The largest number of maximal independent sets in twinkle graphs

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Abstract

A *maximal independent set* is an independent set that is not a proper subset of any other independent set. A *twinkle graph* W is a connected unicyclic graph with cycle C such that $W - x$ is disconnected for any $x \in V(C)$. In this paper, we determine the largest number of maximal independent sets and characterize those extremal graphs achieving these values among all twinkle graphs. Using the results on twinkle graphs, we give an alternative proof to determine the largest number of maximal independent sets among all connected graphs with at most one cycle.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The cardinality of the set of all maximal independent sets of a graph G is denoted by $mi(G)$. Around 1960, Erdős and Moser proposed the problem of determining the maximum number of $mi(G)$ in the family of graphs of order n and characterizing those extremal graphs achieving the maximum value. Shortly after, Moon and Moser [9] solved the problem. The same problem was investigated for certain families of graphs, including trees [4, 10, 11], forests [4], (connected) graphs with at most one cycle [4], (connected) triangle-free graphs [1, 2].

A *twinkle graph* W is a connected unicyclic graph with cycle C such that $W - x$ is disconnected for any $x \in V(C)$. Additionally, a connected graph G with vertex set $V(G)$ is called a *quasi-tree graph*, if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree. The concept of quasi-tree graphs was mentioned by H. Liu and M. Lu in [8]. Lin [6, 7] determined the largest and the second largest numbers of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order n. M. J. Jou and G. J. Chang [4] found the maximum number of maximal independent sets in connected graphs which contain at most one cycle. Trivially, the set of all connected graphs with at most one cycle is the union of the set of trees, twinkle graphs and quasi-tree graphs with exactly one cycle. In this paper, we determine the largest number of maximal independent sets and characterize those extremal graphs achieving these values among all twinkle graphs. Using the results on twinkle graphs, we give an alternative proof to determine the largest number of maximal independent sets among all connected graphs with at most one cycle.

2 Preliminary

In this section, we present some notations and preliminary results, which will be helpful to the proof of our main results in the next section. For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v in G and the *closed neighborhood* $N_G[v]$ is $\{v\} \cup N_G(v)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. A vertex x is a *leaf* if $\deg_G(x) = 1$. A vertex is called a *support vertex* if it is adjacent to a leaf. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. If $A = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. Denote by K_n a *complete graph* with n vertices and P_n a *path* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$.

Lemma 2.1. ([2, 3]) *If u is a leaf adjacent to v in a graph G, $mi(G) = mi(G - N_G[u]) + mi(G - N_G[v])$.*

Lemma 2.2. ([3]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1) \cdot mi(G_2)$.*

The results on the largest numbers of maximal independent sets for trees and forests are presented in Theorems 2.3 and 2.4, respectively.

Theorem 2.3. ([4]) *If T is a tree of order $n \geq 1$, then $mi(G) \leq t_1(n)$, where*

$$t_1(n) = \begin{cases} r^{n-1}, & \text{if } n \text{ is odd,} \\ r^{n-2} + 1, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ if and only if $T \in T_1(n)$, where

$$T_1(n) = \begin{cases} B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd,} \\ B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even,} \end{cases}$$

where $B(i,j)$ is the set of batons, which are the graphs obtained from a basic path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of P in all possible ways (see Figure 1).

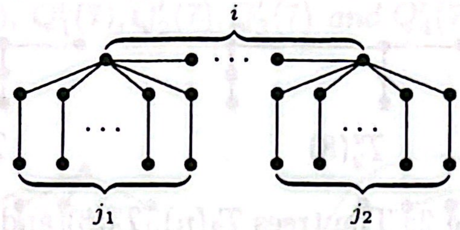

Figure 1: The baton $B(i,j)$ with $j = j_1 + j_2$

Theorem 2.4. ([4]) *If F is a forest of order $n \geq 1$, then $mi(G) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^{n-1}, & \text{if } n \text{ is odd,} \\ r^n, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, $mi(F) = f_1(n)$ if and only if $F \in F_1(n)$, where

$$F_1(n) = \begin{cases} B(1, \frac{n-1-2s}{2}) \cup sP_2, & \text{if } n \text{ is odd,} \\ \frac{n}{2}P_2, & \text{if } n \text{ is even,} \end{cases}$$

where $0 \leq s \leq \frac{n-1}{2}$.

The results on the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.5 and 2.6, respectively.

