The Ring of Support-Classes of $\mathrm{SL}_2(\mathbb{F}_q)$

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Abstract[1]: *We introduce and study a subring* $\mathcal{SC}$ *of* $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$ *obtained by summing elements of* $\mathrm{SL}_2(\mathbb{F}_q)$ *according to their support. The ring* $\mathcal{SC}$ *can be used for the construction of several association schemes.*

1 Main results

Summing elements of the finite group $\mathrm{SL}_2(\mathbb{F}_q)$ according to their support (locations of non-zero matrix coefficients), we get seven elements (six when working over $\mathbb{F}_2$) in the integral group-ring $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$.

Integral linear combinations of these seven elements form a subring $\mathcal{SC}$, called the *ring of support classes*, of the integral group-ring $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$. Supposing $q > 2$, we get thus a 7−dimensional algebra $\mathcal{SC}_{\mathbb{K}} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{K}$ over a field $\mathbb{K}$ when considering $\mathbb{K}$−linear combinations.

This paper is devoted to the definition and the study of a few features of $\mathcal{SC}$.

More precisely, in Section 2 we prove that the ring of support-classes $\mathcal{SC}$ is indeed a ring by computing its structure-constants.

Section 3 describes the structure of $\mathcal{SC}_{\mathbb{Q}} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{Q}$ as a semi-simple algebra independent of q for $q > 2$.

In Section 4 we recall the definition of association schemes and use $\mathcal{SC}$ for the construction of hopefully interesting examples.

Finally, we study a few representation-theoretic aspects in Section 5.

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2 The ring of support-classes

Given subsets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ of a finite field $\mathbb{F}_q$, we denote by

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \sum_{(a,b,c,d)\in\mathcal{A}\times\mathcal{B}\times\mathcal{C}\times\mathcal{D}, ad-bc=1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the element of $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$ obtained by summing all matrices of $\mathrm{SL}_2(\mathbb{F}_q)$ with coefficients $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Identifying 0 with the singleton subset $\{0\}$ of $\mathbb{F}_q$ and denoting by $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ the set of all units in $\mathbb{F}_q$, we consider the seven elements

$$A = \begin{pmatrix} \mathbb{F}_q^* & 0 \\ 0 & \mathbb{F}_q^* \end{pmatrix}, B = \begin{pmatrix} 0 & \mathbb{F}_q^* \\ \mathbb{F}_q^* & 0 \end{pmatrix}, C = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix},$$

$$D_+ = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ 0 & \mathbb{F}_q^* \end{pmatrix}, D_- = \begin{pmatrix} \mathbb{F}_q^* & 0 \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix},$$

$$E_+ = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ \mathbb{F}_q^* & 0 \end{pmatrix}, E_- = \begin{pmatrix} 0 & \mathbb{F}_q^* \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix}.$$

corresponding to all possible supports of matrices in $\mathrm{SL}_2[\mathbb{F}_q]$. The element C is of course missing (and the remaining elements consist simply of all six matrices in $\mathrm{SL}_2(\mathbb{F}_2)$) over $\mathbb{F}_2$. For the sake of concision, we will always assume that q has more than 2 elements in the sequel (there is however nothing wrong with finite fields of characteristic 2 having at least 4 elements).

We denote by

$$\mathcal{SC} = \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}C + \mathbb{Z}D_+ + \mathbb{Z}D_- + \mathbb{Z}E_+ + \mathbb{Z}E_-$$

the free $\mathbb{Z}$−module of rank seven spanned by these seven elements.

The set $\mathcal{SC}$ can also be described as the subset of all elements

$$\sum_{M\in\mathrm{SL}_2(\mathbb{F}_q)} \lambda_{\mathrm{supp}(M)}[M]$$

in $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$ with integral coefficients $\lambda_{\mathrm{supp}(M)}$ depending only on the support of M.

We call $\mathcal{SC}$ the *ring of support-classes of* $\mathrm{SL}_2(\mathbb{F}_q)$, a terminology motivated by our main result:

Theorem 2.1 *$\mathcal{SC}$ is a subring of the integral group-ring $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$.*

The construction of $\mathcal{SC}$ can be carried over to the projective special groups $\mathrm{PSL}_2(\mathbb{F}_q)$ without difficulties by dividing all structure-constants by 2 if q is odd. The obvious modifications are left to the reader.

Another obvious variation is to work with matrices in $GL_2(\mathbb{F}_q)$. This multiplies all structure-constants by $(q-1)$ (respectively by m if working with the subgroup of matrices in $GL_2(\mathbb{F}_q)$ having their determinants in a fixed multiplicative subgroup $M \subset \mathbb{F}_q^*$ with m elements).

We hope to address a few other variations of our main construction in a future paper.

Products among generators of SC are given by

$$
\begin{aligned}
AX &= XA = (q-1)X \text{ for } X \in \{A, B, C, D_\pm, E_\pm\},\\
B^2 &= (q-1)A,\\
BC = CB &= (q-1)C,\\
BD_+ = D_-B &= (q-1)E_-,\\
BD_- = D_+B &= (q-1)E_+,\\
BE_+ = E_-B &= (q-1)D_-,\\
BE_- = E_+B &= (q-1)D_+,\\
C^2 &= (q-1)^2(q-2)(A+B) + (q-1)(q-3)(q-4)C\\
&\quad +(q-1)(q-2)(q-3)(D_+ + D_- + E_+ + E_-),\\
CD_+ = CE_- &= (q-1)(q-3)C + (q-1)(q-2)(D_- + E_+),\\
CD_- = CE_+ &= (q-1)(q-3)C + (q-1)(q-2)(D_+ + E_-),\\
D_+C = E_+C &= (q-1)(q-3)C + (q-1)(q-2)(D_- + E_-),\\
D_-C = E_-C &= (q-1)(q-3)C + (q-1)(q-2)(D_+ + E_+),\\
D_+^2 = E_+E_- &= (q-1)^2A + (q-1)(q-2)D_+,\\
D_+D_- = E_+^2 &= (q-1)(C + E_-),\\
D_-D_+ = E_-^2 &= (q-1)(C + E_+),\\
D_+E_+ = E_+D_- &= (q-1)^2B + (q-1)(q-2)E_+,\\
E_+D_+ = D_+E_- &= (q-1)(C + D_-),\\
E_-D_+ = D_-E_- &= (q-1)^2B + (q-1)(q-2)E_-,\\
D_-^2 = E_-E_+ &= (q-1)^2A + (q-1)(q-2)D_-,\\
D_-E_+ = E_-D_- &= (q-1)(C + D_+).
\end{aligned}
$$

Easy consistency checks of these formulae are given by the antiautomorphisms σ and τ obtained respectively by matrix-inversion and matrix-transposition. Their composition $\sigma \circ \tau = \tau \circ \sigma$ is of course an involutive automorphism of $\mathbb{Z}[SL_2(\mathbb{F}_q)]$ which restricts to an automorphism of SC. It coincides on SC with the action of the inner automorphism $X \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathbb{Z}[SL_2(\mathbb{F}_q)]$, fixes A, B, C and transposes the elements of the two pairs $\{D_+, D_-\}$ and $\{E_+, E_-\}$.

