On the vertex Folkman numbers $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} = 6$ or 7

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Abstract

Let G be a graph and $a_1, ..., a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that for every coloring of the vertices of G in s colors there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i. The vertex Folkman numbers $F_v(a_1, ..., a_s; q)$ are defined by the equality:

$$F_v(a_1, ..., a_s; q) = \min\{|V(G)| : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } K_q \not\subseteq G\}.$$

Let $m = \sum_{i=1}^{s}(a_i - 1) + 1$. It is easy to see that $F_v(a_1, ..., a_s; q) = m$ if $q \geq m + 1$. In [11] it is proved that $F_v(a_1, ..., a_s; m) = m + \max\{a_1, ..., a_s\}$. We know all the numbers $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} \leq 5$ and none of these numbers is known if $\max\{a_1, ..., a_s\} \geq 6$. In this paper we present computer algorithms, with the help of which we compute all Folkman numbers $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} = 6$. We also prove that $F_v(2, 2, 7; 8) = 20$ and obtain new bounds on the numbers $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} = 7$.

Keywords: Folkman number, clique number, independence number, chromatic number

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JCMCC **109** (2019), pp. 213-243

1 Introduction

Only finite, non-oriented graphs without loops and multiple edges are considered in this paper. $G_1 + G_2$ denotes the graph G for which $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x,y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e. G is obtained by connecting every vertex of G_1 to every vertex of G_2. All undefined terms can be found in [32].

Let $a_1, ..., a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that for any coloring of $\mathrm{V}(G)$ in s colors (s-coloring) there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i. In particular, $G \overset{v}{\rightarrow} (a_1)$ means that $\omega(G) \geq a_1$. Further, for convenience, instead of $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r})$ we write $G \overset{v}{\rightarrow} (2_r)$ and instead of $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r}, a_1, ..., a_s)$ we write $G \overset{v}{\rightarrow} (2_r, a_1, ..., a_s)$.

Define:

$\mathcal{H}(a_1, ..., a_s; q) = \left\{G : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } \omega(G) < q\right\}.$

$\mathcal{H}(a_1, ..., a_s; q; n) = \{G : G \in \mathcal{H}(a_1, ..., a_s; q) \text{ and } |\mathrm{V}(G)| = n\}.$

The vertex Folkman number $F_v(a_1, ..., a_s; q)$ is defined by the equality:

$$F_v(a_1, ..., a_s; q) = \min\{|\mathrm{V}(G)| : G \in \mathcal{H}(a_1, ..., a_s; q)\}.$$

The graph G is called an extremal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$ and $|\mathrm{V}(G)| = F_v(a_1, ..., a_s; q)$. We denote by $\mathcal{H}_{extr}(a_1, ..., a_s; q)$ the set of all extremal graphs in $\mathcal{H}(a_1, ..., a_s; q)$.

We say that G is a maximal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$ but $G + e \notin \mathcal{H}(a_1, ..., a_s; q), \forall e \in \mathrm{E}(\overline{G})$, i.e. $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$. G is a minimal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$ but $G - e \notin \mathcal{H}(a_1, ..., a_s; q), \forall e \in \mathrm{E}(G)$, i.e. $G - e \overset{v}{\nrightarrow} (a_1, ..., a_s), \forall e \in \mathrm{E}(G)$.

For convenience, we also define the following term:

Definition 1.1. *The graph G is called a $(+K_t)$-graph if $G + e$ contains a new t-clique for all $e \in \mathrm{E}(\overline{G})$.*

Obviously, $G \in \mathcal{H}(a_1, ..., a_s; q)$ is a maximal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if and only if G is a $(+K_q)$-graph. We shall denote by $\mathcal{H}_{+K_t}(a_1, ..., a_s; q)$ the set of all $(+K_t)$-graphs in $\mathcal{H}(a_1, ..., a_s; q)$, and by $\mathcal{H}_{max}(a_1, ..., a_s; q)$ all maximal K_q-free graphs in this set. The sets $\mathcal{H}_{+K_t}(a_1, ..., a_s; q; n)$ and $\mathcal{H}_{max}(a_1, ..., a_s; q; n)$ are defined in the same way as $\mathcal{H}(a_1, ..., a_s; q; n)$.

Remark 1.2. *In the special case $s = 1$ we have*

$\mathcal{H}(a_1; q; n) = \{G : a_1 \leq \omega(G) < q \text{ and } |\mathrm{V}(G)| = n\}.$

If $a_1 \le n \le q-1$ *then* $K_n \in \mathcal{H}_{max}(a_1;q;n)$, *and if* $n \ge q-1 \ge a_1$, *then* $\mathcal{H}_{max}(a_1;q;n) = \mathcal{H}_{max}(q-1;q;n)$.

Folkman proves in [8] that:

$$(1.1) \qquad F_v(a_1,...,a_s;q) \text{ exists } \Leftrightarrow q > \max\{a_1,...,a_s\}.$$

Other proofs of (1.1) are given in [7] and [14]. In the special case $s = 2$, a very simple proof of this result is given in [22] with the help of corona product of graphs.
Obviously $F_v(a_1,...,a_s;q)$ is a symmetric function of $a_1,...,a_s$, and if $a_i = 1$, then

$$F_v(a_1,...,a_s;q) = F_v(a_1,...,a_{i-1},a_{i+1},...,a_s;q).$$

Therefore, it is enough to consider only such Folkman numbers $F_v(a_1,...,a_s;q)$ for which

$$(1.2) \qquad 2 \le a_1 \le ... \le a_s.$$

We call the numbers $F_v(a_1,...,a_s;q)$ for which the inequalities (1.2) hold canonical vertex Folkman numbers.
In [15] for arbitrary positive integers $a_1,...,a_s$ the following terms are defined

$$(1.3) \quad m(a_1,...,a_s) = m = \sum_{i=1}^{s}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1,...,a_s\}.$$

It is easy to see that $K_m \overset{v}{\rightarrow} (a_1,...,a_s)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1,...,a_s)$. Therefore

$$F_v(a_1,...,a_s;q) = m, \quad q \ge m+1.$$

The following theorem for the numbers $F_v(a_1,...,a_s;m)$ is true:

Theorem 1.3. *Let* $a_1,...,a_s$ *be positive integers and let* m *and* p *be defined by the equalities (1.3). If* $m \ge p+1$, *then:*

(a) $F_v(a_1,...,a_s;m) = m+p$, *[15],[14].*

(b) $K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$

is the only extremal graph in $\mathcal{H}(a_1,...,a_s;m)$, *[14].*

The condition $m \ge p+1$ is necessary according to (1.1). Other proofs of Theorem 1.3 are given in [24] and [25].

Very little is known about the numbers $F_v(a_1, ..., a_s; m-1)$. According to (1.1) we have

$$F_v(a_1, ..., a_s; m-1) \text{ exists } \Leftrightarrow m \geq p+2. \tag{1.4}$$

The following bounds are known:

$$m+p+2 \leq F_v(a_1, ..., a_s; m-1) \leq m+3p, \tag{1.5}$$

where the lower bound is true if $p \geq 2$ and the upper bound is true if $p \geq 3$. The lower bound is obtained in [24] and the upper bound is obtained in [12]. In the border case $m = p+2$ the upper bounds in (1.5) are significantly improved in [31].

When $p = \max\{a_1, ..., a_s\} \leq 5$ we have

$$F_v(a_1, ..., a_s, m-1) = \begin{cases} m+4, & \textit{if } p=2 \textit{ and } m \geq 6, \textit{[20]} \\ m+6, & \textit{if } p=3 \textit{ and } m \geq 6, \textit{[26]} \\ m+7, & \textit{if } p=4 \textit{ and } m \geq 6, \textit{[26]} \\ m+9, & \textit{if } p=5 \textit{ and } m \geq 7, \textit{[1]}. \end{cases} \tag{1.6}$$

In the cases $p=2$ and $p=3$ we also know the numbers: $F_v(2,2,2;3) = 11$, [18] and [4], $F_v(2,2,2,2;4) = 11$, [21] (see also [23]), $F_v(2,2,3;4) = 14$, [24] and [5], $F_v(3,3;4) = 14$, [19] and [27]. These numbers and the numbers (1.6) are all the numbers in the form $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} \leq 5$. We do not know any of these numbers when $\max\{a_1, ..., a_s\} \geq 6$. In [1] we prove that

$$m+9 \leq F_v(a_1, ..., a_s; m-1) \leq m+10, \tag{1.7}$$

when $\max\{a_1, ..., a_s\} = 6$.

In this paper we present two computer algorithms (Algorithm 3.4 and Algorithm 3.7) for finding all maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$. With the help of these algorithms we obtain the following results:

Theorem 1.4. *Let $a_1, ..., a_s$ be positive integers, such that*

$$2 \leq a_1 \leq ... \leq a_s = 6,$$

and $m = \sum_{i=1}^{s}(a_i - 1) + 1 \geq 8$. Then

(a) $F_v(a_1, ..., a_s; m-1) = m+9$, *if $a_1 = ... = a_{s-1} = 2$.*

(b) $F_v(a_1, ..., a_s; m-1) = m+10$, *if $a_{s-1} \geq 3$.*

We obtain the following bounds on numbers of the form $F_v(a_1, ..., a_s; 7)$ where $\max\{a_1, ..., a_s\} = 6$:

Theorem 1.5. *Let $a_1, ..., a_s$ be positive integers such that $\max\{a_1, ..., a_s\} = 6$ and $m = \sum_{i=1}^{s}(a_i - 1) + 1 \geq 9$. Then*

$$F_v(a_1, ..., a_s; 7) \geq F_v(2_{m-6}, 6; 7) \geq 3m - 5.$$

In particular, $F_v(6, 6; 7) \geq 28$.

