The s-bipartite Ramsey numbers involving $K_{2,3}$ and $K_{3,3}$

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Abstract. A *complete bipartite graph* with the number of two partitions s and t is denoted by $K_{s,t}$. For a positive integer s and two bipartite graphs G and H, the s-bipartite Ramsey number $BR_s(G, H)$ of G and H is the smallest integer t such that every 2-coloring of the edges of $K_{s,t}$ contains the a copy of G with the first color or a copy of H with the second color. In this paper, by using an integer linear program and the solver Gurobi Optimizer 8.0, we determine all the exact values of $BR_s(K_{2,3}, K_{3,3})$ for all possible s. More precisely, we show that $BR_s(K_{2,3}, K_{3,3}) = 13$ for $s \in \{8, 9\}$, $BR_s(K_{2,3}, K_{3,3}) = 12$ for $s \in \{10, 11\}$, $BR_s(K_{2,3}, K_{3,3}) = 10$ for $s = 12$, $BR_s(K_{2,3}, K_{3,3}) = 8$ for $s \in \{13, 14\}$, $BR_s(K_{2,3}, K_{3,3}) = 6$ for $s \in \{15, 16, \cdots, 20\}$, and $BR_s(K_{2,3}, K_{3,3}) = 4$ for $s \geq 21$. This extends the results presented in [Zhenming Bi, Drake Olejniczak and Ping Zhang, "The s-Bipartite Ramsey Numbers of Graphs $K_{2,3}$ and $K_{3,3}$", Journal of Combinatorial Mathematics and Combinatorial Computing 106, (2018) 257–272].

1 Introduction

In this paper, we shall only consider graphs without multiple edges or loops. For a graph $G = (V, E)$, we denote by $V(G)$ and $E(G)$ the vertex set and

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edge set of G, respectively. A *complete bipartite graph* with the number of two partitions m and n is denoted by $K_{m,n}$. Please consult [1] for more notion and notation of graph theory.

For a positive integer s and two bipartite graphs G and H, the s-bipartite Ramsey number $BR_s(G,H)$ of G and H is the smallest integer t such that every 2-edge coloring of the edges of $K_{s,t}$ contains the a copy of G with the first color or a copy of H with the second color. For convenience, we use the $\{1,2\}$ to color the edges of $K_{s,t}$. Given two 2-edge colorings f_1, f_2 of a bipartite graph $K_{m,n}$ with partitions A and B, we say f_1 and f_2 are equivalent if there is a 1-1 mapping $g : V(K_{m,n}) \to V(K_{m,n})$ such that $g(A) = A$, $g(B) = B$ and $f_2(g(x)g(y)) = f_1(xy)$ for any $xy \in E(K_{m,n})$. A 2-edge colored graph F of $K_{s,t}$ is called a $(G,H;s;t)$-graph if neither F contains neither G with color 1 nor H with color 2. The set of all inequivalent $(G,H;s;t)$-graphs is denoted by $\mathcal{R}(G,H;s;t)$.

In [2], some exact values of two color s-bipartite Ramsey numbers for $K_{2,3}$ and $K_{3,3}$ were obtained. More precisely, they show that $BR_s(K_{2,3}, K_{3,3}) = 21$ for $s = 4,5$; $BR_s(K_{2,3}, K_{3,3}) = 15$ for $s = 6,7$; $BR_s(K_{2,3}, K_{3,3}) \in \{13,14\}$ for $s = 8,9$ and $BR_s(K_{2,3}, K_{3,3}) \leq 14$ for $8 \leq s \leq BR(K_{2,3}, K_{3,3})$. For more information on s-bipartite Ramsey numbers, please consult [3, 4].

In this paper, we will continue the study the values of two color s-bipartite Ramsey numbers for $K_{2,3}$ and $K_{3,3}$. We show that $BR_s(K_{2,3}, K_{3,3}) = 13$ for $s \in \{8,9\}$, $BR_s(K_{2,3}, K_{3,3}) = 12$ for $s \in \{10,11\}$, $BR_s(K_{2,3}, K_{3,3}) = 10$ for $s = 12$, $BR_s(K_{2,3}, K_{3,3}) = 8$ for $s \in \{13,14\}$, and $BR_s(K_{2,3}, K_{3,3}) = 6$ for $s \in \{15,16,\cdots,20\}$, and $BR_s(K_{2,3}, K_{3,3}) = 4$ for $s \geq 21$.