Theorem 2.5. ([5]) *If T is a tree of order $n \geq 4$ with $T \notin T_1(n)$, then $mi(T) \leq t_2(n)$, where*

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 3r^{n-5} + 1, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ if and only if $T \in \{T_2'(8), T_2''(8), P_{10}, T_2(n)\}$, where $T_2(n)$ and $T_2'(8)$, $T_2''(8)$ are shown in Figures 2.

Theorem 2.6. ([5]) *If F is a forest of order $n \geq 4$ with $F \notin F_1(n)$, then $mi(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 7r^{n-7}, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

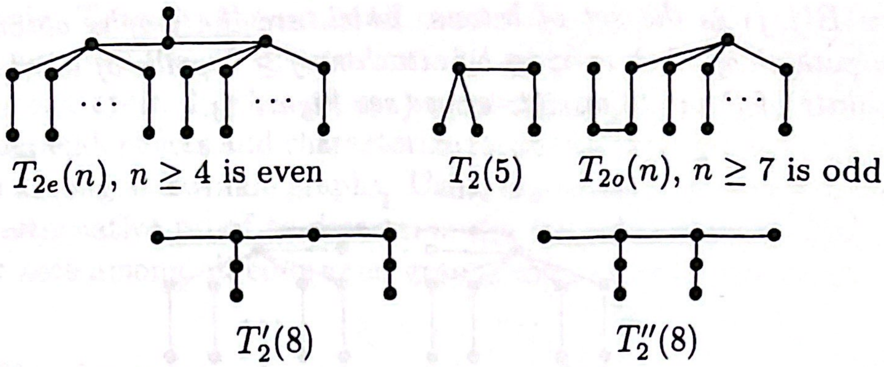

Figure 2: The trees $T_2(n)$, $T_2'(8)$ and $T_2''(8)$

Furthermore, $mi(F) = f_2(n)$ if and only if $F \in \mathcal{F}_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even,} \\ T_2(5) \text{ or } P_1 \cup P_4, & \text{if } n = 5, \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The result on the largest and the second largest numbers of maximal independent sets among all quasi-trees are described in Theorems 2.7 and 2.8, respectively.

Theorem 2.7. ([6]) *If Q is a quasi-tree graph of order $n \geq 5$, then $mi(Q) \leq q_1(n)$, where*

$$q_1(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ r^{n-1} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q_1(n)$ if and only if $Q \in \{Q_1(n), C_5\}$, where $Q_1(n)$ is shown in Figure 3.

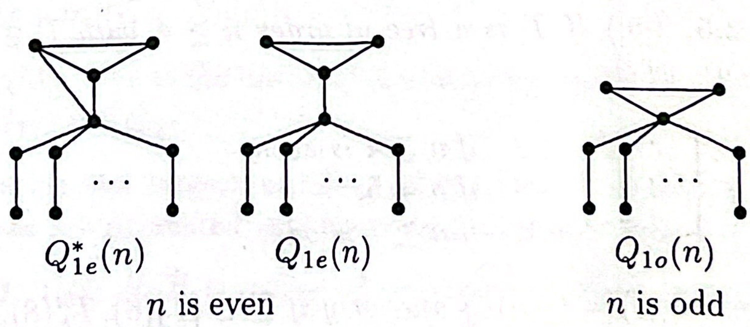

Figure 3: The graph $Q_1(n)$

Theorem 2.8. ([7]) *If Q is a quasi-tree graph of order $n \geq 7$ with $Q \notin \{Q_1(n), C_5\}$, then $mi(Q) \leq q_2(n)$, where*

$$q_2(n) = \begin{cases} 5r^{n-6} + 1, & \text{if } n \geq 8 \text{ is even,} \\ r^{n-1}, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q_2(n)$ if and only if $Q \in \mathcal{Q}_2(n)$, where

$$\mathcal{Q}_2(n) = \begin{cases} Q_{2e}(n) \text{ or } Q^*_{2e}(n), & \text{if } n \geq 8 \text{ is even,} \\ Q'_1(7), Q'_2(7), Q'_3(7), Q'_4(7) \text{ or } B(1, \frac{n-1}{2}), & \text{if } n \geq 7 \text{ is odd,} \end{cases}$$

where $Q_{2e}(n)$, $Q^*_{2e}(n)$, $Q'_1(7), Q'_2(7), Q'_3(7)$ and $Q'_4(7)$ are shown in Figure 4.

