Strong Chromatic Connectivity of Complete Bipartite Graphs

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Abstract

Let G be an edge-colored connected graph. For vertices u and v of G, a shortest $u-v$ path P in G is a $u-v$ geodesic and P is a proper $u-v$ geodesic if no two adjacent edges in P are colored the same. An edge coloring of a connected graph G is called a proper k-geodesic coloring of G for some positive integer k if for every two nonadjacent vertices u and v of G, there exist at least k internally disjoint proper $u-v$ geodesics in G. The minimum number of the colors required in a proper k-geodesic coloring of G is the strong proper k-connectivity $\text{spc}_k(G)$ of G. Sharp lower bounds are established for the strong proper k-connectivity of complete bipartite graphs $K_{r,s}$ for all integers k, r, s with $2 \le k \le r \le s$ and it is shown that the strong proper 2-connectivity of $K_{r,s}$ is $\text{spc}_2(K_{r,s}) = \lceil \sqrt[r-1]{s} \rceil$ for $2 \le r \le s$.

Key Words: edge colorings, strong-path colorings, strong connectivity, complete bipartite graphs.

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1 Introduction

Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G. The minimum number of colors required for a proper-path coloring of G is called the *proper connection number* $\text{pc}(G)$ of G (see [2]). The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the minimum of the lengths of the $u-v$ paths in G. A $u-v$ path of length $d(u, v)$ is called a $u-v$ *geodesic*. Let

c be an edge coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper $u-v$ geodesic* in G is a proper $u-v$ path of length $d(u,v)$. If there is a proper $u-v$ geodesic for every two vertices u and v of G, then c is called a *strong proper-path coloring* of G. The minimum number of colors required to produce a strong proper-path coloring of G is called the *strong proper connection number* $\mathrm{spc}(G)$ of G (see [1]). In a *proper edge coloring* of a nonempty graph G, no two adjacent edges are colored the same. The minimum number of colors required for a proper edge coloring of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. In general, if G is a nontrivial connected graph, then $1 \leq \mathrm{pc}(G) \leq \mathrm{spc}(G) \leq \chi'(G)$. These concepts were inspired by rainbow colorings introduced in [3]. Recently, much research has been done on these concepts (see [1, 2, 10] for example). In fact, there is a book [9] on rainbow colorings and a dynamic survey [8] on proper-path colorings in graphs.

The *connectivity* $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. The connectivity is a common measure of connectedness for a graph. A graph G is *ℓ-connected* for some positive integer ℓ if $\kappa(G) \geq \ell$. That is, G is ℓ-connected if the removal of fewer than ℓ vertices from G results in neither a disconnected nor a trivial graph. Suppose that G is an ℓ-connected graph for some positive integer ℓ. It then follows from a well-known theorem of Whitney [11] that for every integer k with $1 \leq k \leq \ell$ and every two distinct vertices u and v of G, the graph G contains k internally disjoint $u-v$ paths.

Let G be a graph with connectivity $k \geq 1$. The *chromatic connectivity* $\kappa_\chi(G)$ of G is the minimum number of colors needed in an edge coloring of G such that every two distinct vertices u and v of G are connected by k internally disjoint proper $u-v$ paths. For a graph G with $\kappa_\chi(G) \geq 3$, there are *intermediate concepts* between the proper connection number $\mathrm{pc}(G)$ and the chromatic connectivity $\kappa_\chi(G)$ of the graph G. This also leads to the following more general concept. An edge coloring of a connected graph G is called a *proper k-path coloring* of G for some positive integer k if for every two distinct vertices u and v of G, there exist at least k internally disjoint proper $u-v$ paths. The minimum number of colors required for a proper k-path coloring of G is the *proper k-connectivity* $\mathrm{pc}_k(G)$ of G. Thus, $\mathrm{pc}_1(G) = \mathrm{pc}(G)$ is the proper connection number of G. If $\kappa(G) = \kappa$, then $\mathrm{pc}_\kappa(G) = \kappa_\chi(G)$ is the chromatic connectivity of G. The concept of $\mathrm{pc}_k(G)$ has been studied in [2, 10] and the related concept on rainbow colorings has been studied by many (see [4, 9], for example).