Remark 2.2 *The construction of the ring SC described by Theorem 2.1 does not generalise to the matrix-algebra of all* 2×2 *matrices over* $\mathbb{F}_q$. *Indeed,* $\begin{pmatrix} \mathbb{F}_q^* & 0 \\ \mathbb{F}_q^* & 0 \end{pmatrix}\begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ 0 & 0 \end{pmatrix}$ *equals, up to a factor* $(q-1)$, *to the sum of all* $(q-1)^3$ *possible rank* 1 *matrices with all four coefficients in* $\mathbb{F}_q^*$.

Square rank one matrices of any size behave however rather well: The set of all $(2^n-1)^2$ *possible sums of rank* 1 *matrices of size* $n \times n$ *with prescribed support is a* $\mathbb{Z}$*-basis of a ring (defined by extending bilinearly the matrix product) after identifying the zero matrix with* 0.

2.1 Proof of Theorem 2.1

We show that the formulae for the products are correct. We are however not going to prove all $7^2 = 49$ possible identities but all omitted cases are similar and can be derived by symmetry arguments, use of the antiautomorphisms given by matrix-inversion and transposition, or (left/right)-multiplication by B.

Products with A or B are easy and left to the reader.

We start with the easy product D_+^2 (the products

$$D_+E_+, E_+D_-, E_-D_+, D_-E_-, D_-^2, E_-E_+$$

are similar and left to the reader). Since $A + D_+$ is the sum of elements over the full group of all $q(q-1)$ unimodular upper-triangular matrices, we have $(A+D_+)^2 = q(q-1)(A+D_+)$ showing that

$$\begin{aligned} D_+^2 &= (A+D_+)^2 - 2AD_+ - A^2 \\ &= q(q-1)(A+D_+) - 2(q-1)D_+ - (q-1)A \\ &= (q-1)^2 A + (q-1)(q-2)D_+. \end{aligned}$$

For D_+D_- we consider

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ b_2 & 1/a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1b_2 & b_1/a_2 \\ b_2/a_1 & 1/(a_1a_2) \end{pmatrix}.$$

Every unimodular matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with β, γ, δ in $\mathbb{F}_q^*$ can be realised as a summand in the product D_+D_- in exactly $(q-1)$ different ways by choosing a_1 freely in $\mathbb{F}_q^*$ and by setting

$$b_1 = \frac{\beta}{a_1\delta}, a_2 = \frac{1}{a_1\delta}, b_2 = a_1\gamma.$$

This shows $D_+D_- = (q-1)(C+E_-)$. The products

$$E_+^2, D_-D_+, E_-^2, E_+D_+, D_+E_-, D_-E_+, E_-D_-$$

are similar.

In order to compute D_+C, we consider $(D_+ + A)(C + D_- + E_+)$. Since

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & (1+b_2c_2)/a_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1(1+b_2c_2)/a_2 \\ c_2/a_1 & (1+b_2c_2)/(a_1a_2) \end{pmatrix},$$

every unimodular matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\gamma \in \mathbb{F}_q^*$ can be realised in exactly $(q-1)^2$ ways as a summand in $(D_+ + A)(C + D_- + E_+)$ by choosing a_1, a_2 freely in $\mathbb{F}_q^*$ and by setting

$$b_1 = \frac{\alpha - a_1a_2}{a_1\gamma}, b_2 = \frac{a_1a_2\delta - 1}{a_1\gamma}, c_2 = a_1\gamma.$$

This shows $(D_+ + A)(C + D_- + E_+) = (q-1)^2(B + C + D_- + E_+ + E_-)$. We get thus

$$\begin{aligned} & D_+C \\ = \; & (D_+ + A)(C + D_- + E_+) - A(C + D_- + E_+) - D_+D_- - D_+E_+ \\ = \; & (q-1)^2(B + C + D_- + E_+ + E_-) - (q-1)(C + D_- + E_+) \\ & -(q-1)(C + E_-) - (q-1)^2B - (q-1)(q-2)E_+ \\ = \; & (q-1)(q-3)C + (q-1)(q-2)(D_- + E_-). \end{aligned}$$

The products

$$CD_+, CE_-, CD_-, CE_+, E_+C, D_-C, E_-C$$

are similar.

Using all previous products, the formula for C^2 can now be recovered from $(A + B + C + D_+ + D_- + E_+ + E_-)^2 = (q^3 - q)(A + B + C + D_+ + D_- + E_+ + E_-)$. We have indeed

$$C^2 = (A + B + C + D_+ + D_- + E_+ + E_-)^2 - \sum_{(X,Y)\neq(C,C)} XY$$

where the sum is over all elements of $\{A, B, C, D_+, D_-, E_+, E_-\}^2 \setminus (C, C)$. All products of the right-hand-side are known and determine thus C^2. Equivalently, structure-constants of C^2 have to be polynomials of degree at most 3 in q. They can thus also be computed by interpolating the coefficients in 4 explicit examples. (Using divisibility by $q - 1$, computing 3 examples is in fact enough.)

The existence of these formulae proves Theorem 2.1. $\square$

2.2 Matrices for left-multiplication by generators

Left-multiplications by generators with respect to the basis $A, B, C, D_+, D_-, E_+, E_-$ of $\mathcal{S}C$ are encoded by the matrices

$$M_B = (q-1)\begin{pmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}$$

$$M_C = (q-1)\begin{pmatrix} 0&0&(q-1)(q-2)&0&0&0&0\\ 0&0&(q-1)(q-2)&0&0&0&0\\ 1&1&(q-3)(q-4)&q-3&q-3&q-3&q-3\\ 0&0&(q-2)(q-3)&0&q-2&q-2&0\\ 0&0&(q-2)(q-3)&q-2&0&0&q-2\\ 0&0&(q-2)(q-3)&q-2&0&0&q-2\\ 0&0&(q-2)(q-3)&0&q-2&q-2&0 \end{pmatrix}$$

$$M_{D_+} = (q-1)\begin{pmatrix} 0&0&0&q-1&0&0&0\\ 0&0&0&0&0&q-1&0\\ 0&0&q-3&0&1&0&1\\ 1&0&0&q-2&0&0&0\\ 0&0&q-2&0&0&0&1\\ 0&1&0&0&0&q-2&0\\ 0&0&q-2&0&1&0&0 \end{pmatrix}$$

$$M_{E_+} = (q-1)\begin{pmatrix} 0&0&0&0&0&0&q-1\\ 0&0&0&0&q-1&0&0\\ 0&0&q-3&1&0&1&0\\ 0&1&0&0&0&0&q-2\\ 0&0&q-2&1&0&0&0\\ 1&0&0&0&q-2&0&0\\ 0&0&q-2&0&0&1&0 \end{pmatrix}$$

The remaining matrices are given by $M_A = \frac{1}{q-1}(M_B)^2$, $M_{D_-} = \alpha M_{D_+}\alpha$ and $M_{E_-} = \alpha M_{E_+}\alpha$ where

$$\alpha = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix}$$

is the matrix corresponding to the automorphism $\sigma \circ \tau$.