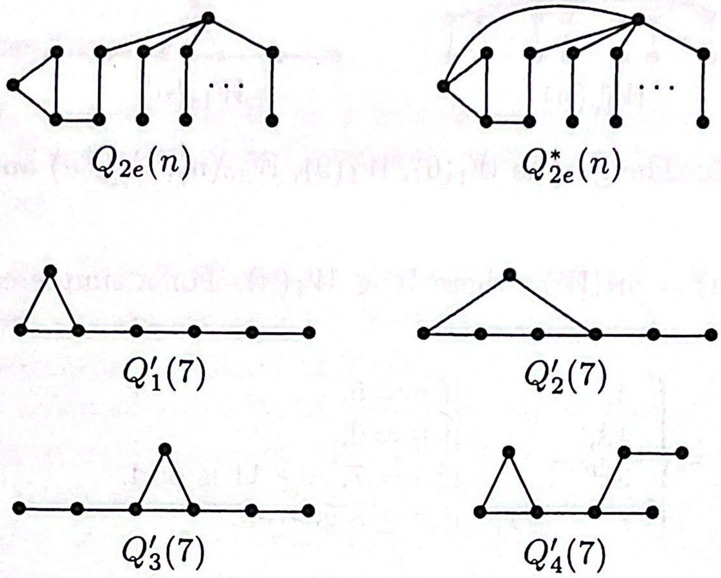

Figure 4: The graphs $Q_{2e}(n)$, $Q^*_{2e}(n)$, $Q'_1(7), Q'_2(7), Q'_3(7)$ and $Q'_4(7)$

3 Main results

In this section, we determine the largest number of maximal independent sets among all twinkle graphs. We also characterize those extremal graphs achieving this maximum value. As a consequence, the results for connected graphs with at most one cycle is given.

Define the graph $W_1(n)$ of order $n \geq 6$ as follows.

$$W_1(n) = \begin{cases} W_1(6), & \text{if } n = 6, \\ W_1(9), & \text{if } n = 9, \\ W_{1o}(n), & \text{if } n = 7,\ n \geq 11 \text{ is odd,} \\ W^*_{1e}(n) \text{ or } W_{1e}(n), & \text{if } n \geq 8 \text{ is even,} \end{cases}$$

where $W_1(6)$, $W_1(9)$, $W_{1o}(n)$, $W^*_{1e}(n)$ and $W_{1e}(n)$ are shown in Figure 5.

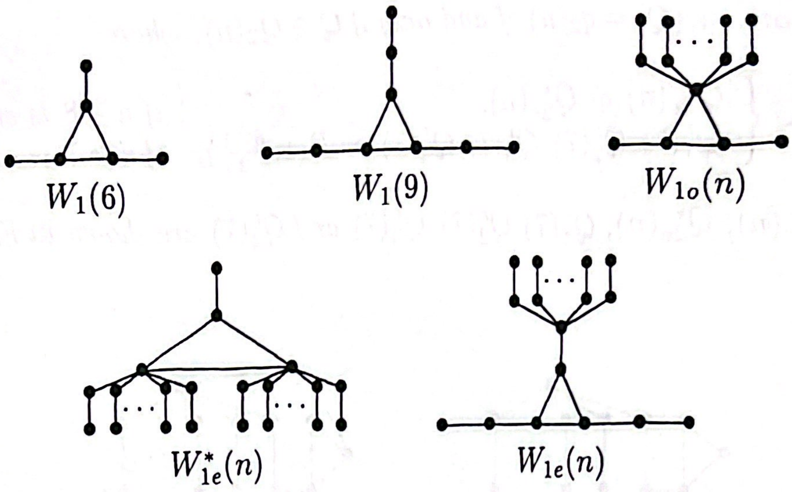

Figure 5: The graphs $W_1(6)$, $W_1(9)$, $W_{1o}(n)$, $W^*_{1e}(n)$ and $W_{1e}(n)$

Let $w_1(n) = mi(W)$, where $W \in \mathcal{W}_1(n)$. For a simple calculation, we have that

$$w_1(n) = \begin{cases} 4, & \text{if } n = 6, \\ 13, & \text{if } n = 9, \\ 3r^{n-5}, & \text{if } n = 7,\ n \geq 11 \text{ is odd}, \\ r^{n-2} + 1, & \text{if } n \geq 8 \text{ is even}. \end{cases}$$

In this paper, we will prove the following result.