The proper connectivity of complete bipartite graphs $K_{r,s}$ for $2 \leq r \leq s$ has been studied in [2, 10] where the exact value of the proper 2-connectivity $\mathrm{pc}_2(K_{r,s})$ has been determined. In [2], Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza showed for complete bipartite graphs $K_{r,s}$

where $2 \leq r \leq s$ that $\text{pc}_k(K_{r,s}) = 2$ when $k \geq 2$ and $2k \leq r \leq s$ and $\text{pc}_k(K_{r,s}) > 2$ when $r = 2k - 1$ and $s \geq 2^r$. Sharp lower bounds have been established for $\text{pc}_k(K_{r,s})$ when $r \leq s$ and $2 \leq k \leq r < 2k$ and the exact values for $\text{pc}_k(K_{r,s})$ have been determined when $2 \leq r \in \{2k-1, 2k-2\}$ and s is sufficiently large (see [10]).

In this work, we introduce the concept of strong proper connectivity resulting from strong proper-path colorings and investigate this concept for complete bipartite graphs. We refer to the book [5] for graph theory notation and terminology not described in this paper.

2 Proper k-Geodesic Colorings

For every nontrivial connected graph G and every two distinct vertices u and v of G, there is at least one $u - v$ path of length $d(u, v)$, namely a $u - v$ geodesic. If u and v are adjacent vertices in G, then (u, v) is the unique $u-v$ geodesic in G; while if u and v are not adjacent, then there may be more than one $u - v$ geodesic in G. For example, if $G = K_{2,s}$ with partite sets $\{u, v\}$ and $\{w_1, w_2, \ldots, w_s\}$ where $s \geq 2$, then there are s internally disjoint $u - v$ geodesics in G, namely (u, w_i, v) for $1 \leq i \leq s$ and for $1 \leq i < j \leq s$, there are two internally disjoint $w_i - w_j$ geodesics in G, namely (w_i, u, w_j) and (w_i, v, w_j). Hence, there are at least two internally disjoint geodesics connecting every two nonadjacent vertices in G. However, it is also possible that there is a unique $u - v$ geodesic connecting two nonadjacent vertices u and v in a nontrivial connected graph. For example, if P is the Petersen graph, then $d(u, v) = 2$ for every two nonadjacent vertices u and v of P. Since the girth of P is 5, there is a unique $u - v$ geodesic connecting *every* two nonadjacent vertices u and v in P.

A connected graph G of diameter at least 2 is *ℓ-geodesic connected* for some positive integer ℓ if for every two *nonadjacent* vertices u and v of G, there exist at least ℓ internally disjoint $u - v$ geodesics in G. For example, the Petersen graph is 1-geodesic connected and not ℓ-geodesic connected for any $\ell \geq 2$. For positive integers r and s with $r \leq s$, the complete bipartite $K_{r,s}$ is r-geodesic connected. In general, if $G = K_{n_1,n_2,\ldots,n_p}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_p$, is a complete p-partite graph for some integer $p \geq 2$ and $\ell = n_1 + n_2 + \cdots + n_{p-1}$, then G is ℓ-geodesic connected.

If G is an ℓ-geodesic connected graph for some positive integer ℓ and k is a positive integer with $k \leq \ell$, then an edge coloring of a connected graph G is called a *proper k-geodesic coloring* (or a *strong proper k-path coloring*) of G if for every two nonadjacent vertices u and v of G, there exist at least k internally disjoint proper $u - v$ geodesics in G. The minimum number of the colors required for a proper k-geodesic coloring of G is the *strong proper k-connectivity* $\text{spc}_k(G)$ of G. In particular, $\text{spc}_1(G) = \text{spc}(G)$ is the *strong proper connection number* of G. If G is an ℓ-geodesic connected graph and the edges of G are properly colored, then every two nonadjacent vertices of G are connected by at least ℓ internally disjoint proper geodesics. Therefore, we have the following observation.

Observation 2.1 *If G is an ℓ-geodesic connected graph for some positive integer ℓ and k is a positive integer with $k \leq \ell$, then* $\mathrm{spc}_k(G)$ *exists and*

$$\mathrm{spc}_k(G) \leq \chi'(G). \tag{1}$$

Furthermore, if k and k' are positive integers with $k \leq k' \leq \ell$, then

$$\mathrm{spc}_k(G) \leq \mathrm{spc}_{k'}(G).$$

We investigate the following problem. For integers k, r and s with $2 \leq k \leq r \leq s$, what is $\mathrm{spc}_k(K_{r,s})$? Since $\mathrm{diam}(K_{r,s}) = 2$, for each pair u, v of vertices of $K_{r,s}$, each $u - v$ geodesic P is a $u - v$ path of length 1 or 2. Therefore, every proper geodesic is also a rainbow geodesic. The question here is to determine the minimum number of the colors required for a proper k-geodesic coloring of $K_{r,s}$ such that every two nonadjacent vertices of $K_{r,s}$ are connected by at least k internally disjoint rainbow geodesics in $K_{r,s}$. In order to do this, we first present some related concepts and useful preliminary information on proper k-geodesic colorings of $K_{r,s}$ for $2 \leq r \leq s$.