The map $\{A, B, C, D_+, D_-, E_+, E_-\} \ni X \longmapsto M_X$ extends of course to an isomorphisme between $\mathcal{SC}$ and

$$\mathbb{Z}M_A + \mathbb{Z}M_B + \mathbb{Z}M_C + \mathbb{Z}M_{D_+} + \mathbb{Z}M_{D_-} + \mathbb{Z}M_{E_+} + \mathbb{Z}M_{E_-}$$

which is a ring. Computations are easier and faster in this matrix-ring than in the subring $\mathcal{SC}$ of $\mathbb{Z}[\mathrm{SL}_2(\mathbb{F}_q)]$.

3 Algebraic properties of $\mathcal{SC}_{\mathbb{Q}}$

3.1 $\mathcal{SC}_{\mathbb{Q}}$ as a semisimple algebra

Theorem 3.1 *The algebra $\mathcal{SC}_{\mathbb{Q}} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semi-simple algebra isomorphic to $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q})$.*

The structure of $\mathcal{SC}_{\mathbb{K}} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{K}$ is of course easy to deduce for any field of characteristic 0.

The algebra $\mathcal{SC}_{\mathbb{K}}$ is also semi-simple (and has the same structure) over most finite fields.

Theorem 3.1 is an easy consequence of the following computations:

The center of $\mathcal{SC}_{\mathbb{Q}}$ has rank 4. It is spanned by the 3 central minimal idempotents

$$\begin{aligned}
\pi_1 &= \frac{1}{q^3-q}(A+B+C+D_+ + D_- + E_+ + E_-),\\
\pi_2 &= \frac{q-2}{2(q^2-q)}(A+B)\\
&\quad + \frac{1}{q(q-1)^2}C - \frac{q-2}{2q(q-1)^2}(D_+ + D_- + E_+ + E_-),\\
\pi_3 &= \frac{1}{2(q+1)}(A-B) + \frac{1}{2(q^2-1)}(-D_+ - D_- + E_+ + E_-)
\end{aligned}$$

and by the central idempotent

$$\pi_4 = \frac{2(q-1)A - 2C + (q-2)(D_+ + D_-) - (E_+ + E_-)}{(q+1)(q-1)^2}$$

which is non-minimal among all idempotents. The three idempotents π_1, π_2 and π_3 induce three different characters (homomorphisms from $\mathcal{SC}_{\mathbb{Q}}$ into $\mathbb{Q}$). Identifying $1 \in \mathbb{Q}$ with π_i in each case, the three homomorphisms are given by

	A	B	C	$D_\pm$	$E_\pm$
π_1	$q-1$	$q-1$	$(q-1)^2(q-2)$	$(q-1)^2$	$(q-1)^2$
π_2	$q-1$	$q-1$	$2(q-1)$	$1-q$	$1-q$
π_3	$q-1$	$1-q$	0	$1-q$	$q-1$

The idempotent π_1 is of course simply the augmentation map counting the number of matrices involved in each generator.

The idempotent $\pi_1 + \pi_2 + \pi_3 + \pi_4 = \frac{1}{q-1} A$ is the identity of $\mathcal{SC}_{\mathbb{Q}}$.

The idempotent π_4 projects $\mathcal{SC}_{\mathbb{Q}}$ homomorphically onto a matrix-algebra of 2×2 matrices.

π_4 can be written (not uniquely) as a sum of two minimal non-central idempotents. We have for example $\pi_4 = M_{1,1} + M_{2,2}$ where

$$\begin{aligned} M_{1,1} &= \frac{1}{q^2-1}(A-B) + \frac{1}{2(q^2-1)}(D_+ + D_- - E_+ - E_-), \\ M_{2,2} &= \frac{1}{q^2-1}(A+B) - \frac{2}{(q+1)(q-1)^2} C + \\ &\quad + \frac{q-3}{2(q+1)(q-1)^2}(D_+ + D_- + E_+ + E_-). \end{aligned}$$

Considering also

$$\begin{aligned} M_{1,2} &= \frac{1}{2(q-1)^2}(D_+ - D_- + E_+ - E_-), \\ M_{2,1} &= \frac{1}{2(q^2-1)}(D_+ - D_- - E_+ + E_-), \end{aligned}$$

the elements $M_{i,j}$ behave like matrix-units and the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto aM_{1,1} + bM_{1,2} + cM_{2,1} + dM_{2,2} \tag{1}$$

defines thus an isomorphism from the ring of integral 2×2 matrices into $\mathcal{SC}_{\mathbb{Q}} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{Q}$. If $\mathbb{F}_q$ has odd characteristic, the elements $M_{i,j}$ can be realised in $\mathcal{SC}_{\mathbb{F}_q} = \mathcal{SC} \otimes_{\mathbb{Z}} \mathbb{F}_q$. In particular, Formula (1) gives an "exotic", non-unital embedding of $\mathrm{SL}_2[\mathbb{F}_q]$ into the group-algebra $\mathbb{F}_q[\mathrm{SL}_2(\mathbb{F}_q)]$ (in fact, (1) gives an embedding of $\mathrm{SL}_2(\mathbb{F}_r)$ into $\mathbb{F}_r[\mathrm{SL}_2(\mathbb{F}_q)]$ whenever the prime power r is coprime to $2(q^2-1)$).

The nice non-central idempotent

$$\pi_3 + M_{1,1} = \frac{1}{2(q-1)}(A-B)$$

projects $\mathcal{SC}_{\mathbb{Q}}$ onto the eigenspace of eigenvalue $1-q$ of the map $X \longmapsto BX$.

The projection of $\mathcal{SC}_{\mathbb{Q}}$ onto the eigenspace of eigenvalue $q-1$ of the map $X \longmapsto BX$ is similarly given by

$$\pi_1 + \pi_2 + M_{2,2} = \frac{1}{2(q-1)}(A+B).$$

3.2 A few commutative subalgebras of $\mathcal{SC}_{\mathbb{Q}}$

The previous section shows that the dimension of a commutative subalgebra of $\mathcal{SC}_{\mathbb{Q}}$ cannot exceed 5.