Theorem 1.6. *Let $a_1, ..., a_s$ be positive integers such that $\max\{a_1, ..., a_s\} = 6$ and $m = \sum_{i=1}^{s}(a_i - 1) + 1$. Then:*

(a) $22 \leq F_v(a_1, ..., a_s; 7) \leq F_v(4, 6; 7) \leq 35$ *if* $m = 9$.

(b) $28 \leq F_v(a_1, ..., a_s; 7) \leq F_v(6, 6; 7) \leq 70$ *if* $m = 11$.

We also obtain the following results related to the numbers $F_v(a_1, ..., a_s; m-1)$ where $\max\{a_1, ..., a_s\} = 7$.

Theorem 1.7. $F_v(2, 2, 7; 8) = 20$.

Theorem 1.8. *Let $a_1, ..., a_s$ be positive integers, such that $\max\{a_1, ..., a_s\} = 7$ and $m = \sum_{i=1}^{s}(a_i - 1) + 1 \geq 9$. Then*

$$m + 10 \leq F_v(a_1, ..., a_s; m-1) \leq m + 12.$$

Remark 1.9. *According to (1.4) the condition $m \geq 8$ from Theorem 1.4 and the condition $m \geq 9$ from Theorem 1.8 are necessary.*

This paper is organized in sections. In the first section the necessary definitions are given, an overview of the known results is provided and at the end we formulate the obtained new results. In the second section we formulate some auxiliary propositions. In the third section we present computer algorithms for finding the maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$. In the fourth section we find all extremal graphs in $\mathcal{H}(2, 2, 6; 7)$ and compute the numbers $F_v(2, 2, 6; 7) = 17$ and $F_v(3, 6; 7) = 18$. In the fifth section we prove Theorem 1.4 (a), and in the sixth section we prove Theorem 1.4 (b). In the seventh section we find all graphs in $\mathcal{H}(2, 2, 6; 7; 18)$ and we prove Theorem 1.5 and Theorem 1.6. In the eight section we show that $F_v(2, 2, 7; 8) = 20$ (Theorem 1.7) and we find some extremal graphs in $\mathcal{H}(2, 2, 7; 8)$. In the last ninth section we prove Theorem 1.8.

This paper has a previous version (arXiv:1512.02051). In the current version we use new faster algorithms with the help of which we reproduce the results from the previous version and we also prove new theorems (Theorem 1.5, Theorem 1.6, Theorem 1.7 and Theorem 1.8).

2 Some auxiliary results

Let $a_1, ..., a_s$ be positive integers and $m = \sum_{i=1}^{s}(a_i - 1) + 1$. Obviously, if $a_i \geq t \geq 2$, then

(2.1) $$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (a_1, ..., a_{i-1}, t, a_i - t + 1, a_{i+1}, ..., a_s).$$

By repeatedly applying (2.1) we obtain

(2.2) $$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (2_{m-1}).$$

Since $G \overset{v}{\rightarrow} (2_{m-1}) \Leftrightarrow \chi(G) \geq m$, from (2.2) it follows

(2.3) $$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow \chi(G) \geq m.$$ [25]

This fact will be necessary in the proof of Theorem 6.1. Very simple examples of graphs for which equality is reached in (2.3) are obtained in [22].

Conjecture 2.1. *If* $G \in \mathcal{H}_{extr}(a_1, ..., a_s; m - 1)$, *then* $\chi(G) \leq m + 1$.

For all known examples of extremal graphs, including the extremal graphs obtained in this paper, this inequality holds.

Let the numbers $a_1, ..., a_s$ satisfy the inequalities

$$2 \leq a_1 \leq ... \leq a_s = p.$$

As before, by repeatedly applying (2.1) we obtain that if $a_{s-1} \geq 3$

(2.4) $$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (2_{m-p-2}, 3, p),$$

and therefore it is true that

(2.5) $$F_v(2_{m-p-2}, 3, p; m - 1) \leq F_v(a_1, ..., a_s; m - 1).$$

Further, we will use the following obvious

Proposition 2.2. *Let G be a graph, $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ and $A \subseteq V(G)$ be an independent set. Then if $a_i \geq 2$*

$$G - A \overset{v}{\rightarrow} (a_1, ..., a_{i-1}, a_i - 1, a_{i+1}, ..., a_s).$$

Let $G \in \mathcal{H}(a_1, ..., a_s; m-1; n)$ and A be an independent set of vertices of G such that $|A| = \alpha(G)$. According to Proposition 2.2, $G - A \in \mathcal{H}(a_1 - 1, ..., a_s; m-1; n-|A|)$ and by Theorem 1.3, $n - |A| \geq m - 1 + p$. Thus, we proved

$$G \in \mathcal{H}(a_1, ..., a_s; m-1; n) \Rightarrow \alpha(G) \leq n - m - p + 1. \tag{2.6}$$

We will also need the following improvement of the lower bound in (1.5)

Theorem 2.3. *[26] Let $a_1, ..., a_s$ be positive integers, let m and p be defined by the equalities (1.3), $p \geq 3$ and $m \geq p + 2$. If $F_v(2, 2, p; p+1) \geq 2p + 5$, then*

$$F_v(a_1, ..., a_s; m-1) \geq m + p + 3.$$

3 Algorithms

In this section we present algorithms for finding all graphs in $\mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with the help of a computer. The remaining graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$ can be obtained by removing edges from the maximal graphs. The algorithms are modifications of the algorithm from [3]. However, we will present them in detail since they are essential to this paper. The idea for these algorithms comes from [27] (see Algorithm A1). Similar algorithms are used in [5], [33], [13], [29], [1] and [2]. Also with the help of a computer, results for Folkman numbers are obtained in [9], [31], [30] and [6]. Let us also note the important role of the *nauty* programs [17] in this work. We use them for fast generation of non-isomorphic graphs and graph isomorph rejection.

Let $2 \leq a_1 \leq ... \leq a_s = p$ be positive integers, $m = \sum_{i=1}^{s}(a_i - 1) + 1$, and let $G \in \mathcal{H}(a_1, ..., a_s; m-1; n), n \geq m-1$. It is clear that $\alpha(G) \geq 2$, and according to (2.6), $\alpha(G) \leq n - m - p + 1$. As we will see further in the proofs of the results of this paper, it is computationally most difficult to obtain the graphs G for which $\alpha(G) = 2$. Therefore, first we present in detail the Algorithm 3.4 for finding all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; m-1; n)$ with $\alpha(G) = 2$, even though it is a special case of the more general Algorithm 3.7. Algorithm 3.4 is based on the following propositions, which are easy to prove:

Proposition 3.1. *Let $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$, A be an independent set of vertices of G and $H = G - A$. Then $H \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, ..., a_s; q; n - |A|)$.*

Proof. According to Proposition 2.2, $G - A \in \mathcal{H}(a_1 - 1, ..., a_s; q; n - |A|)$. From the maximality of G it follows that $G - A$ is a $(+K_{q-1})$-graph. □

Proposition 3.2. *Let G be a maximal K_q-free graph and v_1, v_2 be non-adjacent vertices of G. Then*
$K_{q-2} \subseteq N_G(v_1) \cap N_G(v_2)$.

Proposition 3.3. *Let G be a graph, v_1, v_2 be non-adjacent vertices of G and $H = G - \{v_1, v_2\}$. Then $\alpha(G) = 2$ if and only if the following three conditions are satisfied:*

(a) $\alpha(H) \leq 2$.

(b) $\alpha(H - N_G(v_j)) \leq 1$, $j = 1, 2$, i.e. either $N_G(v_j) = \mathrm{V}(H)$ or $H - N_G(v_j)$ is a clique.

(c) $N_G(v_1) \cup N_G(v_2) = \mathrm{V}(H)$.

Further, we will prove the more general Proposition 3.6.

Now, we formulate the first of the two important algorithms in this paper which finds all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $\alpha(G) = 2$.

Algorithm 3.4. *The input of the algorithm is the set $\mathcal{A}$ of all graphs in $\mathcal{H}_{max}(a_1 - 1, ..., a_s; q; n - 2)$ with independence number not greater than 2. The output of the algorithm is the set $\mathcal{B}$ of all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $\alpha(G) = 2$.*

1. *By removing edges from the graphs in $\mathcal{A}$ obtain the set* $\mathcal{A}' = \{H \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, ..., a_s; q; n - 2) : \alpha(H) \leq 2\}$.

2. *For each graph $H \in \mathcal{A}'$:*

2.1. *Find the family $\mathcal{M}(H) = \{M_1, ..., M_t\}$ of all maximal K_{q-1}-free subsets of $\mathrm{V}(H)$.*

2.2. *Find all pairs $N = \{M_{i_1}, M_{i_2}\}$ of elements of $\mathcal{M}(H)$ (it is possible that $M_{i_1} = M_{i_2}$), which fulfill the conditions:*

(a) $K_{q-2} \subseteq M_{i_1} \cap M_{i_2}$.

(b) $\alpha(H - M_{i_j}) \leq 1$, $j = 1, 2$.

(c) $M_{i_1} \cup M_{i_2} = \mathrm{V}(H)$.

2.3. *For each pair $N = \{M_{i_1}, M_{i_2}\}$ of elements of $\mathcal{M}(H)$ found in step 2.2 construct the graph $G = G(N)$ by adding new non-adjacent vertices v_1, v_2 to $\mathrm{V}(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, 2$. If $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$, then add G to $\mathcal{B}$.*

3. *Remove the isomorph copies of graphs from $\mathcal{B}$.*

4. *Remove from the obtained in step 3 set $\mathcal{B}$ all graphs G for which $G \overset{v}{\not\rightarrow} (a_1, ..., a_s)$.*

Theorem 3.5. *[3] After the execution of Algorithm 3.4, the obtained set $\mathcal{B}$ coincides with the set of all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $\alpha(G) = 2$.*

Proof. Suppose that after the execution of Algorithm 3.4 the graph $G \in \mathcal{B}$. Then, $G = G(N)$ where N and the following notations are the same as in step 2.3. Since $H = G - \{v_1, v_2\} \in \mathcal{A}'$, we have $\omega(H) < q$. Since $N_G(v_1)$ and $N_G(v_2)$ are K_{q-1}-free sets, it follows that $\omega(G) < q$. The check at the end of step 2.3 guarantees that G is a maximal K_q-free graph and the check in step 4 guarantees that $G \overset{v}{\rightarrow} (a_1, ..., a_s)$, therefore $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$. Again, by $H \in \mathcal{A}'$, we have $\alpha(H) \leq 2$ and from the conditions (b) and (c) in step 2.2 and Proposition 3.3 it follows that $\alpha(G) = 2$.