For integers r and n, where $r, n \geq 2$, let $\mathcal{F}(r,n)$ denote the set of all functions $f : [r] \to [n]$. For a positive integer k with $k \leq r$, a subset $S \subseteq \mathcal{F}(r,n)$ is called a *k-distinct subset* if for each pair f, g of functions in S, there are at least k distinct elements $p \in [r]$ such that $f(p) \neq g(p)$. For positive integers k, r and n with $k \leq r$, let $M(r,n,k)$ denote the maximum size of a k-distinct subset of $\mathcal{F}(r,n)$; that is,

$$M(r,n,k) = \max\{|S| : S \text{ is a } k\text{-distinct subset of } \mathcal{F}(r,n)\}.$$

It should be mentioned that the concepts of k-distinct subsets of the set $\mathcal{F}(r,n)$ of all functions $f : [r] \to [n]$ and the maximum size $M(r,n,k)$ of a k-distinct subset of $\mathcal{F}(r,n)$ can be expressed under the context of *error-correcting codes* in coding theory (see [6, pp.477, 683]). In terms of error-correcting codes, the set $\mathcal{F}(r,n)$ is the set of all r-tuples where each coordinate is an element of $[n]$. A collection C of r-tuples is called a *code* and each element in C is called a *code word of length r*. For two code words x and y in a code C, the *distance* $d(x,y)$ between x and y is the number of coordinates at which x and y differ. This distance is referred to as the *Hamming distance* between x and y. For a collection C of code words of length r, the *distance of C* is defined as $d(C) = \min\{d(x,y) : x, y \in C\}$. Thus, a k-distinct subset of the set $\mathcal{F}(r,n)$ is, in fact, a set C of code words of length r (each of whose coordinates is an element of $[n]$) such that $d(C) \geq k$ and $M(r, n, k)$ is the maximum size of a code C of code words of length r (each of whose coordinates is an element of $[n]$) distance is at least k. The following known result will be very useful to us.

Lemma 2.2 [7] *If r, k and n are integers with $2 \leq k \leq r$ and $n \geq 2$, then*

$$M(r, n, k) \leq n^{r-k+1}.$$

In particular, if $k = 2$, then $M(r, n, 2) = n^{r-1}$.

By Lemma 2.2 then, $M(3, 2, 2) = 2^2 = 4$ and $M(3, 3, 2) = 3^2 = 9$. For example, the set

$$S = \{(112), (121), (211), (222)\} \tag{2}$$

is a 2-distinct subset of $\mathcal{F}(3, 2)$ and the set

$$S = \{(111), (222), (333), (123), (132), (213), (231), (312), (321)\} \tag{3}$$

is a 2-distinct subset of $\mathcal{F}(3, 3)$. The k-distinct subsets of $\mathcal{F}(r, n)$ play an important role in constructing proper k-geodesic colorings of $K_{r,s}$ using the colors from the set $[n]$. For example, if $S = \{f_1, f_2, \ldots, f_s\}$ is a 2-distinct subset of $\mathcal{F}(r, n)$, then we can use S to construct a proper 2-geodesic coloring of $K_{r,s}$ using colors in the set $[n]$. We now illustrate this by providing a proper 2-geodesic coloring of $K_{3,4}$ and $K_{3,9}$.

We begin with the graph $K_{3,4}$. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4\}$ be the partite sets of $K_{3,4}$. Let $S = \{(112), (121), (211), (222)\}$ be the 4-element 2-distinct subset of $\mathcal{F}(3, 2)$ described in (2). For each integer i with $1 \leq i \leq 4$, if $f_i = (a_i\ b_i\ c_i)$, then define the colors of the three edges incident with v_i by $c(u_1 v_i) = a_i$, $c(u_2 v_i) = b_i$ and $c(u_3 v_i) = c_i$. This coloring c is shown in Figure 1. Since c is a proper 2-geodesic coloring c of $K_{3,4}$ and $\text{spc}_2(K_{3,4}) \geq 2$, it follows that $\text{spc}_2(K_{3,4}) = 2$.