2 The lower bounds

In combinatorial problems, many tools were used to search a graph with given property [5], including SAT testing [6, 7], integer programming [8], and constraint programming [9]. In this paper, we use an integer programming to establish the lower bounds for some s-bipartite Ramsey numbers.

For integers $s, t \geq 3$, we apply an integer linear programming to determine if there exists a 2-coloring of the edges of $K_{s,t}$ such that there exists no copies of $K_{2,3}$ with the first color and no copies of $K_{3,3}$ with the second color.

Assume the vertex set of $K_{s,t}$ is $A \cup B$, where $A = \{a_1, a_2, \cdots, a_s\}$, $B = \{b_1, b_2, \cdots, b_t\}$, and A and B are both independent sets. For each edge $\{u,v\}$ ($u \in A, v \in B$) we introduce boolean variables $x_{u,v,c}$ ($1 \leq c \leq 2$) and let $x_{u,v,c} = 1$ if and only of $\{u,v\}$ is colored with c. Then for each

$uv \in E(K_{s,t})$ we have

$$x_{u,v,1} + x_{u,v,2} = 1. \tag{1}$$

Now, we enumerate all possible $K_{2,3}$s, and forbidden $K_{2,3}$ with color 1. So for each $K_{2,3}$ with vertices a_1, a_2, b_1, b_2, b_3, we have

$$x_{a_1,b_1,1} + x_{a_1,b_2,1} + x_{a_1,b_3,1} + x_{a_2,b_1,1} + x_{a_2,b_2,1} + x_{a_2,b_3,1} \leq 5. \tag{2}$$

Next, we enumerate all possible $K_{3,3}$s, and forbidden $K_{3,3}$ with color 2. So for each $K_{3,3}$ with vertex $a_1, a_2, a_3, b_1, b_2, b_3$, we have

$$\sum_{1 \leq i,j \leq 3} x_{a_i,b_j,2} \leq 8. \tag{3}$$

If we find a solution subject to (1), (2) and (3), then we find a lower bound for $BR_s(K_{2,3}, K_{3,3})$ and conclude that $BR_s(K_{2,3}, K_{3,3}) \geq t+1$. By using the solver Gurobi Optimizer 8.0 [10], we are able to find a $(K_{2,3}, K_{3,3}; s; t)$-graph within several seconds. We succeed to find the lower bounds for $BR_s(K_{2,3}, K_{3,3})$ for different s, which are presented in Figures 1-5. In order to represent a $(K_{2,3}, K_{3,3}; s; t)$-graph, we will use an adjacency matrix with $s+t$ rows and columns and entries 0,1,2. The graph induced by 1 contains no $K_{2,3}$ and the one induced by 2 contains no $K_{3,3}$. As an example, Figure 5 is the corresponding graph whose adjacency matrix is depicted in Figure 1. The graph depicted in Figure 5 contains no $K_{2,3}$ and its complements of $K_{11,11}$ contains no $K_{3,3}$.

Proposition 1 $BR_s(K_{2,3}, K_{3,3}) \geq 12$ *for* $s \in \{10, 11\}$.

Proof. Figure 1 presents a $(K_{2,3}, K_{3,3}; 11; 11)$-graph, which implies that $BR_s(K_{2,3}, K_{3,3}) \geq 12$ for $s = 11$. By the definition of s-bipartite Ramsey number, we have $BR_{10}(K_{2,3}, K_{3,3}) \geq BR_{11}(K_{2,3}, K_{3,3})$. Therefore, Proposition 1 holds. □

Proposition 2 $BR_s(K_{2,3}, K_{3,3}) \geq 10$ *for* $s = 12$.