Let $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ and $\alpha(G) = 2$. We will prove that, after the execution of Algorithm 3.4, $G \in \mathcal{B}$. Let v_1, v_2 be non-adjacent vertices in G and $H = G - \{v_1, v_2\}$. Then $\alpha(H) \leq 2$ and according to Proposition 3.1, $H \in \mathcal{A}'$. Since G is a maximal K_q-free graph, $N_G(v_1)$ and $N_G(v_2)$ are maximal K_{q-1}-free subsets of $V(H)$, and therefore $N_G(v_i) \in \mathcal{M}(H), i = 1, 2$ (see step 2.1). Let $N = \{N_G(v_1), N_G(v_2)\}$. By Proposition 3.2, N fulfills the condition (a), and by Proposition 3.3, N also fulfills (b) and (c). Thus, we showed that N fulfills all conditions in step 2.2, and since $G = G(N)$ is a maximal K_q-free graph, in step 2.3 G is added to $\mathcal{B}$. Clearly, after step 4 the graph G remains in $\mathcal{B}$. □

We shall now generalize Algorithm 3.4 to find all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; m-1; n)$ for which $r \leq \alpha(G) \leq t$. We will need the following proposition, which is a generalization of Proposition 3.3 (in the special case $t = 2$, when G is not a complete graph Proposition 3.6 coincides with Proposition 3.3).

Proposition 3.6. *Let A be an independent set of vertices of G and $H = G - A$. Then,*

$$\alpha(G) \leq t \Leftrightarrow \alpha(H - \bigcup_{v \in A'} N_G(v)) \leq t - |A'|, \ \forall A' \subseteq A.$$

Proof. Let $\alpha(G) \leq t$. Suppose that for some $A' \subseteq A$ we have $\alpha(H - \bigcup_{v \in A'} N_G(v)) > t - |A'|$. Consequently, there exists an independent set A'' of vertices of $H - \bigcup_{v \in A'} N_G(v)$ such that $|A''| > t - |A'|$. We obtained that the independent set $A' \cup A''$ has more than t vertices, which is a contradiction.

Now, let $\alpha(H - \bigcup_{v \in A'} N_G(v)) \leq t - |A'|, \ \forall A' \subseteq A$. Let $\widetilde{A}$ be an independent set of vertices of G and $|\widetilde{A}| = \alpha(G)$. Then, $\widetilde{A} = A_1 \cup A_2$ where $A_1 \subseteq A$ and A_2 is an independent set in $H - \bigcup_{v \in A_1} N_G(v)$. Since $|A_2| \leq \alpha(H - \bigcup_{v \in A_1} N_G(v)) \leq t - |A_1|$, we obtain $\alpha(G) = |\widetilde{A}| = |A_1| + |A_2| \leq t$. □

Now we move on to the formulation of the second important Algorithm 3.7, which is a generalization of Algorithm 3.4 and finds all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $r \leq \alpha(G) \leq t$.

Algorithm 3.7. *The input of the algorithm is the set $\mathcal{A}$ of all graphs in $\mathcal{H}_{max}(a_1 - 1, ..., a_s; q; n - r)$ with independence number not greater than t. The output of the algorithm is a set $\mathcal{B}$ of all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $r \leq \alpha(G) \leq t$.*

1. *By removing edges from the graphs in $\mathcal{A}$ obtain the set* $\mathcal{A}' = \{H \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, ..., a_s; q; n - r) : \alpha(H) \leq t\}$.

2. *For each graph $H \in \mathcal{A}'$:*

2.1. *Find the family $\mathcal{M}(H) = \{M_1, ..., M_l\}$ of all maximal K_{q-1}-free subsets of $V(H)$.*

2.2. *Find all r-element multisets $N = \{M_{i_1}, M_{i_2}, ..., M_{i_r}\}$ of elements of $\mathcal{M}(H)$, which fulfill the conditions:*

(a) $K_{q-2} \subseteq M_{i_j} \cap M_{i_k}$ for every $M_{i_j}, M_{i_k} \in N$.

(b) $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq t - |N'|$ for subtuple N' of N.

2.3. *For each r-element multiset $N = \{M_{i_1}, M_{i_2}, ..., M_{i_r}\}$ of elements of $\mathcal{M}(H)$ found in step 2.2 construct the graph $G = G(N)$ by adding new independent vertices $v_1, v_2, ..., v_r$ to $V(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, ..., r$. If $\omega(G + e) = q, \forall e \in E(\overline{G})$, then add G to $\mathcal{B}$.*

3. *Remove the isomorph copies of graphs from $\mathcal{B}$.*

4. *Remove from the obtained in step 3 set $\mathcal{B}$ all graphs G for which $G \overset{v}{\nrightarrow} (a_1, ..., a_s)$.*

Theorem 3.8. *[3] After the execution of Algorithm 3.4, the obtained set $\mathcal{B}$ coincides with the set of all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $r \leq \alpha(G) \leq t$.*

Proof. Suppose that after the execution of Algorithm 3.7, $G \in \mathcal{B}$. In the same way as in the proof of Theorem 3.5, we prove that $G \in \mathcal{H}_{max}(a_1, ..., a_s; q)$. From Proposition 3.6 and the condition (b) in step 2.2 it follows that $\alpha(G) \leq t$, and step 2.3 guaranties that $\alpha(G) \geq r$.

Now let $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$, $r \leq \alpha(G) \leq t$, let A be an independent set of r vertices of G and $H = G - A$. Then obviously, $\alpha(H) \leq t$ and according to Proposition 3.1, $H \in \mathcal{A}'$ (see step 1). By repeating the reasoning of the second part of the proof of Theorem 3.5, we prove that after the execution of Algorithm 3.7, $G \in \mathcal{B}$. □

At the end of this section, we will propose a method to improve Algorithm 3.4 and Algorithm 3.7 which is based on the following proposition which is easy to prove:

Proposition 3.9. *Let $G \in \mathcal{H}(2, 2, p; p + 1)$ and $v \in V(G)$. Then, all non-neighbors of v induce a graph with chromatic number greater than 2. In particular, from $G \in \mathcal{H}(2, 2, p; p + 1)$ it follows that $\Delta(G) \leq |V(G)| - 4$.*

As we will see further (see Table 1 and Table 2), the inequality $\Delta(G) \leq |V(G)| - 4$ is exact. In some special cases, for example the proofs of Theorem 4.3, Theorem 7.1 and Theorem 1.7, we can use the inequality $\Delta(G) \leq |V(G)| - 4$ to speed up computations in some parts of the proofs. We used this inequality only to make sure that the obtained results are correct.

All computations were done on a personal computer. The proofs of Theorem 7.1 and Theorem 1.7 were the most time consuming, each taking about a month to complete.

4 Finding all graphs in $\mathcal{H}(2,2,6;7;17)$ and computation of the numbers $F_v(2,2,6;7)$ and $F_v(3,6;7)$

Let $a_1,...,a_s$ be positive integers and let m and p be defined by (1.3). According to (1.4), $F_v(a_1,...,a_s;m-1)$ exists if and only if $m \geq p+2$. In the border case $m = p+2$, $p \geq 3$, there are only two canonical numbers in the form $F_v(a_1,...,a_s;m-1)$, namely $F_v(2,2,p;p+1)$ and $F_v(3,p;p+1)$. The computation of the numbers $F_v(a_1,...,a_s;m-1)$ when $\max\{a_1,...,a_s\} = 6$, i.e. the proof of Theorem 1.4, will be done with the help of the numbers $F_v(2,2,6;7)$ and $F_v(3,6;7)$. Because of this, we will first compute these two numbers by proving

Theorem 4.1. *$F_v(2,2,6;7) = 17$ and $F_v(3,6;7) = 18$.*

From (2.1) it is easy to see that

$$G \overset{v}{\rightarrow} (3,p) \Rightarrow G \overset{v}{\rightarrow} (2,2,p)$$

and therefore $F_v(2,2,p;p+1) \leq F_v(3,p;p+1)$. In [11] the following problem is formulated:

Problem 4.2. *[11] Does there exist a positive integer p for which $F_v(2,2,p;p+1) \neq F_v(3,p;p+1)$?*

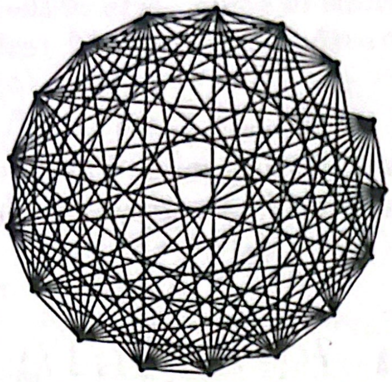
Theorem 4.1 gives a positive answer to Problem 4.2. Since

$$F_v(2,2,p;p+1) = F_v(3,p;p+1), \quad p \leq 5$$

(see [1]), it becomes clear that $p = 6$ is the smallest positive integer for which

$$F_v(2,2,p;p+1) \neq F_v(3,p;p+1)$$

For the proof Theorem 4.1 we will need the following



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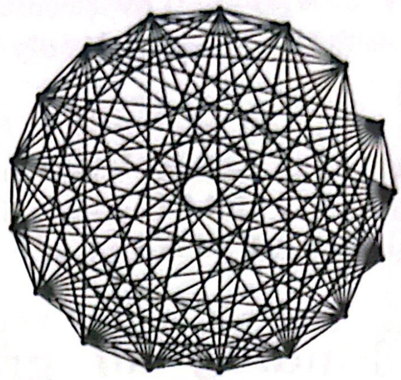
G_1