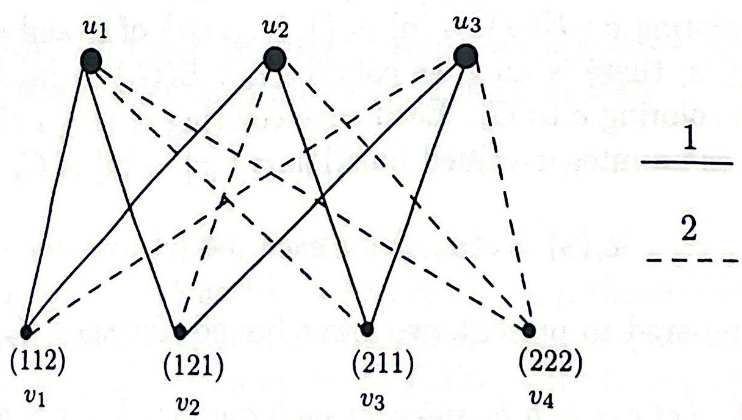

Figure 1: A proper 2-geodesic coloring of $K_{3,4}$ using colors 1 and 2

For the graph $K_{3,9}$, let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, \ldots, v_9\}$ be the partite sets of $K_{3,9}$. Consider the 9-element 2-distinct subset $S = \{f_1, f_2, \ldots, f_9\}$ of $\mathcal{F}(3,3)$ described in (3). We now construct an edge coloring $c : E(G) \to [3] = \{1, 2, 3\}$ of G using the 9 functions in S. For each integer i with $1 \leq i \leq 9$, if $f_i = (a_i\ b_i\ c_i)$, then define the colors of the three edges incident with v_i by $c(u_1v_i) = a_i$, $c(u_2v_i) = b_i$ and $c(u_3v_i) = c_i$. This coloring is shown in Figure 2, where each fine edge is colored 1, each dashed edge is colored 2 and each bold edge is colored 3. Since $c : E(K_{3,9}) \to [3]$ is a proper 2-geodesic coloring of $K_{3,9}$, it follows that $\mathrm{spc}_2(K_{3,9}) \leq 3$. In fact, $\mathrm{spc}_2(K_{3,9}) = 3$ as we will see in Section 4.

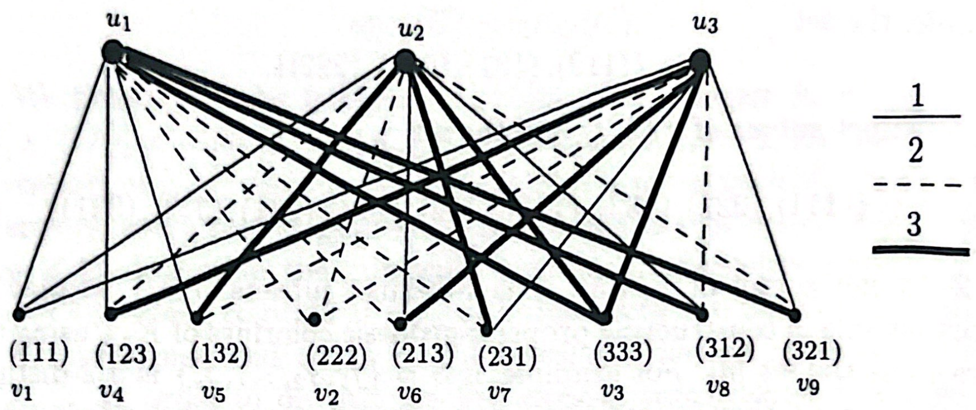

Figure 2: A proper 2-geodesic coloring of $K_{3,9}$ using colors 1, 2, 3

3 Lower Bounds for $\mathrm{spc}_k(K_{r,s})$

In this section, we establish lower bounds for $\mathrm{spc}_k(K_{r,s})$. Let $G = K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ where $2 \leq r \leq s$. For each integer i with $1 \leq i \leq s$, let $G_i = G[U, v_i] \cong K_{1,r}$ be the subgraph induced by the set $[U, v_i]$ of all edges incident with v_i in G. For an edge coloring $c : E(G) \to [n] = \{1, 2, \ldots, n\}$ of G and each integer i with $1 \leq i \leq s$, there is an edge coloring $c_i : E(G_i) \to [n]$ obtained by restricting the coloring c to G_i. Each edge coloring c_i $(1 \leq i \leq s)$ can also be considered as an integer-valued function $c_i : [r] \to [n]$ of G_i defined by

$$c_i(p) = c(u_pv_i) \text{ for each } p \in [r]. \tag{4}$$

We are now prepared to present two lower bounds for $\mathrm{spc}_k(K_{r,s})$.