Proof. Figure 2 presents a $(K_{2,3}, K_{3,3}; 12; 9)$-graph, which implies that $BR_s(K_{2,3}, K_{3,3}) \geq 10$ for $s = 12$. □

$$\begin{bmatrix}
0000000000022212211121\\
0000000000022122111212\\
0000000000012212122112\\
0000000000011222221211\\
0000000000012121221122\\
0000000000012221112221\\
0000000000022111222211\\
0000000000021211121222\\
0000000000011112212222\\
0000000000021122122121\\
0000000000021221212112\\
2211112212200000000000\\
2221222111100000000000\\
2122121211200000000000\\
1212221112200000000000\\
2222111122100000000000\\
2112212121200000000000\\
1122212212100000000000\\
1121122122200000000000\\
1212122221100000000000\\
2111221222100000000000\\
1221211221200000000000
\end{bmatrix}.$$

Figure 1: An adjacency matrix of a $(K_{2,3}, K_{3,3}; 11; 11)$-graph.

$$\begin{bmatrix}
000000000000212112212\\
000000000000221212221\\
000000000000212211122\\
000000000000121221122\\
000000000000122212112\\
000000000000222122121\\
000000000000122111222\\
000000000000211222112\\
000000000000121122212\\
000000000000112222221\\
000000000000211121222\\
000000000000222221211\\
222112121122000000000\\
121222212112000000000\\
212122211212000000000\\
122221121212000000000\\
111212122222000000000\\
221122122211000000000\\
221111212222000000000\\
122212211221000000000\\
212221222121000000000
\end{bmatrix}.$$

Figure 2: An adjacency matrix of a $(K_{2,3}, K_{3,3}; 12; 9)$-graph.

Proposition 3 *$BR_s(K_{2,3}, K_{3,3}) \geq 8$ for $s \in \{13, 14\}$.*

Proof. Figure 3 presents a $(K_{2,3}, K_{3,3}; 14; 7)$-graph, which implies that $BR_s(K_{2,3}, K_{3,3}) \geq 8$ for $s = 14$. By the definition of s-bipartite Ramsey number, we have $BR_{13}(K_{2,3}, K_{3,3}) \geq BR_{14}(K_{2,3}, K_{3,3})$. Therefore, Proposition 3 holds. □

$$
\begin{bmatrix}
000000000000002211221 \\
000000000000001211222 \\
000000000000001122221 \\
000000000000002212121 \\
000000000000001222121 \\
000000000000001222112 \\
000000000000001212212 \\
000000000000002221112 \\
000000000000001121222 \\
000000000000002221211 \\
000000000000002121122 \\
000000000000002112122 \\
000000000000002112212 \\
000000000000002122211 \\
211211121222220000000 \\
221222221211110000000 \\
112122122221120000000 \\
112222211112220000000 \\
222111212211220000000 \\
222221112122110000000 \\
121112222122210000000
\end{bmatrix}.
$$

Figure 3: An adjacency matrix of a $(K_{2,3}, K_{3,3}; 14; 7)$-graph.

Proposition 4 *$BR_s(K_{2,3}, K_{3,3}) \geq 6$ for $s \in \{15, 16, \cdots, 20\}$.*

Proof. Figure 4 presents a $(K_{2,3}, K_{3,3}; 20; 5)$-graph, which implies that $BR_s(K_{2,3}, K_{3,3}) \geq 6$ for $s = 20$. By the definition of s-bipartite Ramsey number, we have $BR_i(K_{2,3}, K_{3,3}) \geq BR_{i+1}(K_{2,3}, K_{3,3})$ for $i \in \{15, 16, \cdots, 19\}$. Therefore, Proposition 3 holds. □