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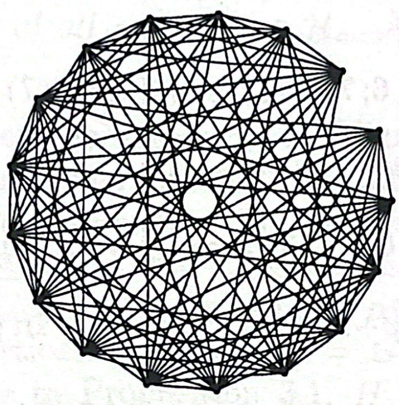
G_2



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1	1	0	1	1	1	1	0	1	1	0	1	1	1	1	1	0

G_3

Figure 1: All 3 graphs in $\mathcal{H}(2,2,6;7;17)$

Theorem 4.3. $|\mathcal{H}(2,2,6;7;17)| = 3$ *and* $\mathcal{H}_{extr}(2,2,6;7) = \mathcal{H}(2,2,6;7;17) = \{G_1, G_2, G_3\}$ *(see Figure 1).*

Proof. We will find all graphs in $\mathcal{H}(2,2,6;7;17)$ with the help of a computer. Let $G \in \mathcal{H}(2,2,6;7;17)$. Clearly $\alpha(G) \geq 2$, and according to (2.6), $\alpha(G) \leq 4$.

First, we prove that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 4. According to Theorem 1.3, $\overline{C}_{13}$ is the only graph in $\mathcal{H}(2,6;7;13)$. Starting from $\mathcal{H}_{max}(2,6;7;13) = \{\overline{C}_{13}\}$, by applying Algorithm 3.7 ($r = 4; t = 4$) we do not obtain any graphs, and from Theorem 3.8 it follows that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 4.

Now, we shall prove that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 3. It is clear that K_6 is the only graph in $\mathcal{H}_{max}(3;7;6)$. Starting from $\mathcal{H}_{max}(3;7;6) = \{K_6\}$ by applying Algorithm 3.7 ($r = 2; t = 3$) we obtain all graphs with independence number not greater than 3 in $\mathcal{H}_{max}(4;7;8)$. In the same way, we successively obtain all graphs with independence number not greater than 3 in $\mathcal{H}_{max}(5;7;10)$, $\mathcal{H}_{max}(6;7;12)$, $\mathcal{H}_{max}(2,6;7;14)$. In the end, no graphs are produced by applying Algorithm 3.7 ($r = 3; t = 3$) to the obtained graphs in $\mathcal{H}_{max}(2,6;7;14)$ with independence number not greater than 3, and from Theorem 3.8 we conclude that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 3.

The last part of the proof is to find all graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 2. It is clear that $K_7 - e$ is the only graph in $\mathcal{H}_{max}(3;7;7)$. By applying Algorithm 3.4 we successively obtain all graphs with independence number 2 in $\mathcal{H}_{max}(4;7;9)$, $\mathcal{H}_{max}(5;7;11)$, $\mathcal{H}_{max}(6;7;13)$, $\mathcal{H}_{max}(2,6;7;15)$ and $\mathcal{H}_{max}(2,2,6;7;17)$. As a result, we obtain the graph $G_1 \in \mathcal{H}_{max}(2,2,6;7;17)$, which is shown on Figure 1. According to Theorem 3.5, G_1 is the only graph in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 2. Since there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number greater than 2, we proved that $\mathcal{H}_{max}(2,2,6;7;17) = \{G_1\}$

The number of maximal K_7-free graphs and $(+K_6)$-graphs obtained in each step of the proof is shown in Table 3. By removing edges from G_1 we find that there are only two other graphs in $\mathcal{H}(2,2,6;7;17)$, which we will denote by G_2 and G_3 (see Figure 1). Let us also note, that $G_1 \supset G_2 \supset G_3$ and for the graphs G_1, G_2 and G_3 the inequality (2.3) is strict (see Conjecture 2.1). It is clear that G_3 is the only minimal graph in $\mathcal{H}(2,2,6;7;17)$. Some properties of the graphs G_1, G_2 and G_3 are given in Table 1. From (1.7) ($m = 8, p = 6$) we obtain

$$17 \leq F_v(2,2,6;7) \leq F_v(3,6;7) \leq 18. \tag{4.1}$$

The inequality $F_v(2,2,6;7) \geq 17$ also follows from the fact, that the graphs G_1, G_2 and G_3 have no isolated vertices. The inequality $F_v(3,6;7) \leq 18$ was first proved in [31].

From (4.1) it follows $\mathcal{H}_{extr}(2,2,6;7) = \mathcal{H}(2,2,6;7;17) = \{G_1, G_2, G_3\}$. Thus, we finish the proof of Theorem 4.3. □

Graph	$\|E(G)\|$	$\delta(G)$	$\Delta(G)$	$\alpha(G)$	$\chi(G)$	$\|Aut(G)\|$
G_1	108	12	13	2	9	2
G_2	107	11	13	2	9	4
G_3	106	11	13	2	9	40

Table 1: The graphs in $\mathcal{H}(2,2,6;7;17)$ and some of their properties

Proof of Theorem 4.1

The equality $F_v(2,2,6;7) = 17$ follows from Theorem 4.3. According to (4.1), it remains to be proved that $F_v(3,6;7) \neq 17$. Since $\mathcal{H}(3,6;7) \subseteq \mathcal{H}(2,2,6;7)$ (see (2.1)), but $G_1 \notin \mathcal{H}(3,6;7)$, we come to the conclusion that $H_v(3,6;7;17) = \emptyset$ and $F_v(3,6;7) = 18$. □

5 Proof of Theorem 1.4 (a)

We will do the proof with the help of the following

Theorem 5.1. *[1] Let $r_0'(p) = r_0'$ be the smallest positive integer for which*

$$\min_{r \geq 2} \{F_v(2_r, p; r+p-1) - r\} = F_v(2_{r_0'}, p; r_0' + p - 1) - r_0'.$$

Then:

(a) $F_v(2_r, p; r+p-1) = F(2_{r_0'}, p; r_0' + p - 1) + r - r_0', \quad r \geq r_0'.$

(b) *if $r_0' = 2$, then*

$$F_v(2_r, p; r+p-1) = F_v(2,2,p;p+1) + r - 2, \quad r \geq 2$$

(c) *if $r_0' > 2$ and G is an extremal graph in $\mathcal{H}(2_{r_0'}, p; r_0' + p - 1)$,*

then $G \overset{v}{\rightarrow} (2, r_0' + p - 2)$.

(d) $r_0' < F_v(2,2,p;p+1) - 2p$.

Theorem 5.1 is proved in [1] as Theorem 5.2. We will note that the proof of Theorem 5.1 is analogous to that of Theorem 6.1 from this paper.

In relation to Theorem 5.1(b) in [1] we formulate

Conjecture 5.2. *If* $p \geq 4$, *then*

$$\min_{r\geq 2} \{F_v(2_r, p; r+p-1) - r\} = F_v(2,2,p;p+1) - 2,$$

and therefore

$$F_v(2_r, p; r+p-1) = F_v(2,2,p;p+1) + r - 2, \quad r \geq 2.$$

It is not difficult to see that Conjecture 5.2 is true if and only if the sequence $\{F_v(2_r, p; r+p-1)\}$ for fixed p is strictly increasing with respect to $r \geq 2$. Since $F_v(2,2,3;4) = 14$ [24] and [5], and $F_v(a_1, ..., a_s; m-1) = m+6$, if $p=3$ and $m \geq 6$ [26], we have $r'_0(3) = 3$. Since $F_v(2,2,4;5) = 13$ [26], from Theorem 5.4 it follows that $r'_0(4) = 2$. The equality $r'_0(5) = 2$ is also true, but it does not follow from Theorem 5.4. It is proved with the help of a computer in [1] as Theorem 6.1. Therefore, Conjecture 5.2 is true when $p = 4$ and $p = 5$. We will prove that when $p = 6$ Conjecture 5.2 is also true. More specifically, we will prove

Theorem 5.3. $r'_0(6) = 2$ *and therefore* $F_v(2_r, 6; r+5) = r + 15$, $r \geq 2$.

Before proving Theorem 5.3 we will prove

Theorem 5.4. *Let* $F_v(2,2,p;p+1) \leq 2p+5$. *Then* $r'_0(p) = 2$ *and*

$$F_v(2_r, p; r+p-1) = F_v(2,2,p;p+1) + r - 2, \ r \geq 2.$$

Proof. From Theorem 5.1(b) it follows that it is enough to prove the equality $r'_0(p) = 2$. According to (1.5), $F_v(2,2,p;p+1) \geq 2p+4$. Therefore, only the following two cases are possible:

Case 1. $F_v(2,2,p;p+1) = 2p+4$. According to (1.5)
$F_v(2_r, p; r+p-1) \geq m+p+2 = r+2p+2$.
Therefore,
$F_v(2_r, p; r+p-1) - r \geq 2p+2 = F_v(2,2,p;p+1) - 2$, $r \geq 2$,
and we have $r'_0(p) = 2$.