Theorem 3.1 *Let r, s, k, n be integers such that $2 \leq k \leq r \leq s$ and $n \geq 2$.*

If $k > r + 1 - \log_n(s)$, then $\mathrm{spc}_k(K_{r,s}) > n$.

Proof. First, we verify the following statement:

$$\text{If } \operatorname{spc}_k(K_{r,s}) = n, \text{ then } k \le r + 1 - \log_n(s). \tag{5}$$

Let $G = K_{r,s}$ and let $c : E(G) \to [n]$ be a strong proper k-path coloring of G. Then $S = \{c_1, c_2, \ldots, c_s\}$ is a k-distinct subset of $\mathcal{F}(r, n)$, where each c_i ($1 \le i \le s$) is the integer-valued function $c_i : [r] \to [n]$ of the subgraph G_i as described in (4). Hence, $s = |S| \le M(r, n, k) \le n^{r-k+1}$ by Lemma 2.2 and so $s \le n^{r-k+1}$. Thus,

$$k \le r + 1 - \log_n(s).$$

Restating the implication in (5) in its contrapositive form, we obtain the following statement:

$$\text{If } k > r + 1 - \log_n(s), \text{ then } \operatorname{spc}_k(K_{r,s}) \ne n. \tag{6}$$

Since the statement in (6) holds for all integers m with $2 \le m < n$, it follows that if $k > r + 1 - \log_n(s)$, then $\operatorname{spc}_k(K_{r,s}) > n$, as desired. ■

The following is a consequence of Observation 2.1 and Theorem 3.1.

Corollary 3.2 *If r and s are integers with $2 \le r \le s$ and $s \ge 3$, then*

$$\operatorname{spc}_r(K_{r,s}) = s.$$

Proof. Since $\log_{s-1}(s) > 1$ for each integer $s \ge 3$, it follows that $r > r+1-\log_{s-1}(s)$. Thus, $\operatorname{spc}_r(K_{r,s}) > s-1$ or $\operatorname{spc}_r(K_{r,s}) \ge s$ by Theorem 3.1. On the other hand, $\operatorname{spc}_r(K_{r,s}) \le \chi'(K_{r,s}) = s$ by Observation 2.1 and so $\operatorname{spc}_r(K_{r,s}) = s$. ■

Next, we present a lower bound for $\operatorname{spc}_k(K_{r,s})$ with a connection to the maximum size of a k-distinct subset of the set $\mathcal{F}(r, n)$ of all functions from $[r]$ to $[n]$.

Theorem 3.3 *Let r, s, k be integers such that $2 \le k \le r \le s$. If N is the smallest positive integer such that $M(r, N, k) \ge s$, then*

$$\operatorname{spc}_k(K_{r,s}) \ge N.$$

Proof. For integers r and s with $2 \le r \le s$, let $G = K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$. Since $\operatorname{diam}(G) = 2$, if x and y are nonadjacent vertices of G, then each $x - y$ geodesic has length 2. Suppose that $\operatorname{spc}_k(G) = n$ and let $c : E(G) \to [n]$ be a proper k-geodesic coloring of G using n colors. For each integer i with $1 \le i \le s$, let c_i be the integer-valued function defined by (4). Now, let $S = \{c_1, c_2, \ldots, c_s\}$. We claim that S is a k-distinct subset of $\mathcal{F}(r, n)$; that is, if $c_i, c_j \in S$ where $i \ne$

j, then there are at least k distinct elements $p \in [r]$ such that $c_i(p) \neq c_j(p)$. For $v_i, v_j \in V$ where $i \neq j$, let $Q_1, Q_2, \ldots, Q_k$ be k internally disjoint proper $v_i - v_j$ geodesics in G. We may assume, without loss of generality, that $Q_p = (v_i, u_p, v_j)$ for $p = 1, 2, \ldots, k$. Since Q_p is properly colored, it follows that $c(v_i u_p) \neq c(u_p v_j)$ and so $c_i(p) \neq c_j(p)$ for $p = 1, 2, \ldots, k$. Therefore, as claimed, S is a k-distinct subset of $\mathcal{F}(r, n)$. If $M(r, n, k)$ is the maximum size of a k-distinct subset of $\mathcal{F}(r, n)$, then $M(r, n, k) \geq |S| = s$. Since N is the smallest positive integer such that $M(r, N, k) \geq s$, it follows that c must use at least N colors and so $n \geq N$. Therefore, $\text{spc}_k(G) = n \geq N$. ∎

With the aid of Theorem 3.3 and Lemma 2.2, we are able to establish an additional lower bound for $\text{spc}_k(K_{r,s})$ in terms of k, r and s.

