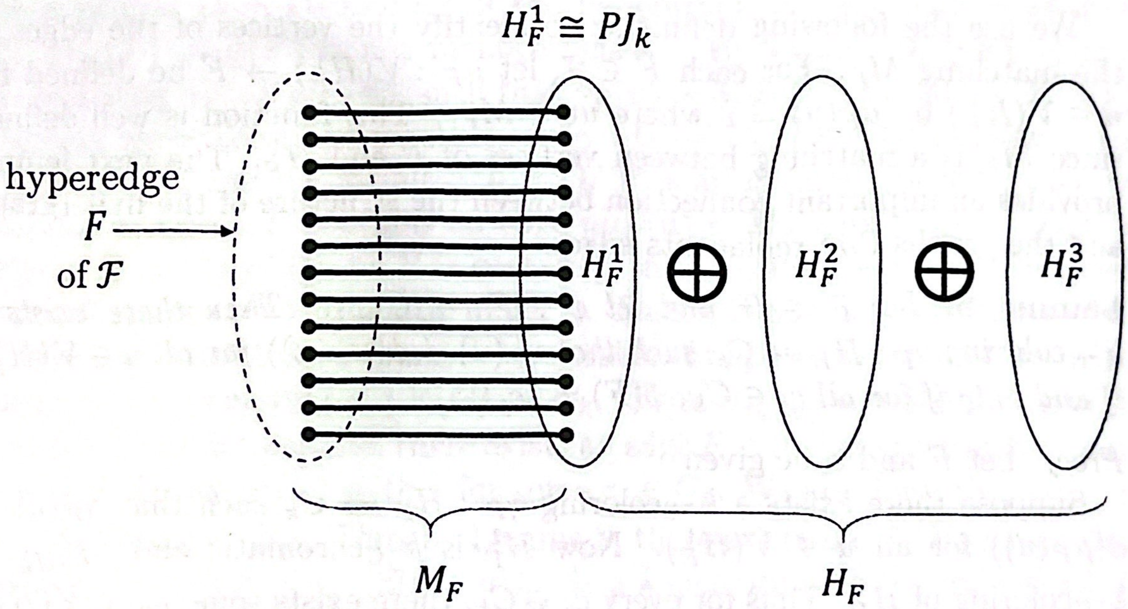

Figure 3. The subgraphs M_F and H_F of the graph G from Construction 6 obtained from a hyperedge $F \in \mathcal{F}$.

Example 7.

Let $k \geq 2$, $n = |V(PJ_k)| = \frac{1}{2}(3^k - 1)$ and $m = |E(PJ_k)| = 3^{k-1}(3^k - 2k - 1)/4$ edges. Let H be a set of $h = (n-1)k + 1 = \frac{3}{2}k(3^{k-1} - 1) + 1$ vertices, and $\mathcal{F}$ be the set all subsets of size n from H. Then $\mathcal{F}$ is an n-uniform $(k+1)$-critical hypergraph.[3] For each $F \in \mathcal{F}$ and $i \in [3]$, let $H_F^i \cong PJ_k$. Then graph G from Construction 6 has $h + \binom{h}{n}(3n)$ vertices and $\binom{h}{n}(3m + n + 2n^2)$ edges. For $k = 2$, this G has 427 vertices and 1575 edges. □

The next two lemmas will be used to show that the graphs from 6 are $(k+1)$-ecritical.

Lemma 8. *If G is a graph constructed using Construction 6, and P is an induced P_4 contained in G, then either*

(1) *P contains an edge of SU, or*
(2) *P is contained in H_F^i for some $F \in \mathcal{F}$ and for some $i \in [3]$.*

Proof. Let P be an induced P_4 of the given graph G. Suppose P does not contain an edge of SU. Since every edge of G incident with a vertex of $\mathcal{F}$ is an edge of SU, all of the vertices of P are contained in H. Moreover, H is the disjoint union of the H_F subgraphs it contains and thus P is contained in H_F for some $F \in \mathcal{F}$. By definition $H_F = PJ(H_F^1, H_F^2, H_F^3)$ and so by Lemma 2, P is contained in H_F^i for some $i \in [3]$. □

We use the following definition to identify the vertices of the edges of the matching M_F. For each $F \in \mathcal{F}$, let $\mu_F : V(H_F^1) \to F$ be defined for $u \in V(H_F^1)$ by $\mu_F(u) = v$ where $uv \in M_F$. This function is well-defined since M_F is a matching between vertices of F and H_F^1. The next lemma provides an important connection between the structure of the hypergraph and the graphs that replace its edges.