The center of $\mathcal{SC}_{\mathbb{Q}}$ is of course of rank 4 and spanned by the four minimal central idempotents $\pi_1, \dots, \pi_4$ of the previous Section.

Splitting $\pi_4 = M_{1,1} + M_{2,2}$, we get a maximal commutative subalgebra of rank 5 by considering the vector space spanned by the minimal idempotents $\pi_1, \pi_2, \pi_3, M_{1,1}, M_{2,2}$. Equivalently, this vector space is spanned by $A, B, C, D = D_+ + D_-, E = E_+ + E_-$, as shown by the formulae for π_i and $M_{1,1}, M_{2,2}$.

In terms of A, B, C, D, E, minimal idempotents $\pi_1, \pi_2, \pi_3, M_{1,1}, M_{2,2}$ are given by

$$\begin{aligned}
\pi_1 &= \frac{1}{q^3-q}(A+B+C+D+E),\\
\pi_2 &= \frac{q-2}{2(q^2-q)}(A+B) + \frac{1}{q(q-1)^2}C - \frac{q-2}{2q(q-1)^2}(D+E),\\
\pi_3 &= \frac{1}{2(q+1)}(A-B) + \frac{1}{2(q^2-1)}(-D+E),\\
M_{1,1} &= \frac{1}{q^2-1}(A-B) + \frac{1}{2(q^2-1)}(D-E),\\
M_{2,2} &= \frac{1}{q^2-1}(A+B) - \frac{2}{(q+1)(q-1)^2}C +\\
&\quad + \frac{q-3}{2(q+1)(q-1)^2}(D+E).
\end{aligned}$$

They define five characters given by $\mathbb{Q}A + \cdots + \mathbb{Q}E \longrightarrow \mathbb{Q}$ given by

	A	B	C	D	E
π_1	$q-1$	$q-1$	$(q-1)^2(q-2)$	$2(q-1)^2$	$2(q-1)^2$
π_2	$q-1$	$q-1$	$2(q-1)$	$2(1-q)$	$2(1-q)$
π_3	$q-1$	$1-q$	0	$2(1-q)$	$2(q-1)$
$M_{1,1}$	$q-1$	$1-q$	0	$(q-1)^2$	$-(q-1)^2$
$M_{2,2}$	$q-1$	$q-1$	$2(1-q)(q-2)$	$(q-1)(q-3)$	$(q-1)(q-3)$

Moreover, $A, B, C, F = D + E$ span a commutative 4-dimensional subalgebra. Minimal idempotents are

$$\begin{aligned}
\pi_1 &= \frac{1}{q^3-q}(A+B+C+F), \\
\pi_2 &= \frac{q-2}{2(q^2-q)}(A+B)+\frac{1}{q(q-1)^2}C-\frac{q-2}{2q(q-1)^2}F, \\
\pi_3+M_{1,1} &= \frac{1}{2(q-1)}(A-B), \\
M_{2,2} &= \frac{1}{q^2-1}(A+B)-\frac{2}{(q+1)(q-1)^2}C+ \\
&\quad +\frac{q-3}{2(q+1)(q-1)^2}F.
\end{aligned}$$

with character-table

	A	B	C	F
π_1	$q-1$	$q-1$	$(q-1)^2(q-2)$	$4(q-1)^2$
π_2	$q-1$	$q-1$	$2(q-1)$	$4(1-q)$
$\pi_3+M_{1,1}$	$q-1$	$1-q$	0	0
$M_{2,2}$	$q-1$	$q-1$	$2(1-q)(q-2)$	$2(q-1)(q-3)$

$I=A+B, C, F=E+D$ span a commutative 3−dimensional subalgebr of $\mathcal{SC}_{\mathbf{C}}$. Generators of this last algebra are sums of elements in $\mathrm{SL}_2(\mathbb{F}_q)$ wit supports of given cardinality. I is the sum of all elements with two non-zer coefficients, C contains all elements having only non-zero coefficients an F contains all elements with three non-zero coefficients.

Products are given by $IX=XI=2(q-1)X$ for $X\in\{I,C,F\}$ and

$$\begin{aligned}
C^2 &= (q-1)^2(q-2)I+(q-1)(q-3)(q-4)C \\
&\quad +(q-1)(q-2)(q-3)F, \\
CF &= FC=4(q-1)(q-3)C+2(q-1)(q-2)F, \\
F^2 &= 4(q-1)^2I+8(q-1)C+2(q-1)^2F.
\end{aligned}$$

Idempotents are given by

$$\begin{aligned}
\pi_1 &= \frac{1}{q^3-q}(I+C+F), \\
\pi_2 &= \frac{q-2}{2(q^2-q)}I+\frac{1}{q(q-1)^2}C-\frac{q-2}{2q(q-1)^2}F, \\
M_{2,2} &= \frac{1}{q^2-1}I-\frac{2}{(q+1)(q-1)^2}C+ \\
&\quad +\frac{q-3}{2(q+1)(q-1)^2}F.
\end{aligned}$$

with character-table

	I	C	F
π_1	$2(q-1)$	$(q-1)^2(q-2)$	$4(q-1)^2$
π_2	$2(q-1)$	$2(q-1)$	$4(1-q)$
$M_{2,2}$	$2(q-1)$	$2(1-q)(q-2)$	$2(q-1)(q-3)$

Working over $\mathbb{C}$ (or over a suitable extension of $\mathbb{Q}$) and setting

$$\tilde{I} = \frac{1}{2(q-1)} I, \tilde{C} = \frac{1}{\sqrt{2(q-1)^3(q-2)}} C, \tilde{F} = \frac{1}{\sqrt{8(q-1)^3}} F$$

we get products $XY = \sum_{Z \in \{\tilde{I}, \tilde{C}, \tilde{F}\}} N_{X,Y,Z} Z$ which are defined by symmetric structure-constants $N_{X,Y,Z} = N_{Y,X,Z} = N_{X,Z,Y}$ for all $X, Y, Z \in \{\tilde{I}, \tilde{C}, \tilde{F}\}$. Up to symmetric permutations, the structure-constants are given by

$$\begin{aligned} N_{\tilde{I},X,Y} &= \delta_{X,Y} \\ N_{\tilde{C},\tilde{C},\tilde{C}} &= \frac{(q-3)(q-4)}{\sqrt{2(q-1)(q-2)}}, \\ N_{\tilde{C},\tilde{C},\tilde{F}} &= (q-3)\sqrt{\frac{2}{q-1}} \\ N_{\tilde{C},\tilde{F},\tilde{F}} &= \sqrt{\frac{2(q-2)}{q-1}}, \\ N_{\tilde{F},\tilde{F},\tilde{F}} &= \sqrt{\frac{q-1}{2}} \end{aligned}$$

where $\delta_{X,Y} = 1$ if and only if $X = Y$ and $\delta_{X,Y} = 0$ otherwise. The evaluation at $q = 3$ leads to particularly nice structure constants with values in $\{0, 1\}$.