Theorem 3.4 *If r, s and k are integers with $2 \leq k \leq r \leq s$, then*

$$\text{spc}_k(K_{r,s}) \geq \lceil \sqrt[r-k+1]{s} \,\rceil. \tag{7}$$

Proof. Let N be the smallest positive integer such that $M(r, N, k) \geq s$. It then follows by Theorem 3.3 that $\text{spc}_k(K_{r,s}) \geq N$. Since $M(r, N, k) \leq N^{r-k+1}$ by Lemma 2.2, it follows that $s \leq M(r, N, k) \leq N^{r-k+1}$ and so $N \geq \lceil \sqrt[r-k+1]{s} \,\rceil$. Therefore, $\text{spc}_k(K_{r,s}) \geq N \geq \lceil \sqrt[r-k+1]{s} \,\rceil$, as desired. ∎

If $k = r = s = 2$, then $\text{spc}_2(K_{2,2}) = 2$ and so (7) holds. If $k = r$ and $s \geq 3$, then equality in (7) holds by Corollary 3.2. In [3] it was shown that $\text{src}(K_{r,s}) = \lceil \sqrt[r]{s} \,\rceil$ for all integers r and s with $2 \leq r \leq s$. Hence, equality in (7) holds for $k = 1$. Furthermore, equality in (7) also holds for $k = 2$, as we will see in Section 4. The following is a consequence of Theorem 3.4.

Corollary 3.5 *If r and k are integers with $2 \leq k \leq r$, then*

$$\lim_{s \to \infty} \text{spc}_k(K_{r,s}) = \infty.$$

4 Strong Proper 2-Connectivity of $K_{r,s}$

With the aid of Lemma 2.2 and Theorem 3.4, we are able to determine the exact values of the strong proper 2-connectivity of $K_{r,s}$ where $2 \leq r \leq s$. We consider two situations, namely (i) $s \geq 2^{r-1}$ and (ii) $r \leq s \leq 2^{r-1}$, beginning with (i).

Theorem 4.1 *Let r and s be integers with $2 \leq r \leq s$. If $s \geq 2^{r-1}$, then*

$$\text{spc}_2(K_{r,s}) = \lceil \sqrt[r-1]{s} \,\rceil.$$

Proof. By Theorem 3.4, $\mathrm{spc}_2(K_{r,s}) \geq \lceil \sqrt[r-1]{s} \rceil$. Thus, it remains to show that $\mathrm{spc}_2(K_{r,s}) \leq \lceil \sqrt[r-1]{s} \rceil$. Let $G = K_{r,s}$. If $r = 2$, then $\lceil \sqrt[r-1]{s} \rceil = s = \chi'(G)$. Since $\mathrm{spc}_2(G) \leq \chi'(G)$ by Observation 2.1, it follows that $\mathrm{spc}_2(G) \leq \lceil \sqrt[r-1]{s} \rceil$ and so $\mathrm{spc}_2(G) = \lceil \sqrt[r-1]{s} \rceil$ when $r = 2$. Thus, we may assume that $r \geq 3$ and $s \geq 2^{r-1} \geq 4$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G. Furthermore, let $N = \lceil \sqrt[r-1]{s} \rceil \geq 2$ and let A be a 2-distinct subset of maximum size in $\mathcal{F}(r, N)$. Since $k = 2$, it follows by Lemma 2.2 that $|A| = N^{r-1} \geq s$. For each $f \in A$, let f' be the restriction of f to $[r-1]$; that is, $f'(x) = f(x)$ for each $x \in [r-1]$. Now let $A' = \{f' : f \in A\}$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, it follows that if $f \neq g \in A$, then there are two distinct elements p_1 and p_2 in $[r]$ such that $f(p_1) \neq g(p_1)$ and $f(p_2) \neq g(p_2)$. We may assume that $p_1 \neq r$. Hence, $f'(p_1) \neq g'(p_1)$ and so $f' \neq g'$ in T'. Therefore, $|A'| = |A| = N^{r-1}$. Since $A' \subseteq \mathcal{F}(r-1, N)$ and $|\mathcal{F}(r-1, N)| = N^{r-1} = |A'|$, it follows that $A' = \mathcal{F}(r-1, N)$. Let $A' = \{c'_1, c'_2, \ldots, c'_{N^{r-1}}\}$ such that the ranges of the first 2^{r-1} elements $c'_1, c'_2, \ldots, c'_{2^{r-1}}$ in A' belong to $[2]$. Now, let $A = \{c_1, c_2, \ldots, c_{N^{r-1}}\}$, where then the restriction of c_i to $[r-1]$ is $c'_i \in A'$ for $1 \leq i \leq N^{r-1}$.