Proposition 5 *$BR_s(K_{2,3}, K_{3,3}) \geq 4$ for any $s \geq 21$.*

Proof. Let $K_{s,3}$ have vertex partitions $A = \{a_1, a_2, \cdots, a_s\}$ and $B = \{b_1, b_2, b_3\}$. Now we consider a 2-edge coloring f with $f(a_ib_1) = 1$ and $fa_ib_2 = f(a_ib_2) = 2$ for any i. Then f induces a $(K_{2,3}, K_{3,3}; s; 3)$-graph and thus $BR_s(K_{2,3}, K_{3,3}) \geq 4$. □

$$
\begin{bmatrix}
0000000000000000000021122 \\
0000000000000000000012122 \\
0000000000000000000011222 \\
0000000000000000000011222 \\
0000000000000000000022121 \\
0000000000000000000022112 \\
0000000000000000000012221 \\
0000000000000000000012212 \\
0000000000000000000021212 \\
0000000000000000000022211 \\
0000000000000000000022211 \\
0000000000000000000021221 \\
0000000000000000000021212 \\
0000000000000000000022121 \\
0000000000000000000022112 \\
0000000000000000000021122 \\
0000000000000000000012221 \\
0000000000000000000012122 \\
0000000000000000000012212 \\
0000000000000000000021221 \\
2111221122222222111200000 \\
1211222212211221222100000 \\
1122112222222111212200000 \\
2222212111121212221200000 \\
2222121221112122122100000
\end{bmatrix}
$$

Figure 4: An adjacency matrix of a $(K_{2,3}, K_{3,3}; 20; 5)$-graph.

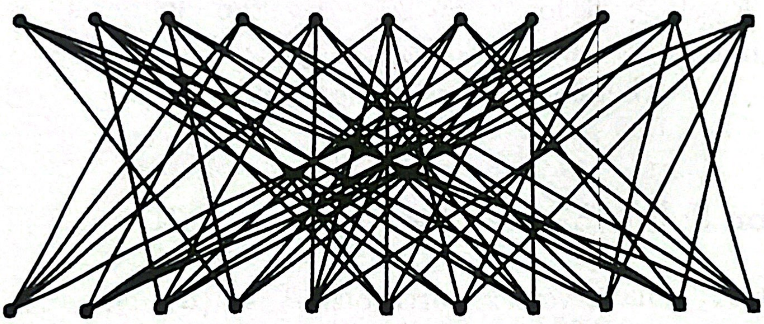

Figure 5: The $(K_{2,3}, K_{3,3}; 11; 11)$-graph depicted in Figure 1.

3 The upper bounds

The equivalence of two such colorings can be tested by using tool *nauty* [11] on two constructed graphs. Since the solver Gurobi 8.0 failed to exhaustively search all the 2-edge coloring of $K_{s,t}$ for the desired values of s and t, we can not confirm the corresponding upper bounds via this approach. However, we succeed to use the tool *nauty* to test all inequivalent of 2-edge colorings of complete bipartite graphs up to 21 vertices.

By the definition of $BR_s(K_{2,3}, K_{3,3})$, we will extend $\mathcal{R}(K_{2,3}, K_{3,3}; s; t)$ to $\mathcal{R}(K_{2,3}, K_{3,3}; s; t+1)$ by increasing t until $\mathcal{R}\sqcup(K_{2,3}, K_{3,3}; s; t+1) = \emptyset$. The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; s; t)|$ for some $4 \leq s \leq 21$ are presented in Tables 1-13. It can be seen that the s-bipartite Ramsey numbers for $s \in \{4, 5, 6, 7\}$ are the same as known results in [2], which supports the correctness of this statistics. From the results of Tables 5-13, we have

Theorem 1 *(i)* $BR_s(K_{2,3}, K_{3,3}) \leq 13$ *for* $s \in \{8, 9\}$;
(ii) $BR_s(K_{2,3}, K_{3,3}) \leq 12$ *for* $s \in \{10, 11\}$;
(iii) $BR_s(K_{2,3}, K_{3,3}) \leq 10$ *for* $s \in \{12\}$;
(iv) $BR_s(K_{2,3}, K_{3,3}) \leq 8$ *for* $s \in \{13, 14\}$;
(v) $BR_s(K_{2,3}, K_{3,3}) \leq 6$ *for* $s \in \{15, 16, \cdots, 20\}$;
(vi) $BR_s(K_{2,3}, K_{3,3}) \leq 4$ *for* $s \geq 21$.