Case 2. $F_v(2,2,p;p+1) = 2p+5$. From Theorem 2.3 we have $F_v(2_r, p; r+p-1) \geq r+2p+3$, $r \geq 2$. From this inequality we obtain
$F_v(2_r, p; r+p-1) - r \geq 2p+3 = F_v(2,2,p;p+1) - 2$, $r \geq 2$.
Therefore, in this case we also have $r'_0(p) = 2$. □

Remark 5.5. *It is unknown whether the first case is possible, i.e. if* $F_v(2,2,p;p+1) = 2p+4$ *for some* p. *If* $p \leq 7$ *this equality is not true.*

Proof of Theorem 5.3

According to Theorem 4.1, $F_v(2,2,6;7) = 17$. From this fact and Theorem 5.4 we obtain $r_0'(6) = 2$ and the equality $F_v(2_r,6;r+5) = r+15$, $r \geq 2$. □

Proof of Theorem 1.4 (a)

Since $a_1 = ... = a_{s-1} = 2$ and $a_s = 6$ we have $m = s+5$ and therefore

$$F_v(a_1,...,a_s;m-1) = F_v(2_{s-1},6;m-1) = F_v(2_{m-6},6;m-1).$$

From Theorem 5.3 it now follows that $F_v(a_1,...,a_s;m-1) = m+9$. □

6 Proof of Theorem 1.4 (b)

We will need the following

Theorem 6.1. *Let $r_0''(p) = r_0''$ be the smallest positive integer for which*

$$\min\{F_v(2_r,3,p;r+p+1) - r\} = F_v(2_{r_0''},3,p;r_0''+p+1) - r_0''$$

Then

(a) $F_v(2_r,3,p;r+p+1) = F(2_{r_0''},3,p;r_0''+p+1) + r - r_0'', \quad r \geq r_0''.$

(b) *if* $r_0'' = 0$, *then*

$$F_v(2_r,3,p;r+p+1) = F_v(3,p;p+1) + r, \quad r \geq 0$$

(c) *if* $r_0'' > 0$ *and* G *is an extremal graph in* $\mathcal{H}(2_{r_0''},3,p;r_0''+p+1)$,

then $G \overset{v}{\to} (2, r_0''+p)$.

(d) $r_0'' < F_v(3,p;p+1) - 2p - 2$.

Proof. (a) According to the definition of $r_0'' = r_0''(p)$ we have

$$F_v(2_r,3,p;r+p+1) \geq F_v(2_{r_0''},3,p;r_0''+p+1) + r - r_0'', \ r \geq 0.$$

Now we will prove that if $r \geq r_0''$ the opposite inequality is also true. Let us note that if $G \overset{v}{\to} (a_1,...,a_s)$, then $K_1 + G \overset{v}{\to} (2,a_1,...,a_s)$. It follows

$$G \overset{v}{\to} (a_1,...,a_s) \Rightarrow K_t + G \overset{v}{\to} (2_t,a_1,...,a_s). \tag{6.1}$$

Let $G \in \mathcal{H}_{extr}(2_{r_0''},3,p;r_0''+p+1)$. Then from (6.1) it follows that $K_{r-r_0''} + G \in \mathcal{H}(2_r,3,p;r+p+1)$, $r \geq r_0''$. Therefore

$F_v(2_r,3,p;r+p+1) \leq |V(K_{r-r_0''}+G)| = F_v(2_{r_0''},3,p;r_0''+p+1) + r - r_0''$, $r \geq r_0''$.

Thus, (a) is proved.

(b) If $r_0''(p) = 0$, then obviously the equality (b) follows from (a).

(c) Assume the opposite is true and let G be an extremal graph in $\mathcal{H}(2_{r_0''}, 3, p; r_0''+p+1)$, such that $V(G) = V_1 \cup V_2$ where V_1 is an independent set and V_2 does not contain $(r_0'' + p)$-clique. We can assume that $V_1 \neq \emptyset$. Let $G_1 = G[V_2] = G - V_1$. Then $\omega(G_1) < r_0'' + p$ and since $r_0'' \geq 1$, from Proposition 2.2 it follows that $G_1 \stackrel{v}{\rightarrow} (2_{r_0''-1}, 3, p)$. Therefore $G_1 \in \mathcal{H}(2_{r_0''-1}, 3, p; r_0'' + p)$ and
$|V(G)| - 1 \geq |V(G_1)| \geq F_v(2_{r_0''-1}, 3, p; r_0'' + p)$.
Since $|V(G)| = F_v(2_{r_0''}, 3, p; r_0'' + p + 1)$ we obtain
$F_v(2_{r_0''-1}, 3, p; r_0'' + p) - (r_0'' - 1) \leq F_v(2_{r_0''}, 3, p; r_0'' + p + 1) - r_0''$,
which contradicts the definition of r_0''.

(d) According to (1.5) $F_v(3, p; p+1) \geq 2p + 4$ and therefore in the case $r_0'' = 0$ the inequality holds. Let $r_0'' > 0$ and G be an extremal graph in $\mathcal{H}(2_{r_0''}, 3, p; r_0'' + p + 1)$. According to (2.3)

$$\chi(G) \geq r_0'' + p + 2. \tag{6.2}$$

According to (c) and Theorem 1.3
$|V(G)| \geq 2r_0'' + 2p + 1$.
Since $\chi(\overline{C}_{2r_0''+2p+1}) = r_0'' + p + 1$, from (6.2) it follows $G \neq \overline{C}_{2r_0''+2p+1}$. By Theorem 1.3(b)
$|V(G)| = F_v(2_{r_0''}, 3, p; r_0'' + p + 1) \geq 2r_0'' + 2p + 2$.
Since $r_0'' > 0$, we have
$F_v(2_{r_0''}, 3, p; r_0'' + p + 1) - r_0'' < F_v(3, p; p + 1)$.
From these inequalities we can easily see that
$r_0'' < F_v(3, p; p + 1) - 2p - 2$. □

Since $F_v(3, 3; 4) = 14$, from (1.6) we obtain $r_0''(3) = 1$. Also from (1.6) we see that $r_0''(4) = 0$ and $r_0''(5) = 0$. We suppose the following conjecture is true

Conjecture 6.2. *If $p \geq 4$, then*

$$\min\{F_v(2_r, 3, p; r + p - 1) - r\} = F_v(3, p; p + 1),$$

and therefore

$$F_v(2_r, 3, p; r + p + 1) = F_v(3, p; p + 1) + r.$$

It is not difficult to see that Conjecture 6.2 is true if and only if the sequence $\{F_v(2_r, 3, p; r + p + 1)\}$ for fixed p is strictly increasing with respect to r. We will prove that when $p = 6$ Conjecture 6.2 is also true. The Theorem 1.4 (b) follows easily from this fact.

Theorem 6.3. $r_0''(6) = 0$.

Proof. From Theorem 6.1 (d) we obtain $r_0''(6) < 4$. Therefore we have to prove $r_0''(6) \neq 1$, $r_0''(6) \neq 2$ and $r_0''(6) \neq 3$. Since $F_v(3,6;7) = 18$, we have to prove the inequalities $F_v(2,3,6;8) > 18$, $F_v(2,2,3,6;9) > 19$ and $F_v(2,2,2,3,6;10) > 20$. We will prove these inequalities with the help of a computer. From (6.1) (t = 1) it is easy to see that $F_v(2_{r-1},3;p) + 1 \geq F_v(2_r,3;p+1)$ and therefore it is enough to prove $F_v(2,2,2,3,6;10) > 20$. We shall present the proof of this inequality only, but we also checked the other two inequalities in the same way with a computer in order to obtain more information, which is presented in Appendix A.

Similarly to the proof of Theorem 4.3, we shall use Algorithm 3.4 and Algorithm 3.7 to prove that $\mathcal{H}(2,2,2,3,6;10;20) = \emptyset$. According to (2.6), there are no graphs in $\mathcal{H}(2,2,2,3,6;10;20)$ with independence number greater than 4.

By Theorem 1.3, $K_3 + \overline{C}_{13}$ is the only graph in $\mathcal{H}(2,2,3,6;10;16)$. Starting from $\mathcal{H}_{max}(2,2,3,6;10;16) = \{K_3 + \overline{C}_{13}\}$, by applying Algorithm 3.7 ($r = 4; t = 4$) we show that there are no graphs in $\mathcal{H}_{max}(2,2,2,3,6;10;20)$ with independence number 4.

The next step is to prove that there are no graphs in $\mathcal{H}_{max}(2,2,2,3,6;10;20)$ with independence number 3. The only graph in $\mathcal{H}_{max}(6;10;9)$ is K_9. Starting from $\mathcal{H}_{max}(6;10;9) = \{K_9\}$ by applying Algorithm 3.7 ($r = 2; t = 3$) we successively obtain all graphs with independence number not greater than 3 in $\mathcal{H}_{max}(2,6;10;11)$, $\mathcal{H}_{max}(3,6;10;13)$, $\mathcal{H}_{max}(2,3,6;10;15)$, $\mathcal{H}_{max}(2,2,3,6;10;17)$. In the end, we apply Algorithm 3.7 ($r = 3; t = 3$) to the obtained graphs in $\mathcal{H}_{max}(2,2,3,6;10;17)$ with independence number not greater than 3 to show that there are no graphs in $\mathcal{H}_{max}(2,2,2,3,6;10;20)$ with independence number 3.

Finally, we prove that there are no graphs in $\mathcal{H}_{max}(2,2,2,3,6;10;20)$ with independence number 2. The only graph in $\mathcal{H}_{max}(6;10;10)$ is $K_{10} - e$. Starting from $\mathcal{H}_{max}(6;10;10) = \{K_{10} - e\}$ by applying Algorithm 3.4 we successively obtain all graphs with independence number 2 in $\mathcal{H}_{max}(2,6;10;12)$, $\mathcal{H}_{max}(3,6;10;14)$, $\mathcal{H}_{max}(2,3,6;10;16)$, $\mathcal{H}_{max}(2,2,3,6;10;18)$ and $\mathcal{H}_{max}(2,2,2,3,6;10;20)$. As a result, no graphs in $\mathcal{H}_{max}(2,2,2,3,6;10;20)$ with independence number 2 were obtained.