We define an edge coloring $c : E(G) \to [N]$ by using the first s integer-valued functions $c_1, c_2, \ldots, c_s \in A$ as follows. For each integer i with $1 \leq i \leq s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. It remains to show that c is a strong proper 2-path N-coloring of G. Let x and y be two nonadjacent vertices of G. First, suppose that $x = v_i$ and $y = v_j$, where $1 \leq i < j \leq s$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, there are $p, q \in [r]$ such that $c_i(p) \neq c_j(p)$ and $c_i(q) \neq c_j(q)$. Thus, (v_i, u_p, v_j) and (v_i, u_q, v_j) are two internally disjoint properly colored $v_i - v_j$ geodesics in G. Next, suppose that $x = u_i$ and $y = u_j$, where $1 \leq i < j \leq r$. Since $r \geq 3$, it follows that $[r] - \{i, j\} \neq \emptyset$. Let $x \in [r] - \{i, j\}$. For each i with $1 \leq i \leq 2^{r-1}$, let c_i^* be the restriction of c_i to $[r] - \{x\}$. Now let $B' = \{c_1^*, c_2^*, \ldots, c_{2^{r-1}}^*\}$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, it follows that $|B'| = 2^{r-1}$. Since the range of each c_i belongs to $[2]$ for $1 \leq i \leq 2^{r-1}$, it follows that $B' \subseteq \mathcal{F}(r, 2)$. Again, $|\mathcal{F}(r, 2)| = 2^{r-1}$ and so $B' = \mathcal{F}(r, 2)$. Let $B = \{c_1, c_2, \ldots, c_{2^{r-1}}\}$ where the restriction of c_i to $[r] - \{x\}$ is c_i^* for $1 \leq i \leq 2^{r-1}$. Since B is a 2-distinct subset of $\mathcal{F}(r, N)$, we saw that $|B| = |B'| = 2^{r-1}$. There are at least $2^{r-2} \geq 2$ elements f in B such that $f(i) \neq f(j)$, say $c_1(i) \neq c_1(j)$ and $c_2(i) \neq c_2(j)$. Then (u_i, v_1, u_j) and (u_i, v_2, u_j) are two internally disjoint properly colored $u_i - u_j$ geodesics in G. In each case, c is a strong proper 2-path N-coloring of G and so $\mathrm{spc}_2(G) \leq N$. Therefore,

$$\mathrm{spc}_2(K_{r,s}) = N = \lceil \sqrt[r-1]{s} \rceil$$

when $s \geq 2^{r-1} \geq 2$. ■

Next, we determine the strong proper 2-connectivity of $K_{r,s}$ when $2 \leq r \leq s \leq 2^{r-1}$. If r and s are integers with $2 \leq r \leq s \leq 2^{r-1}$, then $1 < \sqrt[r-1]{s} \leq 2$ and so $\lceil \sqrt[r-1]{s} \rceil = 2$. Therefore, we show that $\text{spc}_2(K_{r,s}) = 2$ in this case.

Theorem 4.2 *If r and s are integers with $2 \leq r \leq s \leq 2^{r-1}$, then*

$$\text{spc}_2(K_{r,s}) = 2.$$

Proof. Let $G = K_{r,s}$. It suffices to show that G has a strong proper 2-path coloring using the colors 1 and 2. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G. We consider two cases, according to whether $r = s$ or $r < s$.

Case 1. $r = s$. Define an edge coloring $c : E(G) \to \{1, 2\}$ by

$$c(u_i v_j) = \begin{cases} 1 & \text{if } i \neq j \\ 2 & \text{if } i = j. \end{cases} \tag{8}$$

We show that c is a strong proper 2-path coloring of G. Let x and y be two nonadjacent vertices of G. First, assume that $x = u_i$ and $y = u_j$ where $1 \leq i < j \leq r$. Then (u_i, v_i, u_j) and (u_i, v_j, u_j) are two internally disjoint properly colored $u_i - u_j$ geodesics in G. Next, assume that $x = v_i$ and $y = v_j$ where $1 \leq i < j \leq s$. By symmetry, there are two internally disjoint properly colored $v_i - v_j$ geodesics in G. However, we provide a different argument which will be useful for the general case, namely Case 2.