Table 1: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 4; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 4; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 4; t)\|$
1	5	2	19
3	60	4	167
5	378	6	713
7	1102	8	1451
9	1595	10	1518
11	1241	12	890
13	549	14	304
15	143	16	62
17	23	18	8
19	2	20	1
21	0		

Table 2: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 5; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 5; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 5; t)\|$
1	6	2	27
3	106	4	378
5	1173	6	2876
7	5352	8	7599
9	8073	10	6578
11	4197	12	2176
13	922	14	350
15	115	16	38
17	12	18	4
19	1	20	1
21	0		

Table 3: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 6; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 6; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 6; t)\|$
1	7	2	37
3	171	4	713
5	2876	6	9157
7	20605	8	30587
9	28027	10	15080
11	4532	12	702
13	43	14	4
15	0		

Table 4: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 7; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 7; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 7; t)\|$
1	7	2	37
3	171	4	713
5	2876	6	9157
7	20605	8	30587
9	28027	10	15080
11	4532	12	702
13	43	14	4
15	0		

Table 5: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 8; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 8; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 8; t)\|$
1	9	2	61
3	352	4	1451
5	7599	6	30587
7	77837	8	93272
9	42344	10	5590
11	342	12	14
13	0		

Table 6: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 9; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 9; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 9; t)\|$
1	10	2	75
3	468	4	1595
5	8073	6	28027
7	56743	8	42344
9	8615	10	223
11	10	12	3
13	0		

Table 7: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 10; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 10; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 10; t)\|$
1	11	2	91
3	603	4	1518
5	6578	6	15080
7	18511	8	5590
9	223	10	8
11	1	12	0

Table 8: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 11; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 11; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 11; t)\|$
1	12	2	108
3	754	4	1241
5	4197	6	4532
7	2724	8	342
9	10	10	1
11	1	12	0

Table 9: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 12; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 12; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 12; t)\|$
1	13	2	127
3	924	4	890
5	2176	6	702
7	162	8	14
9	3	10	0

Table 10: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 13; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 13; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 13; t)\|$
1	14	2	147
3	1110	4	549
5	922	6	43
7	2	8	0

Table 11: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 14; t)|$

t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 14; t)\|$	t	$\|\mathcal{R}(K_{2,3}, K_{3,3}; 14; t)\|$
1	15	2	169
3	1315	4	304
5	350	6	4
7	1	8	0

Table 12: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 15; t)|$

t	$|\mathcal{R}(K_{2,3}, K_{3,3}; 15; t)|$	t	$|\mathcal{R}(K_{2,3}, K_{3,3}; 15; t)|$
1	16	2	192
3	1536	4	143
5	115	6	0

Table 13: The statistics of $|\mathcal{R}(K_{2,3}, K_{3,3}; 21; t)|$

t	$|\mathcal{R}(K_{2,3}, K_{3,3}; 21; t)|$	t	$|\mathcal{R}(K_{2,3}, K_{3,3}; 21; t)|$
1	22	2	363
3	3234	4	0

Table 14: Exact values of $BR_s(K_{2,3}, K_{3,3})$

$BR_s(K_{2,3}, K_{3,3})$	condition	Ref.
21	$s \in \{4, 5\}$	[2]
15	$s \in \{6, 7\}$	[2]
13	$s \in \{8, 9\}$	this work
12	$s \in \{10, 11\}$	this work
10	$s = 12$	this work
8	$s \in \{13, 14\}$	this work
6	$s \in \{15, 16, \cdots, 20\}$	this work
4	$s \geq 21$	this work

4 Conclusion

In [2], Bi et al. initial the study of the s-bipartite Ramsey numbers for $K_{2,3}$ and $K_{3,3}$. But they only solved the case for $s \leq 7$. In this paper, we use computational technique to obtain all the cases for this these families of s-bipartite Ramsey numbers and the results are summarized in Table 1. Since finding the Ramsey number is a hard problem, determining the exact values of $BR_s(K_{3,3}, K_{3,3})$ is an interesting and challenging avenue of further research.

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