Algebras with generating systems having symmetric structure-constants and a character taking real positive values on generators (satisfied by π_1 for $q > 2$) are sometimes called *algebraic fusion-algebras*, see for example [3].

4 Association schemes and Bose-Mesner algebras

An *association scheme* is a set of $d+1$ square matrices $C_0, \ldots, C_d$ with coefficients in $\{0, 1\}$ such that C_0 is the identity-matrix, $C_0 + \cdots + C_d$ is the all-one matrix and $\mathbb{Z}C_0 + \cdots + \mathbb{Z}C_d$ is a commutative ring with (necessarily integral) structure constants $p^k_{i,j} = p^k_{j,i}$ defined by $C_iC_j = C_jC_i = \sum_{k=0}^{d} p^k_{i,j}C_k$. An association scheme is *symmetric* if $C_1, \ldots, C_d$ are symmetric matrices. The algebra (over a field) generated by the elements C_i is called a *Bose-Mesner algebra.* See for example the monograph [1] for additional information.

Identifying an element g of $\mathrm{SL}_2(\mathbb{F}_q)$ with the permutation-matrix associated to left-multiplication by g we get a commutative association scheme with $d = 5$ if $q \geq 4$ by setting

$$C_0 = \mathrm{Id}, C_1 = A - \mathrm{Id}, C_2 = B, C_3 = C, C_4 = D_+ + D_-, C_5 = E_+ + E_-$$

where we consider sums of permutation-matrices. All matrices are symmetric. Products with C_0, C_1 are given by $C_0X = XC_0 = X, C_1^2 = (q-2)C_0 + (q-3)C_1, C_1Y = YC_1 = (q-2)Y$ for $X \in \{C_0, \ldots, C_5\}$ and $Y \in \{C_2, \ldots, C_5\}$. The remaining products are given by

$$\begin{aligned}
C_2^2 &= (q-1)(C_0 + C_1),\\
C_2C_3 = C_3C_2 &= (q-1)C_3\\
C_2C_4 = C_4C_2 &= (q-1)C_5\\
C_2C_5 = C_5C_2 &= (q-1)C_4\\
C_3^2 &= (q-1)^2(q-2)(C_0 + C_1 + C_2) + (q-1)(q-3)(q-4)C_3\\
&\quad +(q-1)(q-2)(q-3)(C_4 + C_5)\\
C_3C_4 = C_4C_3 &= 2(q-1)(q-3)C_3 + (q-1)(q-2)(C_4 + C_5)\\
C_3C_5 = C_5C_3 &= 2(q-1)(q-3)C_3 + (q-1)(q-2)(C_4 + C_5)\\
C_4^2 &= 2(q-1)^2(C_0 + C_1) + 2(q-1)C_3\\
&\quad +(q-1)(q-2)C_4 + (q-1)C_5\\
C_4C_5 = C_5C_4 &= 2(q-1)^2C_2 + 2(q-1)C_3\\
&\quad +(q-1)C_4 + (q-1)(q-2)C_5\\
C_5^2 &= 2(q-1)^2(C_0 + C_1) + 2(q-1)C_3\\
&\quad +(q-1)(q-2)C_4 + (q-1)C_5
\end{aligned}$$

The reader should be warned that C_1 behaves not exactly like $(q-2)C_0$.

We leave it to the reader to write down matrices for multiplications with basis-elements and to compute the complete list of minimal idempotents.

Additional association schemes are given by $C_0, C_1, C_2, C_3, C_4 + C_5$ and $C_0, C_1 + C_2, C_3, C_4 + C_5$. It is also possible to split C_1 and/or C_2 according to subgroups of $\mathbb{F}_q^*$ into several matrices (or classes, as they are sometimes called).

We discuss now with a little bit more details the smallest interesting association scheme with classes $\tilde{C}_0, \tilde{C}_1 = C_1 + C_2, \tilde{C}_2 = C_3, \tilde{C}_3 = C_4 + C_5$.

Products are given by $\tilde{C}_0 X = X\tilde{C}_0$ and

$$\begin{aligned}
\tilde{C}_1^2 &= (2q-3)\tilde{C}_0 + 2(q-2)\tilde{C}_1 \\
\tilde{C}_1\tilde{C}_2 = \tilde{C}_2\tilde{C}_1 &= (2q-3)\tilde{C}_2 \\
\tilde{C}_1\tilde{C}_3 = \tilde{C}_3\tilde{C}_1 &= (2q-3)\tilde{C}_3 \\
\tilde{C}_2^2 &= (q-1)^2(q-2)(\tilde{C}_0 + \tilde{C}_1) + (q-1)(q-3)(q-4)\tilde{C}_2 \\
&\quad +(q-1)(q-2)(q-3)\tilde{C}_3 \\
\tilde{C}_2\tilde{C}_3 = \tilde{C}_3\tilde{C}_2 &= 4(q-1)(q-3)\tilde{C}_2 + 2(q-1)(q-2)\tilde{C}_3 \\
\tilde{C}_3^2 &= 4(q-1)^2(\tilde{C}_0 + \tilde{C}_1) + 8(q-1)\tilde{C}_2 + 2(q-1)^2\tilde{C}_3
\end{aligned}$$

Matrices $M_0, \dots, M_3$ corresponding to multiplication by $\tilde{C}_0, \dots, \tilde{C}_3$ are given by

$$\tilde{C}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} 0 & 2q-3 & 0 & 0 \\ 1 & 2(q-2) & 0 & 0 \\ 0 & 0 & 2q-3 & 0 \\ 0 & 0 & 0 & 2q-3 \end{pmatrix}$$

$$\tilde{C}_2 = \begin{pmatrix} 0 & 0 & (q-1)^2(q-2) & 0 \\ 0 & 0 & (q-1)^2(q-2) & 0 \\ 1 & 2q-3 & (q-1)(q-3)(q-4) & 4(q-1)(q-3) \\ 0 & 0 & (q-1)(q-2)(q-3) & 2(q-1)(q-2) \end{pmatrix},$$

$$\tilde{C}_3 = \begin{pmatrix} 0 & 0 & 0 & 4(q-1)^2 \\ 0 & 0 & 0 & 4(q-1)^2 \\ 0 & 0 & 4(q-1)(q-3) & 8(q-1) \\ 1 & 2q-3 & 2(q-1)(q-2) & 2(q-1)^2 \end{pmatrix}$$

and minimal idempotents are given by

$$\begin{aligned}
\beta_0 &= \frac{1}{q^3-q}(\tilde{C}_0 + \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3), \\
\beta_1 &= \frac{q-2}{2(q^2-q)}(\tilde{C}_0 + \tilde{C}_1) + \frac{1}{q(q-1)^2}\tilde{C}_2 - \frac{q-2}{2q(q-1)^2}\tilde{C}_3, \\
\beta_2 &= \frac{2q-3}{2(q-1)}\tilde{C}_0 - \frac{1}{2(q-1)}\tilde{C}_1, \\
\beta_3 &= \frac{1}{q^2-1}(\tilde{C}_0 + \tilde{C}_1) - \frac{2}{(q-1)^2(q+1)}\tilde{C}_2 + \frac{q-3}{2(q-1)^2(q+1)}\tilde{C}_3.
\end{aligned}$$