Lemma 9. *Let $F \in \mathcal{F}$ and let $\phi : F \to C_{k+1}$. Then there exists a k-ecoloring $\gamma_F : H_F \to C_k$ such that $\gamma_F(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$ if and only if for all $c_j \in C_k$, $\phi(F) \neq \{c_j\}$.*

Proof. Let F and ϕ be given.

Suppose there exists a k-ecoloring $\gamma_F : H_F \to C_k$ such that $\gamma_F(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$. Now H_F^1 is k-echromatic and $\gamma_F|_{H_F^1}$ is k-ecoloring of H_F^1. Thus for every $c_j \in C_k$, there exists some $u_{c_j} \in V(H_F^1)$ such that $\gamma_F(u_{c_j}) = \gamma_F|_{H_F^1}(u_{c_j}) = c_j$. Let $v = \mu_F(u_{c_j})$, the vertex of F matched to u_{c_j} under M_F. Thus $c_j = \gamma_F(u_{c_j}) \neq \phi(v)$ and since $\phi(v) \in \phi(F)$, $\phi(F) \neq \{c_j\}$.

---

[3]Note that n-uniform $(k+1)$-critical hypergraphs with t vertices exist if and only if $t \geq (n-1)k + 1$(see Toft [9]), and thus for a fixed k and n, there are an infinite number of them.

Suppose, conversely, for all $c_j \in C_k$, $\phi(F) \neq \{c_j\}$.

If $\phi(F) = \{c_{k+1}\}$, then for $i \in [3]$, let $\gamma_i : V(H_F^i) \to C_k$ be any k-ecoloring, and let γ be the functional extension of γ_1, γ_2 and γ_3. Thus for all $u \in V(H_F^1)$, $\gamma(u) \neq c_{k+1} = \phi(\mu_F(u))$.

If $\phi(F) \neq \{c_{k+1}\}$, then let $c_j \in \phi(F) \cap C_k$ and since $k \geq 2$, let $c_\ell \in C_k$ such that $c_\ell \neq c_j$. Define $m : V(H_F^1) \to C_k$ such that for all $u \in V(H_F^1)$,

$$m(u) = \begin{cases} c_\ell & \text{if } \phi(\mu_F(u)) = c_{k+1} \\ \phi(\mu_F(u)) & \text{otherwise.} \end{cases}$$

Then m is not constant on $V(H_F^1)$ since $\{c_j, c_\ell\} \subseteq m(V(H_F^1))$. By Lemma 5, there is a k-ecoloring $\gamma_1 : H_F^1 \to C_k$ such that $\gamma_1(u) \neq m(u)$ for all $u \in V(H_F^1)$. Thus, for all $u \in V(H_F^1)$, $\gamma_1(u) \neq \phi(\mu_F(u))$. Let γ_2 and γ_3 be any k-ecolorings of H_F^2 and H_F^3 respectively using colors from C_k. Finally, let γ be the functional extension of γ_1, γ_2 and γ_3. By Lemma 2, if P is an induced P_4 of H_F, P is contained in $V(H_F^i)$ for some $i \in [3]$ and hence is not monochromatic under γ_i. Thus γ is a k-ecoloring of H_F with $\gamma_1(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$. ☐

Theorem 10. *A graph G constructed using Construction 6 is $(k+1)$-ecritical.*

Proof. Let G be a graph constructed using Construction 6.

We first show that G is $(k+1)$-ecolorable. Let $\phi : V(\mathcal{F}) \to C_{k+1}$ be a proper $(k+1)$-coloring of the hypergraph $\mathcal{F}$. For each edge $F \in \mathcal{F}$, $\phi|_F(F) = \phi(F)$ is not monochromatic. By Lemma 9, there exist a k-ecoloring $\gamma_F : H_F \to C_k$ such that $\gamma_F(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$. Putting these colorings together, let $\gamma : G \to C_{k+1}$ be the functional extensions of ϕ and γ_F for all $F \in \mathcal{F}$. If P is an induced P_4 of G, then by Lemma 8, either P contains an edge $u\mu_F(u) \in M_F$ with $\gamma(u) = \gamma_F(u) \neq \phi(\mu_F(u)) = \gamma(\mu_F(u))$, or P is contained in H_F for some $F \in \mathcal{F}$. In either case, P is not monochromatic under γ. Thus γ is a $(k+1)$-ecoloring of G.