Theorem 3.1. *If W is a twinkle graph of order $n \geq 6$, then $mi(W) \leq w_1(n)$. Furthermore, the equality holds if and only if $W \in \mathcal{W}_1(n)$.*

Besides, it is straightforward to check that $W_1(6)$, $W_1(7)$ and $W_1(8)$ and $W_1(9)$ have the largest numbers of independent sets among all twinkle graphs of order 6, 7, 8 and 9, respectively. Hence we consider the case for $n \geq 10$ in the sequel.

In the following, let u be a leaf lying on a longest path P joining u and the unique cycle C of a twinkle graph W, say $P = u, v, w, \ldots$, and $\ell(u, C)$ the length from u to C.

We prove Theorem 3.1 by establishing the following three lemmas.

Lemma 3.2. *If $\widetilde{W}$ is a twinkle graph of order $n \geq 10$ with $\ell(u, C) = 1$, then $mi(\widetilde{W}) < w_1(n)$.*

Proof. Since $\ell(u, C) = 1$ and u is a leaf, $\widetilde{W} - N_{\widetilde{W}}[u]$ is the union of $i \geq 0$ isolated vertices and a tree T with $n - 2 - i$ vertices. Thus $mi(T) \leq t_1(n - 2 - i) \leq t_1(n - 2)$. On the other hand, $\widetilde{W} - N_{\widetilde{W}}[v]$ is the union of $i' \geq 2$ isolated vertices and a tree T' with $n - 4 - i - i'$ vertices. Thus

$mi(T') \le t_1(n-4-i-i') \le t_1(n-6)$. Hence, by Lemma 2.1, we have that

$$\begin{aligned} mi(\widetilde{W}) &= mi(\widetilde{W} - N_{\widetilde{W}}[u]) + mi(\widetilde{W} - N_{\widetilde{W}}[v]) \\ &= mi(iK_1) \cdot mi(T) + mi(i'K_1) \cdot mi(T') \\ &\le \begin{cases} r^{(n-2)-2} + 1 + r^{(n-6)-2} + 1, & \text{if } n \text{ is even,} \\ r^{(n-2)-1} + r^{(n-6)-1}, & \text{if } n \text{ is odd,} \end{cases} \\ &< w_1(n). \end{aligned}$$

This completes the proof. □

Lemma 3.3. *Suppose that $\widehat{W}$ is a twinkle graph of order $n \ge 10$ with $\ell(u, C) = 2$. If $x \in V(C)$ is not a support vertex with $\deg_{\widehat{W}}(x) = 3$, then $mi(\widehat{W}) < w_1(n)$.*

Proof. Since $\ell(u, C) = 2$, we assume that the longest path $P : uvw$ joining the leave u and the unique cycle C. Note that $w \in V(C)$ is not a support vertex, by assumption, we have that $\deg_{\widehat{W}}(w) = 3$. It follows that $\widehat{W} - N_{\widehat{W}}[u]$ is the union of $j \ge 0$ isolated vertices and a quasi-tree Q of order $n-2-j$ with exactly one cycle. Note that $B_1(1, \frac{n-3-j}{2})$, $Q^*_{1e}(n-2-j)$, $Q^*_{2e}(n-2-j)$ are not unicyclic and $|\{x \in V(C) : \deg_Q(x) = 2\}| = 1$ implies that $Q \ne Q_1(n-2-j), Q_{2e}(n-2-j), Q^*_{2e}(n-2-j), Q'_1(7), Q'_2(7), Q'_4(7)$. In addition, if $Q = Q'_3(7)$, then $\widehat{W} = W^{(3)}$. See Figure 6. By simple calculation, we have that $mi(W^{(3)}) = 13 < w_1(n)$ for $n \ge 10$. Thus, by Theorem 2.8, we have that

$$mi(Q) \le q_2(n-2-j) - 1 \le q_2(n-2) - 1. \tag{1}$$

On the other hand, $\widehat{W} - N_{\widehat{W}}[v]$ is a tree T'' with $n-3-j$ vertices. Suppose that $T'' = T_1(n-3-j)$. Since $\ell(u, C) = 2$ and $\deg_{\widehat{W}}(w) = 3$, it follows that there are three possibilities for graph W. See Figure 6. By simple calculation, we have $mi(W^{(1)}) = 6$, $mi(W^{(2)}) = 9$ and $mi(W^{(3)}) = 13$. It follows that $mi(W^{(i)}) < w_1(n)$ for $n \ge 10$ and $i = 1, 2, 3$. In addition, $\ell(u, C) = 2$ implies that $T'' \ne T_2(n-3-j)$.