For the edge coloring $c : E(G) \to \{1, 2\}$ defined in (8), consider the induced color functions $c_i : [r] \to [2]$ for each integer i with $1 \leq i \leq r$, where $c_i(x) = c(v_i u_x)$ for each $x \in [r]$. Thus,

$$c_i(x) = \begin{cases} 1 & \text{if } i \neq x \\ 2 & \text{if } i = x. \end{cases} \tag{9}$$

Let $A = \{c_i : 1 \leq i \leq r\}$. We show that A is a 2-distinct subset of $\mathcal{F}(r, 2)$. Let $c_i, c_j \in A$, where $1 \leq i \neq j \leq r$. Since $c_i(i) = 2$ and $c_i(j) = 1$ and $c_j(i) = 1$ and $c_j(j) = 2$, it follows that $c_i(i) \neq c_j(i)$ and $c_i(j) \neq c_j(j)$. Furthermore, $c_i(x) = c_j(x)$ for each $x \in [r] - \{i, j\}$. Hence, (v_i, u_i, v_j) and (v_i, u_j, v_j) are two internally disjoint properly colored $v_i - v_j$ geodesics in G. Thus, c is a strong proper 2-path coloring of G and so $\text{spc}_2(G) = 2$ in this case.

Case 2. $r < s \leq 2^{r-1}$. First, we show that there exists 2-distinct subset T of $\mathcal{F}(r, 2)$ such that $|T| = 2^{r-1}$ and $A \subseteq T$. For each $f \in \mathcal{F}(r, 2)$, let

$$w(f) = |\{x \in [r] : f(x) = 2\}|$$

(which is referred to as the *Hamming weight of* f in coding theory). Thus, if $f \in A$, then $w(f) = 1$. We claim that if $f, g \in \mathcal{F}(r,2)$ such that $w(f)$ and $w(g)$ are of the same parity, then $\{f, g\}$ is a 2-distinct subset of $\mathcal{F}(r,2)$. If this were not the case, then we may assume, without loss of generality, that $f(1) \neq g(1)$ and $f(x) = g(x)$ for all $x \in [r] - \{1\}$. Since $f(1), g(1) \in [2]$ and $f(1) \neq g(1)$, we may assume that $f(1) = 1$ and $g(1) = 2$. Because $f(x) = g(x)$ for each $x \in [r] - \{1\}$, it follows that $w(g) = w(f) + 1$ and so $w(f)$ and $w(g)$ are of opposite parity, which is a contradiction. Now, let

$$T = \{f \in \mathcal{F}(r,2) : \ w(f) \text{ is odd}\}.$$

Since $w(f) = 1$ for each $f \in A$, we have $A \subseteq T$. By the argument above, T is a 2-distinct subset of $\mathcal{F}(r,2)$. Furthermore,

$$|T| = \frac{1}{2}|\mathcal{F}(r,2)| = \frac{2^r}{2} = 2^{r-1}.$$

Since $s \leq 2^{r-1}$, we can choose a s-element subset B of T such that $A \subseteq B$.

Next, suppose that $B = \{c_1, c_2, \ldots, c_s\}$ where c_i is defined in (9) for $1 \leq i \leq r$. We now define an edge coloring $c : E(G) \to [2]$ by using the s integer-valued functions $c_1, c_2, \ldots, c_s \in B$ as follows. For each integer i with $1 \leq i \leq s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. It remains to show that c is a strong proper 2-path 2-coloring of G. Let x and y be two nonadjacent vertices of G. If $x = u_i$ and $y = u_j$, where $1 \leq i < j \leq r$, then the argument in Case 1 shows that there are 2 internally disjoint properly colored $u_i - u_j$ geodesics in G. Next, suppose that $x = v_i$ and $y = v_j$, where $1 \leq i < j \leq s$. There are $p, q \in [r]$ where $p \neq q$ such that $c_i(p) \neq c_j(p)$ and $c_i(q) \neq c_j(q)$. Hence, (v_i, u_p, v_j) and (v_i, u_q, v_j) are 2 internally disjoint properly colored $v_i - v_j$ geodesics in G. Therefore, c is a strong proper 2-path 2-coloring of G and so $\text{spc}_2(G) = 2$. ∎

In summary, Theorems 4.1 and 4.2 give rise to a formula of $\text{spc}_2(K_{r,s})$ for all integers r and s with $2 \leq r \leq s$.

Theorem 4.3 *If r and s are integers with $2 \leq r \leq s$, then*

$$\text{spc}_2(K_{r,s}) = \lceil \sqrt[r-1]{s} \, \rceil.$$

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