We now show that G is not k-ecolorable. Suppose, to the contrary, that there is a k-ecoloring $\sigma : V(G) \to C_k$. Then $\sigma|_{V(\mathcal{F})}$ is a k-coloring of $\mathcal{F}$. Since $\mathcal{F}$ is not k-colorable, there exists an edge $F \in \mathcal{F}$ that is monochromatic under σ. Hence, $\sigma(F) = \{c_j\}$ for some $c_j \in C_k$. We also have that $\sigma|_{H_F}$ is a k-ecoloring of H_F. Thus by Lemma 9, there exists $u \in V(H_F^1)$ such that $\sigma(u) = \sigma(\mu_F(u)) = c_j$. Since $\sigma|_{H_F^i}$ is a k-ecoloring of H_F^i for $i \in \{2, 3\}$ there exists $u_i \in H_F^i$ such that $\sigma(u_i) = c_j$. Thus $u_3u_2u\mu_F(u)$ is an induced P_4 in G that is monochromatic under σ, a contradiction. Thus no such σ exits.

Finally, we show that G is k-ecritical. Let v be a vertex of G.

Case 1: $v \in V(\mathcal{F})$.

If $v \in V(\mathcal{F})$, then since $\mathcal{F}$ is $(k+1)$-critical, there is a proper $(k+1)$-coloring $\phi : V(\mathcal{F}) \to C_{k+1}$ so that v is the only vertex of $\mathcal{F}$ colored c_{k+1}. Since no edge F of $\mathcal{F}$ is monochromatic, by Lemma 9, there exist a k-ɐcoloring $\gamma_F : V(H_F) \to C_k$ such that $\gamma_F(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$. Let $\gamma : G \to C_{k+1}$ be the functional extensions of ϕ and γ_F for all $F \in \mathcal{F}$. As above, γ is a $(k+1)$-ɐcoloring of G. Note, however, that v is the only vertex colored c_{k+1}. Thus $\gamma|_{V(G-v)}$ is a k-ɐcoloring of $G - v$.

Case 2: $v \in V(H_{F'})$ for some $F' \in \mathcal{F}$.

Since $\mathcal{F}$ is $(k+1)$-critical, there is a k-coloring $\phi : V(\mathcal{F}) \to C_k$ so that F' is the only monochromatic edge of $\mathcal{F}$ under ϕ. By rearranging the colors, we may assume, without loss of generality, that $\phi(F') = \{c_k\}$. For $i \in [3]$, let $u_i \in V(H_{F'}^i)$ such that $u_j = v$ for some $j \in [3]$. Since each $H_{F'}^i$ is k-ɐcritical, $i \in [3]$, there exists a k-ɐcoloring $\gamma_{F'}^i : H_{F'}^i \to C_k$ such that the only vertex colored c_k in $H_{F'}^i$ is u_i. Define $\gamma_{F'}$ to be the functional extension of $\gamma_{F'}^1$, $\gamma_{F'}^2$, and $\gamma_{F'}^3$.

For $F \in \mathcal{F}$, $F \neq F'$, $\phi|_F$ is a function $\phi|_F : F \to C_k$ such that $\phi|_F(F)$ is not monochromatic. Thus by Lemma 9, there exists a k-ɐcoloring $\gamma_F : H_F \to C_k$ such that $\gamma_F(u) \neq \phi(\mu_F(u))$ for all $u \in V(H_F^1)$.

Now define $\gamma : V(G) \to C_k$ to be the functional extension of ϕ and γ_F for all $F \in \mathcal{F}$. Although γ is not a k-ɐcoloring of G, the induced P_4s that are monochromatic under γ have an intentionally restricted form. Let P be an induced P_4 of G that is monochromatic under γ. If P is contained in H_F^i for some $F \in \mathcal{F}$ and for some $i \in [3]$, then P is not monochromatic under γ_F^i and hence γ. Thus this cannot be the case and so, by Lemma 8, P contains some edge e of SU. The only edge in SU that is monochromatic under γ is $u_1\mu_{F'}(u_1)$, since for $F \in \mathcal{F}$, if $F \neq F'$, then $\gamma(u) = \gamma_F(u) \neq \phi(\mu_F(u)) = \gamma(\mu_F(u))$, and if $F = F'$, then $\gamma(F) = \{c_k\}$ and the only vertex colored c_k in H_F^1 under γ is u_1. Since P is connected, P is thus contained in $H_{F'} \cup M_{F'}$ with only the edge $u_1\mu_{F'}(u_1)$ from $M_{F'}$. The only other two vertices in $H_{F'}$ colored c_k under γ are u_2 and u_3. Thus $P = u_3u_2u_1\mu_{F'}(u_1)$ is the only induced P_4 in G that is monochromatic under γ. Therefore, $G - v = G - u_j$ has k-ɐcoloring $\gamma|_{G-v}$.