Thus, we proved $\mathcal{H}_{max}(2,2,2,3,6;10;20) = \emptyset$ and therefore $F_v(2,2,2,3,6;10) > 20$ and $r_0''(6) = 0$.

The numbers of graphs obtained in each step are shown in Table 7 (see also Table 5 and Table 6). □

Proof of Theorem 1.4 (b)

According to Theorem 6.3 and Theorem 6.1 (b) we have $F_v(2_{m-s}, 3, 6; m-1) = m + 10$. From (2.5) it now follows $F_v(a_1, ..., a_s; m-1) \geq m + 10$. The opposite inequality is true according to (1.7) (see also the Main Theorem from [2]). □

7 Finding all graphs in $\mathcal{H}(2,2,6;7;18)$ and proofs of Theorem 1.5 and Theorem 1.6

Theorem 7.1. $|\mathcal{H}(2,2,6;7;18)| = 76515$.

Proof. Similarly to the proof of Theorem 4.3, we will find all graphs in $\mathcal{H}(2,2,6;7;18)$ with the help of a computer. Some of the graphs that we obtain in the steps of this proof were already obtained in the proof of Theorem 4.3 (compare Table 3 to Table 4).

Let $G \in \mathcal{H}(2,2,6;7;18)$. Clearly $\alpha(G) \geq 2$, and according to (2.6), $\alpha(G) \leq 5$.

First, we prove that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 5. According to Theorem 1.3, $\overline{C}_{13}$ is the only graph in $\mathcal{H}(2,6;7;13)$. Starting from $\mathcal{H}_{max}(2,6;7;13) = \{\overline{C}_{13}\}$, by applying Algorithm 3.7 ($r = 5; t = 5$) we show that there are no graphs in $\mathcal{H}(2,2,6;7;18)$ with independence number 5.

Now, we shall prove that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;17)$ with independence number 4. Starting from $\mathcal{H}_{max}(3;7;6) = \{K_6\}$, by applying Algorithm 3.7 ($r = 2; t = 4$) we successively obtain all graphs with independence number not greater than 4 in $\mathcal{H}_{max}(4;7;8)$, $\mathcal{H}_{max}(5;7;10)$, $\mathcal{H}_{max}(6;7;12)$, $\mathcal{H}_{max}(2,6;7;14)$. By applying Algorithm 3.7 ($r = 4; t = 4$) to the obtained graphs in $\mathcal{H}_{max}(2,6;7;14)$ with independence number not greater than 4 we conclude that there are no graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 4.

Next, we find all graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 3. Starting from $\mathcal{H}_{max}(3;7;7) = \{K_7 - e\}$, by applying Algorithm 3.7 ($r = 2; t = 3$) we successively obtain all graphs with independence number not greater than 3 in $\mathcal{H}_{max}(4;7;9)$, $\mathcal{H}_{max}(5;7;11)$, $\mathcal{H}_{max}(6;7;13)$, $\mathcal{H}_{max}(2,6;7;15)$. By applying Algorithm 3.7 ($r = 3; t = 3$) to the obtained graphs in $\mathcal{H}_{max}(2,6;7;15)$ with independence number not greater than 3 we obtain all 308 graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 3.

The last, and computationally most difficult step, is to find all graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 2. It is easy to see that $\mathcal{H}_{max}(3;7;8) = \{\overline{K}_3 + K_5, C_4 + K_4\}$ and therefore $C_4 + K_4$ is the only graph in $\mathcal{H}_{max}(3;7;8)$ with independence number 2. Starting from $\{C_4 + K_4\}$, by applying Algorithm 3.4 we successively obtain all graphs with independence number 2 in $\mathcal{H}_{max}(4;7;10)$, $\mathcal{H}_{max}(5;7;12)$, $\mathcal{H}_{max}(6;7;14)$, $\mathcal{H}_{max}(2,6;7;16)$ and $\mathcal{H}_{max}(2,2,6;7;18)$. As a result, we find all 84 graphs in $\mathcal{H}_{max}(2,2,6;7;18)$ with independence number 2.

Thus, we obtained all 392 graphs in $\mathcal{H}_{max}(2,2,6;7;18)$. By removing edges from these graphs we find all 76 515 graphs in $\mathcal{H}(2,2,6;7;18)$. Some properties of these graphs are listed in Table 2. The number of maximal K_7-free graphs and $(+K_6)$-graphs obtained in each step of the proof is shown in Table 4. □

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G)\rvert$	#
106	1	0	3	13	65	2	290	8	84	1	72 335
107	4	1	20	14	76 450	3	76 225	9	76 431	2	3 699
108	19	2	124							4	430
109	88	3	571							8	33
110	369	4	1 943							10	2
111	1 240	5	4 986							16	2
112	3 303	6	9 826							20	6
113	6 999	7	14 896							24	1
114	11 780	8	17 057							36	1
115	15 603	9	14 288							40	6
116	15 956	10	8 397								
117	12 266	11	3 504								
118	6 575	12	876								
119	2 044	13	24								
120	261										
121	7										

Table 2: Some properties of the graphs in $\mathcal{H}(2,2,6;7;18)$

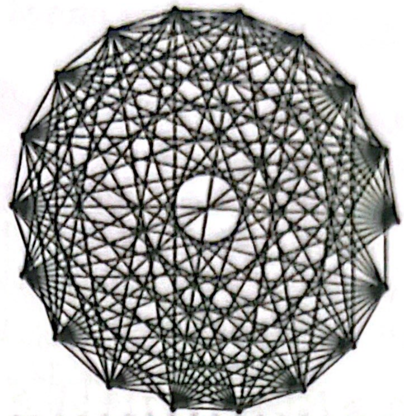
We check with a computer that among the 76 515 graphs in $\mathcal{H}(2,2,6;7;18)$, only the graph G_4 (see Figure 2) belongs to $\mathcal{H}(3,6;7;18)$. This is the graph that gives the upper bound $F_v(3,6;7) \leq 18$ in [31]. Thus, we proved the following

Theorem 7.2. $|\mathcal{H}(3,6;7;18)| = 1$ *and* $\mathcal{H}_{extr}(3,6;7) = \mathcal{H}(3,6;7;18) = \{G_4\}$.

Let us note, that $\chi(G_4) = 9$ and for this graph the inequality (2.3) is strict. However, from Theorem 7.2 it follows that in this special case Conjecture 2.1 is true.

There are two 13-regular graphs in $\mathcal{H}(2,2,6;7;18)$, one of them being G_4. The graph G_4 is the only vertex transitive graph in $\mathcal{H}(2,2,6;7;18)$ and it has 36 automorphisms. The other 13-regular graph has 24 automorphisms.

Let us also note that there are 2 467 vertex critical graphs in $\mathcal{H}(2,2,6;7;18)$. We obtained all 74048 non-critical graphs in another way by adding one vertex to the graphs in $\mathcal{H}(2,2,6;7;17)$. This also testifies to the correctness of our implementation.



0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
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1	0	0	1	1	1	0	1	1	1	1	1	1	1	0	1	1	0
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0	0	1	0	1	1	0	1	1	1	1	1	1	1	1	0	1	1
0	0	1	1	0	1	1	0	1	1	1	1	1	1	1	1	0	1
0	1	0	0	1	1	1	1	0	1	1	1	1	1	1	1	1	0

G_4

Figure 2: The only graph $G_4 \in \mathcal{H}(3,6;7;18)$ from [31]

Proof of Theorem 1.5

According to Theorem 9.2 from this paper, $F_v(a_1,...,a_s;7) \geq F_v(2_{m-6},6;7)$. We shall prove by induction that $F_v(2_{m-6},6;7) \geq 3m - 5, m \geq 9$.

The base case is $m = 9$, i.e. we have to prove that $F_v(2,2,2,6;7) \geq 22$. We will show that $\mathcal{H}(2,2,2,6;7;21) = \emptyset$ From $F_v(2,2,6;7) = 17$ and Proposition 2.2 it follows that there are no graphs in $\mathcal{H}(2,2,2,6;7;21)$ with independence number greater than 4. All graphs in $\mathcal{H}(2,2,6;7;17)$ have independence number 2 (see Table 1) and all graphs in $\mathcal{H}(2,2,6;7;18)$ have independence number 2 or 3 (see Table 2). No graphs are obtained by applying Algorithm 3.7 ($r = 4, t = 4$) to the graphs in $\mathcal{H}_{max}(2,2,6;7;17)$, or by applying Algorithm 3.7 ($r = 3, t = 3$) to the graphs in $\mathcal{H}_{max}(2,2,6;7;18)$. From Theorem 3.8 it follows that there are no graphs in $\mathcal{H}(2,2,2,6;7;21)$ with independence number 3 or 4. It remains to be proved that there are no graphs in $\mathcal{H}(2,2,2,6;7;21)$ with independence number 2. All 21-vertex graphs G for which $\alpha(G) < 3$ and $\omega(G) < 7$ are known and are available on [16]. There are 1 118 436 such graphs G, and with the help of the computer we check that none of these graphs belong to $\mathcal{H}(2,2,2,6;7)$. Thus, we proved $\mathcal{H}(2,2,2,6;7;21) = \emptyset$ and $F_v(2,2,2,6;7) \geq 22$.