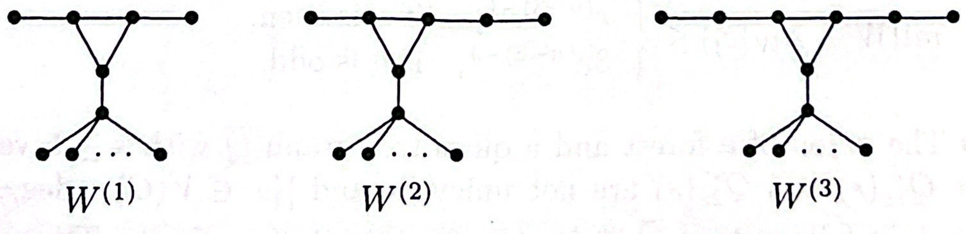

Figure 6: The three possibilities for graph W

Now consider that $T'' \neq T_1(n-3-j), T_2(n-3-j)$. By Theorem 2.5, we have that

$$mi(T'') \leq t_2(n-3-j) - 1 \leq t_2(n-3) - 1. \tag{2}$$

By Lemma 2.1, 2.2 and (1),(2), it follows that

$$\begin{aligned} mi(\widehat{W}) &= mi(\widehat{W} - N_{\widehat{W}}[u]) + mi(\widehat{W} - N_{\widehat{W}}[v]) \\ &= mi(jK_1) \cdot mi(Q) + mi(T'') \\ &\leq \begin{cases} (5r^{(n-2)-6} + 1 - 1) + (3r^{(n-3)-5} + 1 - 1), & \text{if } n \text{ is even}, \\ (r^{(n-2)-1} - 1) + (r^{(n-3)-2} - 1), & \text{if } n \text{ is odd}, \end{cases} \\ &\leq \begin{cases} 5r^{n-8} + 3r^{n-8}, & \text{if } n \text{ is even}, \\ r^{n-3} - 1 + r^{n-5} - 1, & \text{if } n \text{ is odd}, \end{cases} \\ &< w_1(n). \end{aligned}$$

This completes the proof. □

Lemma 3.4. *If W is a twinkle graph of order $n \geq 10$ that W does not satisfy the conditions of either Lemma 3.2 or Lemma 3.3, then $mi(W) \leq w_1(n)$. Furthermore, $mi(W) = w_1(n)$ if and only if $W \in W_1(n)$.*

Proof. We shall prove this case by induction on n. It is true for $n = 10, 11$. Assume that it is true for all $n' < n$. Suppose that W is a twinkle graph of order $n \geq 10$ such that W does not satisfy the conditions of either Lemma 3.2 or Lemma 3.3. Then there exists a leaf u such that $W - N_W[u]$ is the union of $r \geq 0$ isolated vertices and a twinkle graph W' with $n-2-r$ vertices. Thus, by the induction hypothesis and Lemma 2.2, $mi(W - N_W[u]) \leq mi(rK_1) \cdot mi(W') \leq w_1(n-2-r) \leq w_1(n-2)$. Also, $mi(W - N_W[u]) = w_1(n-2)$ implies that $r = 0$ and $W - N_W[u] = W_1(n-2)$ by the induction hypothesis. On the other hand, there are three possibilities for $W - N_W[v]$, which are

• A forest $\overline{F}$ with at most $n-3$ vertices. Suppose that $\overline{F} = F_1(n-3)$, then $W = Q_1$ when n is odd. This is a contradiction to W being a twinkle graph. So, by Theorem 2.4 and 2.6, we have that

$$mi(W - N_W[v]) \leq \begin{cases} r^{(n-3)-1}, & \text{if } n \text{ is even}, \\ 3r^{(n-3)-4}, & \text{if } n \text{ is odd}. \end{cases} \tag{3}$$