The coefficient of $\tilde{C}_0$ multiplied by (q^3-q) gives the dimension of the associated eigenspace. Eigenvalues (with multiplicities) of generators, obtained by evaluating the characters $\beta_0, \ldots, \beta_4$ on $\tilde{C}_0, \ldots, \tilde{C}_3$, are given by

	dim	$\tilde{C}_0$	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$
β_0	1	1	$2q-3$	$(q-1)^2(q-2)$	$4(q-1)^2$
β_1	$\frac{(q+1)(q-2)}{2}$	1	$2q-3$	$2(q-1)$	$4(1-q)$
β_2	$\frac{q(2q-3)(q+1)}{2}$	1	-1	0	0
β_3	q	1	$2q-3$	$2(1-q)(q-2)$	$2(q-1)(q-3)$

Remark 4.1 *There exists a few more exotic variations of this construction. An example is given by partitioning the elements of* $\mathrm{SL}_2(\mathbb{F}_5)$ *according to the* 30 *possible values of the Legendre symbol* $\left(\frac{x}{5}\right)$ *on entries. Since* -1 *is a square modulo* 5, *these classes are well-defined on* $\mathrm{PSL}_2(\mathbb{F}_5)$ *which is isomorphic to the simple group* A_5. *Details (and a few similar examples) will hopefully appear in a future paper.*

Remark 4.2 *E. Bannai constructed in [2] subschemes of group association schemes by considering suitable unions of conjugacy classes in* $\mathrm{PSL}_2(\mathbb{F}_q)$. *This leads to examples which are fairly different from the examples constructed in this section as can be seen as follows: Sizes of classes and structure constants in [2] are quiet different. Moreover, the results of the next section imply that the classes of our examples are very far from being unions of conjugacy classes but slice instead through many different conjugacy classes (as should be expected for classes defined in terms of supports, a notion which is not at all preserved by conjugation). Our examples are thus in some sense "orthogonal" to Bannai's assocation schemes in [2].*

5 Representation-theoretic aspects

In this Section, we work over $\mathbb{C}$ for the sake of simplicity.

5.1 Traces

Left-multiplication by the identity $\frac{1}{q-1}A$ of $SC_{\mathbb{C}}$ defines an idempotent on $\mathbb{Q}[\mathrm{SL}_2(\mathbb{F}_q)]$ whose trace is the dimension $\frac{1}{q-1}(q^3-q) = q(q+1)$ of the non-trivial eigenspace. Indeed, every non-trivial element of $\mathrm{SL}_2(\mathbb{F}_q)$ has trace 0 and the identity-matrix has trace q^3-q since it fixes all q^3-q elements of $\mathrm{SL}_2(\mathbb{F}_q)$. A basis of the non-trivial $q(q+1)$-dimensional eigenspace of $\frac{1}{q-1}A$ is given by sums over all matrices with rows representing two distinct fixed elements of the projective line over $\mathbb{F}_q$.

The traces $\mathrm{tr}(\pi_1), \ldots, \mathrm{tr}(\pi_4)$ of the minimal central projectors $\pi_1, \ldots, \pi_4$ of $SC_{\mathbb{C}}$ are equal to q^3-q times the coefficient of A in π_i. They are thus given by

	π_1	π_2	π_3	π_4
trace	1	$\frac{(q+1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$2q$

and we have

$$q(q+1) = \mathrm{tr}(\pi_1) + \mathrm{tr}(\pi_2) + \mathrm{tr}(\pi_3) + \mathrm{tr}(\pi_4),$$

as expected.

5.2 Characters

Since simple matrix-algebras of $\mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$ are indexed by characters of $\mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$, it is perhaps interesting to understand all irreducible characters involved in idempotents of $\mathcal{SC}_{\mathbb{C}} \subset \mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$.

The algebra $\mathcal{SC}_{\mathbb{C}}$ is in some sense almost "orthogonal" to the center of $\mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$. The algebra $\mathcal{SC}_{\mathbb{C}}$ should thus involve many different irreducible characters of $\mathrm{SL}_2(\mathbb{F}_q)$. We will see that this is indeed the case.

We decompose first the identity $\frac{1}{q-1}A$ according to irreducible characters of $\mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$. We refine this decomposition to the minimal central idempotents $\pi_1, \ldots, \pi_4$ of $\mathcal{SC}_{\mathbb{C}}$ in Section 5.4.

For simplicity we work over $\mathrm{GL}_2(\mathbb{F}_q)$ which has essentially the same character-theory as $\mathrm{SL}_2(\mathbb{F}_q)$. We work over $\mathbb{C}$ and we identify (irreducible) characters with the corresponding (irreducible) representations.

In order to do this, we introduce $F = \sum_{\lambda \in \mathbb{F}_q^*} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \in \mathbb{Z}[\mathrm{GL}_2(\mathbb{F}_q)]$. We have $F^2 = (q-1)F$ and $FX = XF$ for $X \in \{A, B, C, D_\pm, E_\pm\}$, considered as an element of $\mathbb{Z}[\mathrm{GL}_2(\mathbb{F}_q)]$. The map $X \longmapsto \frac{1}{q-1} FX$ preserves traces and defines an injective homomorphism of $\mathcal{SC}$ into $\mathbb{Q}[\mathrm{GL}_2(\mathbb{F}_q)]$.

We use the conventions of Chapter 5 of [4] for conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$. More precisely, we denote by a_x conjugacy classes of central diagonal matrices with common diagonal value x in $\mathbb{F}_q^*$, by b_x conjugacy classes given by multiplying unipotent matrices by a scalar x in $\mathbb{F}_q^*$, by $c_{x,y}$ conjugacy classes with two distinct eigenvalues $x, y \in \mathbb{F}_q^*$ and by d_ξ conjugacy classes with two conjugate eigenvalues $\xi, \xi^q \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q$. The number of conjugacy classes of each type is given by

a_x	b_x	$c_{x,y}$	d_ξ
$q-1$	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$.