Now suppose that for all m' such that $9 \leq m' < m$ we have $F_v(2_{m'-6},6;7) \geq 3m'-5$. Let $G \in \mathcal{H}(2_{m-6},6;7)$ and $|V(G)| = F_v(2_{m-6},6;7)$. From the base case it follows that $F_v(2_{m-6},6;7) > 22$, and since the Ramsey number $R(3,7) = 23$ we have $\alpha(G) \geq 3$. Let A be an independent set of vertices of G and $|A| = 3$. According to Proposition 2.2, $G - A \in \mathcal{H}(2_{m-7},6;7)$ and therefore

$$F_v(2_{m-6},6;7) = |V(G)| \geq F_v(2_{m-7},6;7) + |A| \geq 3(m-1)-5+3 = 3m-5.$$

□

Proof of Theorem 1.6

We will prove only (b), since (a) can be proved in the same way. The lower bound is true according to Theorem 1.5. Clearly, we can assume that $a_s = \max\{a_1,...,a_s\} = 6$. Therefore, from (2.1) we obtain the inclusion $\mathcal{H}(6,6;7) \subseteq \mathcal{H}(a_1,...,a_s;7)$ and it follows that $F_v(a_1,...,a_s;7) \leq F_v(6,6;7)$. Kolev proves in [10] that

$$F_v(a_1,...,a_s;q+1).F_v(b_1,...,b_s;t+1) \geq F_v(a_1.b_1,...,a_s.b_s;qt+1),$$

where $q = \max\{a_1,...,a_s\}$ and $t = \max\{b_1,...,b_s\}$. Since $F_v(2,2;3) = 5$ and $F_v(3,3;4) = 14$, [19] and [27], it follows that $F_v(6,6;7) \leq F_v(2,2;3).F_v(3,3;4) = 70$. In (a) instead of $F_v(3,3;4) = 14$ we use $F_v(2,3;4) = 7$ (see Theorem 1.3). □

8 Proof of Theorem 1.7

Proof of the lower bound $F_v(2,2,7;8) \geq 20$

We can prove that $\mathcal{H}(2,2,7;8;19) = \emptyset$ using the method from the proof of Theorem 4.3. Suppose that $G \in \mathcal{H}(2,2,7;8;19)$. Clearly $\alpha(G) \geq 2$, and according to (2.6), $\alpha(G) \leq 4$. According to Theorem 1.3, $\overline{C}_{15}$ is the only graph in $\mathcal{H}(2,7;8;15)$. By applying Algorithm 3.7($r = 4; t = 4$) to $\mathcal{H}_{max}(2,7;8;15) = \{\overline{C}_{15}\}$ we prove that there are no graphs in $\mathcal{H}_{max}(2,2,7;8;19)$ with independence number 4.

With the help of Algorithm 3.7($r = 2; t = 3$) we successively obtain all graphs with independence number not greater than 3 in $\mathcal{H}_{max}(4;8;8)$, $\mathcal{H}_{max}(5;8;10)$, $\mathcal{H}_{max}(6;8;12)$, $\mathcal{H}_{max}(7;8;14)$, $\mathcal{H}_{max}(2,7;8;16)$. Then, we apply Algorithm 3.7($r = 3; t = 3$) to the obtained graphs in

$\mathcal{H}_{max}(2,7;8;16)$ with independence number not greater than 3 to show that there are no graphs in $\mathcal{H}_{max}(2,2,7;8;19)$ with independence number 3.

In the last and computationally most difficult part of the proof, with the help of Algorithm 3.4 we successively obtain all graphs with independence number 2 in $\mathcal{H}_{max}(4;8;9)$, $\mathcal{H}_{max}(5;8;11)$, $\mathcal{H}_{max}(6;8;13)$, $\mathcal{H}_{max}(7;8;15)$, $\mathcal{H}_{max}(2,7;8;17)$ and $\mathcal{H}_{max}(2,2,7;8;19)$. As a result, no graphs in $\mathcal{H}_{max}(2,2,7;8;19)$ with independence number 2 are obtained.

Thus, we proved $\mathcal{H}_{max}(2,2,7;8;19) = \emptyset$. The number of graphs obtained in each step are shown in Table 8. □

Proof of the upper bound $F_v(2,2,7;8) \leq 20$

We need to construct a 20-vertex graph in $\mathcal{H}(2,2,7,8;20)$. All vertex transitive graphs with up to 31 vertices are known and can be found in [28]. With the help of a computer we check which of these graphs belong to $\mathcal{H}(2,2,7;8)$. In this way, we find one 24-vertex graph, one 28-vertex graph and 6 30-vertex graphs in $\mathcal{H}(2,2,7;8)$.

By removing one vertex from the 24-vertex transitive graph in $\mathcal{H}(2,2,7;8)$ we obtain 3 23-vertex graphs in $\mathcal{H}(2,2,7;8)$, and by removing two vertices we obtain 8 22-vertex graphs in $\mathcal{H}(2,2,7;8)$. We add two edges to one of the 8 22-vertex graphs (the only one with 180 edges) we find one graph in $\mathcal{H}_{max}(2,2,7;8;22)$. Using the following Procedure 8.1 we find 1696 more graphs in $\mathcal{H}_{max}(2,2,7;8;22)$.

By removing one vertex to the obtained graphs in $\mathcal{H}_{max}(2,2,7;8;22)$ we find 22 21-vertex graphs in $\mathcal{H}(2,2,7;8)$. We add edges to these graphs to obtain 22 graphs in $\mathcal{H}_{max}(2,2,7;8;21)$. Then, we apply Procedure 8.1 twice to obtain 15259 more graphs in $\mathcal{H}_{max}(2,2,7;8;21)$.

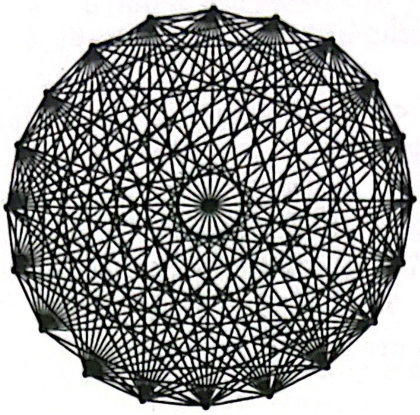
By removing one vertex to the obtained graphs in $\mathcal{H}_{max}(2,2,7;8;21)$ we find 9 20-vertex graphs in $\mathcal{H}(2,2,7;8)$. Again, by successively applying Procedure 8.1 we obtain 39 graphs in $\mathcal{H}_{max}(2,2,7;8;20)$. One of these graphs is the graph G_5 shown on Figure 3. Later, we shall use the graph G_5 in the proof of Theorem 1.8. □

Procedure 8.1. *[1] Extending a set of maximal graphs in $\mathcal{H}(a_1,...,a_s;q;n)$.*

1. Let $\mathcal{A}$ be a set of maximal graphs in $\mathcal{H}(a_1,...,a_s;q;n)$.

2. By removing edges from the graphs in $\mathcal{A}$, find all their subgraphs which are in $\mathcal{H}(a_1,...,a_s;q;n)$. This way a set of non-maximal graphs in $\mathcal{H}(a_1,...,a_s;q;n)$ is obtained.

3. Add edges to the non-maximal graphs to find all their supergraphs which are maximal in $\mathcal{H}(a_1,...,a_s;q;n)$. Extend the set $\mathcal{A}$ by adding the new maximal graphs.



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1	1	1	1	0	1	1	0	1	1	1	0	1	1	1	1	1	1	1	0

G_5

Figure 3: 20-vertex graph $G_5 \in \mathcal{H}(2,2,7;8)$

9 Proof of Theorem 1.8

In [1], we define a modification of the vertex Folkman numbers $F_v(a_1, ..., a_s; q)$ with the help of which we obtain the upper bound in (1.7).

Definition 9.1. *[1] Let G be a graph and let m and p be positive integers. The expression*

$$G \overset{v}{\rightarrow} m\big|_p$$

means that for every choice of positive integers $a_1, ..., a_s$ (s is not fixed), such that $m = \sum_{i=1}^{s}(a_i - 1) + 1$ and $\max\{a_1, ..., a_s\} \le p$, we have

$$G \overset{v}{\rightarrow} (a_1, ..., a_s).$$

In [1] we also define the following notations (see also [2]):

$\widetilde{\mathcal{H}}(m|_p;q) = \left\{G : G \overset{v}{\to} m|_p \text{ and } \omega(G) < q\right\}.$

$\widetilde{F}_v(m|_p;q) = \min\left\{|V(G)| : G \in \widetilde{\mathcal{H}}(m|_p;q)\right\}.$

To prove the upper bound in Theorem 1.8 we shall use the following results from [1].

Theorem 9.2. *[1] Let $a_1, \ldots, a_s$ be positive integers and let m and p be defined by (1.3), $q > p$. Then*

$$F_v(2_{m-p}, p; q) \le F_v(a_1, \ldots, a_s; q) \le \widetilde{F}_v(m|_p; q).$$

Theorem 9.3. *[1] Let m, m_0, p and q be positive integers, $m \ge m_0$ and $q > \min\{m_0, p\}$. Then*

$$\widetilde{F}_v(m|_p; m - m_0 + q) \le \widetilde{F}_v(m_0|_p; q) + m - m_0.$$

Proof of Theorem 1.8

The lower bound in Theorem 1.8 follows from Theorem 1.7 and Theorem 2.3 To prove the upper bound, we shall use the graph $G_6 \in \mathcal{H}(3,7;8) \cap \mathcal{H}(4,6;8) \cap \mathcal{H}(5,5;8)$ (see Figure 4) obtained by adding one vertex to the graph $G_5 \in \mathcal{H}(2,2,7;8;20)$ (see Figure 3). Using (2.1), it is easy to prove that from $G_6 \overset{v}{\to} (3,7)$, $G_6 \overset{v}{\to} (4,6)$ and $G_6 \overset{v}{\to} (5,5)$ it follows $G_6 \overset{v}{\to} 9|_7$. Therefore $G_6 \in \widetilde{\mathcal{H}}(9|_7; 8)$ and $\widetilde{F}_v(9|_7; 8) \le 21$. Now from Theorem 9.2 and Theorem 9.3 we derive

$$F_v(a_1, \ldots, a_s; m-1) \le \widetilde{F}_v(m|_7; m-1) \le \widetilde{F}_v(9|_7; 8) + m - 9 \le m + 12.$$

□

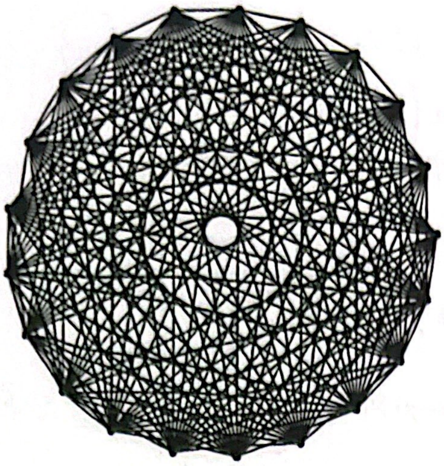
Regarding the number $F_v(3,7;8)$, the following bounds were known:

$$18 \le F_v(3,7;8) \le 22.$$

The lower bound is true according to (1.5) and the upper bound was proved in [31]. Using the results in this paper, we improve these bounds by proving the following

Theorem 9.4. $20 \le F_v(3,7;8) \le 21$.