• The union of a forest and a quasi-tree graph $\overline{Q}$ with $s \geq 5$ vertices. Since $Q^*_{1e}(s)$ and $Q^*_{2e}(s)$ are not unicyclic and $|\{x \in V(C) : \deg_{\overline{Q}}(x) = 2\}| = 1$, it follows that $\overline{Q} \neq Q_{1e}(s)$, $Q_{1o}(s)$, $Q_{2e}(s)$. So, by Theorem 2.7

and 2.8, we have that

$$mi(W-N_W[v]) \leq \begin{cases} (5r^{s-6}+1-1)r^{(n-3)-s-1}, & \text{if } s \text{ is even, } n \text{ is even,} \\ r^{s-1}r^{(n-3)-s}, & \text{if } s \text{ is odd, } n \text{ is even,} \\ (5r^{s-6}+1-1)r^{(n-3)-s}, & \text{if } s \text{ is even, } n \text{ is odd,} \\ r^{s-1}r^{(n-3)-s-1}, & \text{if } s \text{ is odd, } n \text{ is odd.} \end{cases} \tag{4}$$

• The union of a forest and a twinkle graph $\overline{W}$ with $t \geq 6$ vertices. So, by the induction hypothesis, we have that

$$mi(W-N_W[v]) \leq \begin{cases} (r^{t-2}+1)r^{(n-3)-t-1}, & \text{if } t \text{ is even, } n \text{ is even,} \\ 3r^{t-5}r^{(n-3)-t}, & \text{if } t \text{ is odd, } n \text{ is even,} \\ (r^{t-2}+1)r^{(n-3)-t}, & \text{if } t \text{ is even, } n \text{ is odd,} \\ 3r^{t-5}r^{(n-3)-t-1}, & \text{if } t \text{ is odd, } n \text{ is odd.} \end{cases} \tag{5}$$

Thus, by (3), (4) and (5), we have that

$$\begin{aligned} mi(W-N_W[v]) &\leq \begin{cases} \max\{r^{n-4}, 5r^{n-10}, r^{n-4}, r^{n-6}+r^{n-4-t}, 3r^{n-8}\}, \\ \qquad \text{if } n \text{ is even,} \\ \max\{3r^{n-7}, 5r^{n-9}, r^{n-5}, r^{n-5}+r^{n-3-t}, 3r^{n-9}\}, \\ \qquad \text{if } n \text{ is odd,} \end{cases} \\ &\leq \begin{cases} r^{n-4}, & \text{if } n \text{ is even,} \\ 3r^{n-7}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Also, if the equalities hold, then

$$W - N_W[v] = \begin{cases} F_1(n-3) \text{ or } Q_3'(7) \cup \frac{n-10}{2}P_2, & \text{if } n \text{ is even,} \\ F_2(n-3), & \text{if } n \text{ is odd.} \end{cases}$$

Hence, by Lemma 2.1, we obtain that

$$\begin{aligned} mi(W) &= mi(W-N_W[u]) + mi(W-N_W[v]) \\ &\leq \begin{cases} w_1(n-2)+r^{n-4}, & \text{if } n \text{ is even,} \\ w_1(n-2)+3r^{n-7}, & \text{if } n \text{ is odd,} \end{cases} \\ &= \begin{cases} (r^{n-4}+1)+r^{n-4}, & \text{if } n \text{ is even,} \\ 3r^{n-7}+3r^{n-7}, & \text{if } n \text{ is odd,} \end{cases} \\ &= w_1(n). \end{aligned}$$

Furthermore, the equality holding imply that $W - N_W[u] = W_1(n-2)$ and $W - N_W[v]$ is as above. In conclusion, $W \in W_1(n)$. □

Theorem 3.1 now follow from Lemmas 3.2, 3.3 and 3.4.

The set of all connected graphs with at most one cycle is the union of the set of trees, twinkle graphs and quasi-tree graphs with exactly one cycle. By Theorem 2.7, C_5, $Q_{1o}(n)$ and $Q_{1e}(n)$ are quasi-tree graphs with only one cycle. Combined with Theorems 2.3, 2.7, 3.1 and $w_1(n) \le t_1(n) < q_1(n)$, we have

Theorem 3.5. ([4]) *If G is a connected graph with at most one cycle of order $n \ge 5$, then $mi(G) \le g(n)$, where*

$$g(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ r^{n-1} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(G) = g(n)$ if and only if $F \in G(n)$, where

$$G(n) = \begin{cases} Q_{1e}(n), & \text{if } n \text{ is even,} \\ Q_{1o}(n) \text{ or } C_5, & \text{if } n \text{ is odd.} \end{cases}$$

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