The character table of $\mathrm{GL}_2(\mathbb{F}_q)$, copied from [4], is now given by

	1	q^2-1	q^2+q	q^2-q
	a_x	b_x	$c_{x,y}$	d_ξ
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\xi^{q+1})$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\xi^{q+1})$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\xi)+\varphi(\xi^q))$

where α, β are distinct characters of $\mathbb{F}_q^*$ and where φ is a character of $\mathbb{F}_{q^2}^*$ with φ^{q-1} non-trivial. The first row indicates the number of elements in a conjugacy class indicated by the second row. The remaining rows give the character-table. U_α are the one-dimensional representations factoring through the determinant. $V_\alpha = V \otimes U_\alpha$ are obtained from the permutation-representation $V = V_1 + U_1$ describing the permutation-action of $\mathrm{GL}_2(\mathbb{F}_q)$ on all $p+1$ points of the projective line over $\mathbb{F}_q$. $W_{\alpha,\beta}$ (isomorphic to $W_{\beta,\alpha}$) are induced from non-trivial 1-dimensional representations of a Borel subgroup (given for example by all upper triangular matrices). X_φ (isomorphic to X_φ) are the remaining irreducible representations $V_1 \otimes W_{\varphi|_{\mathbb{F}_q^*},1} - W_{\varphi|_{\mathbb{F}_q^*},1} - \mathrm{Ind}_\varphi$ with Ind_φ obtained by inducing a 1-dimensional representation $\varphi \neq \varphi^q$ of a cyclic subgroup isomorphic to $\mathbb{F}_{q^2}^*$.

The trace of the idempotent $\pi = \frac{1}{(q-1)^2} FA$ in irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ is now given by:

$$U_\alpha \ : \ \frac{1}{(q-1)^2}\left(\sum_{x\in\mathbb{F}_q^*}\alpha(x)\right)^2$$

$$V_\alpha \ : \ q\left(\frac{1}{q-1}(\sum_{x\in\mathbb{F}_q^*}\alpha(x^2) + \frac{1}{(q-1)^2}\left(\sum_{x\in\mathbb{F}_q^*}\alpha(x)\right)^2\right)$$

$$W_{\alpha,\beta} \ : \ \frac{q+1}{q-1}\sum_{x\in\mathbb{F}_q^*}\alpha(x)\beta(x) + \frac{2(q+1)}{(q-1)^2}\left(\sum_{x\in\mathbb{F}_q^*}\alpha(x)\right)\left(\sum_{x\in\mathbb{F}_q^*}\beta(x)\right)$$

$$X_\varphi \ : \ (q-1)\sum_{x\in\mathbb{F}_q^*}\varphi(x)$$

The factors $q, q\pm 1$ are the multiplicities of the irreducible representations $V_\alpha, W_{\alpha,\beta}$ and X_φ in the regular (left or right) representation. They are of course equal to the dimensions of $V_\alpha, W_{\alpha,\beta}$ and X_φ.

Irreducible representations U_α are involved in π only if α is the trivial character. This corresponds of course to the central idempotent π_1 of $\mathcal{SC}_\mathbf{Q}$.

For V_α we get $2q$ for α trivial, q for odd q if α is the quadratic character (defined by the Legendre-symbol and existing only for odd q) and 0 otherwise.

For $W_{\alpha,\beta}$ we get $q+1$ if $\beta = \overline{\alpha}, \beta \neq \alpha$ and 0 otherwise. There are $\frac{q-3}{2}$ such representations for odd q and $\frac{q-2}{2}$ such representations for even q.

For X_φ we get $q-1$ if the character φ of $\mathbb{F}_{q^2}^*$ with non-trivial φ^{q-1} restricts to the trivial character of $\mathbb{F}_q^*$ and 0 otherwise. Characters of $\mathbb{F}_{q^2}^*$ with trivial restrictions to $\mathbb{F}_q^*$ are in one-to-one correspondence with characters of the additive group $\mathbb{Z}/(q+1)\mathbb{Z}$. The character φ^{q-1} is trivial for two of them (the trivial and the quadratic one) if q is odd. This gives thus $\frac{q-1}{2}$ such irreducible representations for odd q. For even q we get $\frac{q}{2}$ irreducible representations.

All irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ involved in $\pi = \frac{1}{(q-1)^2}FA$ have irreducible restrictions to $\mathrm{SL}_2(\mathbb{F}_q)$.

The following table sums up contributions of all different irreducible characters to the trace of $\pi = \frac{1}{(q-1)^2}FA$:

q	U	V	W	X
odd	1	$3q$	$(q+1)\frac{q-3}{2}$	$(q-1)\frac{q-1}{2}$
even	1	$2q$	$(q+1)\frac{q-2}{2}$	$(q-1)\frac{q}{2}$

and we have

$$\mathrm{tr}(\pi) = q(q+1) = 1 + 3q + (q+1)\frac{q-3}{2} + (q-1)\frac{q-1}{2},$$

respectively

$$\mathrm{tr}(\pi) = q(q+1) = 1 + 2q + (q+1)\frac{q-2}{2} + (q-1)\frac{q}{2},$$

as expected.

5.3 Decompositions of $\pi_1, \ldots, \pi_4$ for even q

Over a finite field of characteristic 2, conjugacy classes are involved in generators of SC as follows:

	$q-1$ a_x	$q-1$ b_x	$\frac{(q-1)(q-2)}{2}$ $c_{x,y\neq x}$	$\frac{q(q-1)}{2}$ d_ξ
FA	1	0	2	0
FB	0	$q-1$	0	0
FC	0	$(q-1)(q-2)$	$(q-1)(q-4)$	$(q-1)(q-2)$
$FD\pm$	0	$q-1$	$2(q-1)$	0
$FE\pm$	0	0	$q-1$	$(q-1)$
	1	q^2-1	q^2+q	q^2-q

This implies the following decompositions of central idempotents of SC: The idempotent π_1 of rank 1 is involved with multiplicity 1 in the trivial representation of type U. The idempotent π_2 of rank $\frac{(q+1)(q-2)}{2}$ (in $\mathbb{C}[\mathrm{SL}_2(\mathbb{F}_q)]$) is involved with multiplicity $q+1$ in all $\frac{q-2}{2}$ relevant characters of type W. The idempotent π_3 of rank $\frac{q(q-1)}{2}$ is involved with multiplicity $q-1$ in all $\frac{q}{2}$ relevant characters of type X. Finally, the idempotent π_4 is involved with multiplicity q in the irreducible representation V.