□

We now turn to a completely different construction inspired by the 3-ɐcritical outerplanar graph of Figure 2 in [6].

Construction 11. *For $k \geq 1$, let F be a $(k+1)$-critical graph with n vertices, $v_1, \ldots, v_n$, and let $H_1, \ldots, H_n$ be disjoint k-ec-critical graphs with vertices disjoint from F. Define*

$$G = F \cup \left(\bigcup_{i=1}^{n} \{v_i\} \oplus H_i \right).$$

Note that for $k = 1$, the graph G of Construction 11 is just P_4. An example of Construction 11 is drawn in Figure 4.

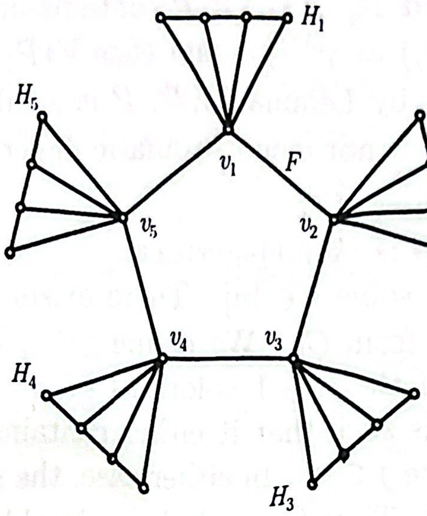

FIGURE 4. Example of 3-ec-critical graph using Construction 11 where F is the 5-cycle and $H_i \cong P_4$, $i \in [5]$.

Lemma 12. *If G is a graph constructed using Construction 11, and P is an induced P_4 contained in G, then either*

(1) *P contains an edge of F, or*
(2) *P is a subgraph of H_j for some $j \in [n]$.*

Proof. Let P be an induced P_4 of a graph G. Suppose P does not contain an edge of F. Then P is a subgraph of $\{v_j\} \oplus H_j$ for some $j \in [n]$. By Lemma 2, P is a subgraph of H_j. □

Theorem 13. *A graph G constructed using Construction 11 is $(k+1)$-ec-critical.*

Proof. Let G be a graph constructed using Construction 11.

We first prove that G is not k-ec-colorable. Suppose, to the contrary, G has a k-ec-coloring γ using colors from C_k. The restriction of γ to $V(F)$ is not a proper k-coloring of F since F is $(k+1)$-chromatic. Thus there exists an edge $f_i f_j$ of F such that $\gamma(f_i) = \gamma(f_j) = c_l$ for some $c_l \in C_k$. Since H_i and H_j are not $(k-1)$-ec-colorable, the restriction of γ to either graph is not a $(k-1)$-ecoloring, and hence H_i has a vertex h_i with $\gamma(h_i) = c_l$ and H_j has a vertex h_j with $\gamma(h_j) = c_l$. The induced path $h_i f_i f_j h_j$ is a P_4 in G whose vertices are all colored c_l by γ, a contradiction. Thus no such γ exists.

We now prove that G is $(k+1)$-ecolorable. The graph F has a proper $(k+1)$-coloring ϕ using colors from $C_{k+1} = C_k \cup \{c_{k+1}\}$. For each H_i, $i \in [n]$, let η_i be a k-ecoloring of H_i using colors from C_k. We define $\gamma^{(1)} : V(G) \to C_{k+1}$ to be the functional extension of the $n+1$ colorings ϕ, $\eta_1, \ldots, \eta_n$.

Let P be an induced P_4 of G. If P contains an edge $f_i f_j$ of F, then $\gamma^{(1)}(f_i) = \phi(f_i) \neq \phi(f_j) = \gamma^{(1)}(f_j)$ and thus $V(P)$ is not monochromatic under $\gamma^{(1)}$. Otherwise, by Lemma 12(2), P is a subgraph of H_j for some $j \in [n]$ and hence $V(P)$ is not monochromatic under $\eta_j = \gamma^{(1)}|_{V(H_j)}$. Thus $\gamma^{(1)}$ is a $(k+1)$-ecoloring of G.