Proof. The upper bound is true according to Theorem 1.8 and the lower bound follows from $F_v(3,7;8) \ge F_v(2,2,7;8) = 20$. □



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0 0 0 0 0 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1
0 0 0 0 1 1 1 1 0 1 1 1 1 0 1 1 1 1 1 1 1
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1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 0 1 1 0 1
1 1 1 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 1 1 0
```

G_6

Figure 4: 21-vertex graph $G_6 \in \widetilde{\mathcal{H}}(9|_7; 8)$

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Appendix A Results of computations

set	independence number	maximal graphs	$(+K_6)$-graphs
$\mathcal{H}(2,6;7;13)$	≤ 4	1	1
$\mathcal{H}(2,2,6;7;17)$	$= 4$	0	
$\mathcal{H}(3;7;6)$	≤ 3	1	2
$\mathcal{H}(4;7;8)$	≤ 3	2	12
$\mathcal{H}(5;7;10)$	≤ 3	6	274
$\mathcal{H}(6;7;12)$	≤ 3	37	78 926
$\mathcal{H}(2,6;7;14)$	≤ 3	20	5 291
$\mathcal{H}(2,2,6;7;17)$	$= 3$	0	
$\mathcal{H}(3;7;7)$	≤ 2	1	3
$\mathcal{H}(4;7;9)$	≤ 2	2	22
$\mathcal{H}(5;7;11)$	≤ 2	5	468
$\mathcal{H}(6;7;13)$	≤ 2	24	97 028
$\mathcal{H}(2,6;7;15)$	≤ 2	473	10 018 539
$\mathcal{H}(2,2,6;7;17)$	$= 2$	1	
$\mathcal{H}(2,2,6;7;17)$		1	

Table 3: Steps in finding all maximal graphs in $\mathcal{H}(2,2,6;7;17)$

set	independence number	maximal graphs	$(+K_6)$-graphs
$\mathcal{H}(2,6;7;13)$	≤ 5	1	1
$\mathcal{H}(3,6;7;18)$	$= 5$	0	
$\mathcal{H}(3;7;6)$	≤ 4	1	2
$\mathcal{H}(4;7;8)$	≤ 4	2	13
$\mathcal{H}(5;7;10)$	≤ 4	7	317
$\mathcal{H}(6;7;12)$	≤ 4	50	102 387
$\mathcal{H}(2,6;7;14)$	≤ 4	20	5 293
$\mathcal{H}(2,2,6;7;18)$	$= 4$	0	
$\mathcal{H}(3;7;7)$	≤ 3	1	4
$\mathcal{H}(4;7;9)$	≤ 3	3	45
$\mathcal{H}(5;7;11)$	≤ 3	12	3 071
$\mathcal{H}(6;7;13)$	≤ 3	168	4 691 237
$\mathcal{H}(2,6;7;15)$	≤ 3	1627	70 274 176
$\mathcal{H}(2,2,6;7;18)$	$= 3$	308	
$\mathcal{H}(3;7;8)$	≤ 2	1	8
$\mathcal{H}(4;7;10)$	≤ 2	3	82
$\mathcal{H}(5;7;12)$	≤ 2	10	5 057
$\mathcal{H}(6;7;14)$	≤ 2	96	2 799 416
$\mathcal{H}(2,6;7;16)$	≤ 2	7509	920 112 878
$\mathcal{H}(2,2,6;7;18)$	$= 2$	84	
$\mathcal{H}(2,2,6;7;18)$		392	

Table 4: Steps in finding all maximal graphs in $\mathcal{H}(2,2,6;7;18)$

set	independence number	maximal graphs	$(+K_7)$-graphs
$\mathcal{H}(3,6;8;14)$	≤ 4	1	1
$\mathcal{H}(2,3,6;8;18)$	$= 4$	0	
$\mathcal{H}(4;8;7)$	≤ 3	1	2
$\mathcal{H}(5;8;9)$	≤ 3	2	12
$\mathcal{H}(6;8;11)$	≤ 3	6	276
$\mathcal{H}(2,6;8;13)$	≤ 3	37	79 749
$\mathcal{H}(3,6;8;15)$	≤ 3	21	3 458
$\mathcal{H}(2,3,6;8;18)$	$= 3$	0	
$\mathcal{H}(4;8;8)$	≤ 2	1	3
$\mathcal{H}(5;8;10)$	≤ 2	2	22
$\mathcal{H}(6;8;12)$	≤ 2	5	489
$\mathcal{H}(2,6;8;14)$	≤ 2	25	119 126
$\mathcal{H}(3,6;8;16)$	≤ 2	509	3 582 157
$\mathcal{H}(2,3,6;8;18)$	$= 2$	0	
$\mathcal{H}(2,3,6;8;18)$		0	

Table 5: Steps in finding all maximal graphs in $\mathcal{H}(2,3,6;8;18)$

set	independence number	maximal graphs	$(+K_8)$-graphs
$\mathcal{H}(2,3,6;9;15)$	≤ 4	1	1
$\mathcal{H}(2,2,3,6;9;19)$	$= 4$	0	
$\mathcal{H}(5;9;8)$	≤ 3	1	2
$\mathcal{H}(6;9;10)$	≤ 3	2	12
$\mathcal{H}(2,6;9;12)$	≤ 3	6	277
$\mathcal{H}(3,6;9;14)$	≤ 3	37	79 901
$\mathcal{H}(2,3,6;9;16)$	≤ 3	21	3 459
$\mathcal{H}(2,2,3,6;9;19)$	$= 3$	0	
$\mathcal{H}(5;9;9)$	≤ 2	1	3
$\mathcal{H}(6;9;11)$	≤ 2	2	22
$\mathcal{H}(2,6;9;13)$	≤ 2	5	496
$\mathcal{H}(3,6;9;15)$	≤ 2	25	121 499
$\mathcal{H}(2,3,6;9;17)$	≤ 2	512	3 585 530
$\mathcal{H}(2,2,3,6;9;19)$	$= 2$	0	
$\mathcal{H}(2,2,3,6;9;19)$		0	

Table 6: Steps in finding all maximal graphs in $\mathcal{H}(2,2,3,6;9;19)$

set	independence number	maximal graphs	$(+K_9)$-graphs
$\mathcal{H}(2,2,3,6;10;16)$	≤ 4	1	1
$\mathcal{H}(2,2,2,3,6;10;20)$	$= 4$	0	
$\mathcal{H}(6;10;9)$	≤ 3	1	2
$\mathcal{H}(2,6;10;11)$	≤ 3	2	12
$\mathcal{H}(3,6;10;13)$	≤ 3	6	277
$\mathcal{H}(2,3,6;10;15)$	≤ 3	37	79 934
$\mathcal{H}(2,2,3,6;10;17)$	≤ 3	21	3 459
$\mathcal{H}(2,2,2,3,6;10;20)$	$= 3$	0	
$\mathcal{H}(6;10;10)$	≤ 2	1	3
$\mathcal{H}(2,6;10;12)$	≤ 2	2	22
$\mathcal{H}(3,6;10;14)$	≤ 2	5	498
$\mathcal{H}(2,3,6;10;16)$	≤ 2	25	121 864
$\mathcal{H}(2,2,3,6;10;18)$	≤ 2	512	3 585 546
$\mathcal{H}(2,2,2,3,6;10;20)$	$= 2$	0	
$\mathcal{H}(2,2,2,3,6;10;20)$		0	

Table 7: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,3,6;10;20)$

set	independence number	maximal graphs	$(+K_7)$-graphs
$\mathcal{H}(2,7;8;15)$	≤ 4	1	1
$\mathcal{H}(2,2,7;8;19)$	$= 4$	0	
$\mathcal{H}(4;8;8)$	≤ 3	1	4
$\mathcal{H}(5;8;10)$	≤ 3	3	45
$\mathcal{H}(6;8;12)$	≤ 3	12	3 104
$\mathcal{H}(7;8;14)$	≤ 3	169	4 776 518
$\mathcal{H}(2,7;8;16)$	≤ 3	34	22 896
$\mathcal{H}(2,2,7;8;19)$	$= 3$	0	
$\mathcal{H}(4;8;9)$	≤ 2	1	8
$\mathcal{H}(5;8;11)$	≤ 2	3	84
$\mathcal{H}(6;8;13)$	≤ 2	10	5 394
$\mathcal{H}(7;8;15)$	≤ 2	102	4 984 994
$\mathcal{H}(2,7;8;17)$	≤ 2	2760	380 361 736
$\mathcal{H}(2,2,7;8;19)$	$= 2$	0	
$\mathcal{H}(2,2,7;8;19)$		0	

Table 8: Steps in finding all maximal graphs in $\mathcal{H}(2,2,7;8;19)$