5.4 Decompositions of $\pi_1, \ldots, \pi_4$ for odd q

Over a finite field of odd characteristic, contributions of the different conjugacy classes to the elements FX (for X a generator of $\mathcal{SC}$) are given by

	$q-1$	$q-1$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)^2}{2}$
	a_x	b_x	$c_{x,-x}$	$c_{x,y\neq\pm x}$	$d_{\xi,\mathrm{tr}(\xi)=0}$	$d_{\xi,\mathrm{tr}(\xi)\neq 0}$
FA	1	0	2	2	0	0
FB	0	0	$q-1$	0	$q-1$	0
FC	0	q^2-4q+3	q^2-4q+3	$(q-1)(q-4)$	$(q-1)^2$	$(q-1)(q-2)$
$FD_\pm$	0	$q-1$	$2(q-1)$	$2(q-1)$	0	0
$FE_\pm$	0	$q-1$	0	$q-1$	0	$q-1$
	1	q^2-1	q^2+q	q^2+q	q^2-q	

and conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ are involved in all four projectors $\tilde{\pi}_1 = \frac{1}{q-1}\pi_1 F, \ldots, \tilde{\pi}_4 = \frac{1}{q-1}\pi_4 F$ with the following coefficients

	$q-1$	$q-1$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)^2}{2}$
	a_x	b_x	$c_{x,-x}$	$c_{x,y\neq\pm x}$	$d_{\xi,\mathrm{tr}(\xi)=0}$	$d_{\xi,\mathrm{tr}(\xi)\neq 0}$
$\tilde{\pi}_1$	$\frac{1}{(q-1)^2 q(q+1)}$	$\frac{1}{q(q-1)}$	$\frac{1}{(q-1)^2}$	$\frac{1}{(q-1)^2}$	$\frac{1}{q^2-1}$	$\frac{1}{q^2-1}$
$\tilde{\pi}_2$	$\frac{q-2}{2q(q-1)^2}$	$\frac{-1}{q(q-1)}$	$\frac{q-3}{2(q-1)^2}$	$\frac{-1}{(q-1)^2}$	$\frac{1}{2(q-1)}$	0
$\tilde{\pi}_3$	$\frac{1}{2(q^2-1)}$	0	$\frac{-1}{2(q-1)}$	0	$\frac{-1}{2(q+1)}$	$\frac{1}{q^2-1}$
$\tilde{\pi}_4$	$\frac{2}{(q-1)^2(q+1)}$	0	$\frac{2}{(q-1)^2}$	$\frac{2}{(q-1)^2}$	$\frac{-2}{q^2-1}$	$\frac{-2}{q^2-1}$

The idempotent $\tilde{\pi}_1$ (or equivalently the idempotent π_1 of $\mathcal{SC}$) is only involved in the trivial representation with multiplicity 1. The idempotent $\tilde{\pi}_4$ appears (with multiplicity q) only in the unique non-trivial q-dimensional irreducible representation V involved in the permutation-representation of $\mathrm{GL}_2(\mathbb{F}_q)$ acting on all $q+1$ points of the projective line over $\mathbb{F}_q$.

The character $V_L = V \otimes \alpha_L$ where α_L is the quadratic character of $\mathbb{F}_q$ given by the Legendre-symbol has mean-values given by

	$q-1$	$q-1$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)^2}{2}$
	a_x	b_x	$c_{x,-x}$	$c_{x,y\neq\pm x}$	$d_{\xi,\mathrm{tr}(\xi)=0}$	$d_{\xi,\mathrm{tr}(\xi)\neq 0}$
V_L	q	0	$\left(\frac{-1}{q}\right)$	$-\frac{1+\left(\frac{-1}{q}\right)}{q-3}$	$\left(\frac{-1}{q}\right)$	$\frac{1-\left(\frac{-1}{q}\right)}{q-1}$

on conjugacy-classes. The character V_L involves thus $\tilde{\pi}_2$ (or π_2) with multiplicity q if $q \equiv 1 \pmod 4$ and $\tilde{\pi}_3$ (or π_3) with multiplicity q if $q \equiv 3 \pmod 4$.

Mean-values of a character $W_{\alpha,\beta}$ with non-real $\beta = \overline{\alpha}$ on conjugacy classes are given by

	$q-1$	$q-1$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)^2}{2}$
	a_x	b_x	$c_{x,-x}$	$c_{x,y\neq\pm x}$	$d_{\xi,\mathrm{tr}(\xi)=0}$	$d_{\xi,\mathrm{tr}(\xi)\neq 0}$
$W_{\alpha,\beta}$	$q+1$	1	$2\alpha(-1)$	$\frac{-2(1+\alpha(-1))}{q-3}$	0	0

They depend on the value $\alpha(-1) \in \{\pm 1\}$. For $q \equiv 1 \pmod 4$ there are $\frac{q-5}{4}$ such characters $W_{\alpha,\beta}$ with $\alpha(-1) = 1$ and $\frac{q-1}{4}$ such characters with $\alpha(-1) = -1$. For $q \equiv 3 \pmod 4$, there are the same number $\frac{q-3}{4}$ of such characters for both possible values of $\alpha(-1)$. Such representations involve $\tilde{\pi}_2$ (or π_2 of $\mathcal{SC}$) with multiplicity $q+1$ if $\alpha(-1) = 1$ and $\tilde{\pi}_3$ with multiplicity $q+1$ otherwise.

We consider now a non-real character φ of $\mathbb{F}_{q^2}^*$ with trivial restriction to $\mathbb{F}_q^*$. Since ξ^2 belongs to $\mathbb{F}_q^*$ if ξ has trace 0, we have $\sigma = \varphi(\xi) \in \{\pm 1\}$.

Mean-values on conjugacy-classes of a character X_φ with φ as above are thus given by

	$q-1$	$q-1$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)^2}{2}$
	a_x	b_x	$c_{x,-x}$	$c_{x,y\neq\pm x}$	$d_{\xi,\mathrm{tr}(\xi)=0}$	$d_{\xi,\mathrm{tr}(\xi)\neq 0}$
X_φ	$q-1$	-1	0	0	-2σ	$\frac{2(1+\sigma)}{q-1}$

For $q \equiv 1 \pmod 4$, there are $\frac{q-1}{4}$ such characters for both possible values of σ. For $q \equiv 3 \pmod 4$, the value $\sigma = 1$ is achieved by $\frac{q-3}{4}$ such representations and the value $\sigma = -1$ by $\frac{q+1}{4}$ representations.

Each such representation with $\sigma = 1$ involves $\tilde{\pi}_3$ (or π_3) with multiplicity $q-1$ and each such representation with $\sigma = -1$ involves $\tilde{\pi}_2$ (or π_2) with multiplicity $q-1$.

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