We now show that G is $(k+1)$-ecritical.

Consider $G - v_i$ for some $i \in [n]$. There exists a proper k-coloring ϕ_i of $F - v_i$ using colors from C_k. We define $\gamma^{(2)} : V(G) \to C_k$ to be the functional extension of the $n+1$ colorings ϕ_i, $\eta_1, \ldots, \eta_n$. The location of any given P_4 in G is such that it either contains an edge of F or is a subgraph of H_j for some $j \in [n]$. In either case, the set of its vertices is not monochromatic by $\gamma^{(2)}$. Thus $G - v_i$ is k-ecolorable.

Finally, consider $G - v$ where $v \in V(H_t)$ for some $t \in [n]$.

Since F is $(k+1)$-critical, F has an edge incident with v_t. Thus, let $e = v_s v_t$ for some $s \in [n]$, $s \neq t$. There exists a proper k-coloring ϵ of $F - e$ using colors from C_k since F is $(k+1)$-critical. Every such coloring colors v_s and v_t the same color. We may suppose therefore that $\epsilon(v_s) = c_l = \epsilon(v_t)$ for some $l \in [k]$. Let η'_t be a $(k-1)$-ecoloring of $H_t - v$ using colors from $C_k \setminus \{c_l\}$. For $i \in [n]$, $i \neq t$, let $\eta'_i = \eta_i$, as defined above. Define $\gamma^{(3)} : V(G) \to C_k$ to be the functional extension of the $n+1$ colorings ϵ, $\eta'_1, \ldots, \eta'_n$.

Suppose P is an induced P_4 of $G - v$. If P is contained in H_j for some $j \in [n]$, then $V(P)$ is not monochromatic under $\gamma^{(3)}$ since $\gamma^{(3)}(V(P)) = \eta'_j(V(P))$. Otherwise, by Lemma 12(1), P contains an edge of F. Since $V(F) = V(F - e)$, the k-coloring ϵ of $F - e$ is a k-coloring of F. Moreover, since ϵ is a proper k-coloring of $F - e$, e is the only edge of F whose vertices are colored the same by ϵ. If P contains an edge of F different from $v_s v_t$, then $V(P)$ must not be monochromatic under ϵ and hence under $\gamma^{(3)}$ since $\gamma^{(3)}(V(P)) = \epsilon(V(P))$. We may assume, therefore, that P contains only one edge from F and that this edge is $v_s v_t$. Since P does not contain a 3-cycle, P can contain at most one vertex of any H_i, for $i \in \{s, t\}$. Thus P must contain a vertex of $H_t - v$. This vertex is not colored c_l by $\gamma^{(3)}$ since $\gamma^{(3)}(V(H_t - v)) = \eta'_t(V(H_t - v)) \subseteq C_k \setminus \{c_l\}$. Now $v_t \in V(P)$ and

$\gamma^{(3)}(v_t) = \epsilon(v_t) = c_l$ and so $V(P)$ is not monochromatic under $\gamma^{(3)}$. Hence $\gamma^{(3)}$ is a k-ecoloring of $G - v$.

Therefore G is $(k+1)$-ecritical.

□

Since the 3-critical graphs are simply odd cycles and the only 2-ecritical graph is P_4, every 3-ecritical graph G from Construction 11 has $15 + 10s$ vertices for some $s \geq 0$ and is not only planar but also outerplanar. Thus, every 4-ecritical graph from Construction 11 is planar when 3-ecritical graphs H_i are produced from Construction 11 as well, and the 4-critical graph F is chosen to be planar (e.g. $F \cong K_4$, amongst many others). Using Construction 11 with F as an odd-wheel on $2(a+1)$ vertices (a 4-critical planar graph) produces 4-ecritical planar graphs with $32(a+1)+10b$ vertices, $a \geq 1$, $b \geq 0$. Starting with F as a 4-critical planar graph on 7 vertices, Construction 11 produces 4-ecritical planar graphs on $112 + 10b$ vertices, $b \geq 0$. Thus there are 4-ecritical planar graphs on $2t$ vertices for all $t \geq 76$.

The 3-ecritical planar graphs produced using Construction 11 have many triangles. There do exist, however, 3-ecritical planar graphs that have no triangles (i.e. see the construction for 3-echromatic triangle-free planar graphs of [5]).

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The second author's M.Sc. thesis [10] contains many of the results found here.

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