Bounds for some generalized vertex Folkman numbers

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Abstract. For a graph G and positive integers $a_1, \cdots, a_r$, if every r-coloring of vertices in $V(G)$ must result in a monochromatic a_i-clique of color i for some $i \in \{1, \cdots, r\}$, then we write $G \rightarrow (a_1, \cdots, a_r)^v$. $F_v(K_{a_1}, \cdots, K_{a_r}; H)$ is the smallest integer n such that there is an H-free graph G of order n, and $G \rightarrow (a_1, \cdots, a_r)^v$. In this paper we study upper and lower bounds for some generalized vertex Folkman numbers of form $F_v(K_{a_1}, \cdots, K_{a_r}; K_4 - e)$, where $r \in \{2, 3\}$ and $a_i \in \{2, 3\}$ for any $i \in \{1, \cdots, r\}$. We prove that $F_v(K_2, K_2, K_2; K_4 - e) = 10$ and $F_v(K_2, K_3; K_4 - e) = 19$ by computing, and prove $F_v(K_3, K_3; K_4 - e) \geq F_v(K_2, K_2, K_3; K_4 - e) \geq 25$.

1 Introduction

All graphs considered in this paper are finite and undirected graphs. For any positive integer t, the complete graph of order t is denoted by K_t. The graph obtained by deleting one edge from K_t is denoted by $K_t - e$ for integer $t \geq 3$. Note that $K_3 - e$ is same to P_3, the path of order 3.

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Suppose that G is a graph and integer $r \geq 2$. For positive integers $a_1, \cdots, a_r$, if every r-coloring of the vertices in $V(G)$ must result in a monochromatic a_i-clique of color i for some $i \in \{1, \cdots, r\}$, then we write $G \rightarrow (a_1, \cdots, a_r)^v$. The generalized vertex Folkman number $F_v(K_{a_1}, \cdots, K_{a_r}; H)$ is the smallest positive integer n that there is an H-free graph G of order n such that $G \rightarrow (a_1, \cdots, a_r)^v$. When H is K_t, $F_v(K_{a_1}, \cdots, K_{a_r}; H)$ is denoted by $F_v(a_1, \cdots, a_r; t)$, where $t > a_i$ holds for any $i \in \{1, \cdots, r\}$. The generalized edge Folkman numbers $F_e(K_{a_1}, \cdots, K_{a_r}; H)$ and $F_e(a_1, \cdots, a_r; t)$ can be defined similarly. A $(G_1, G_2)^v$-coloring of G is a red-blue coloring of $V(G)$ in which there is neither a red G_1 nor a blue G_2. We can define $G \rightarrow (G_1, \cdots, G_r)^v$, $F_v(G_1, \cdots, G_r; H)$ and $(G_1, \cdots, G_r)^v$-coloring similarly.

The generalized Ramsey number $R(G, H)$ is the smallest positive integer n such that $K_n \rightarrow (G, H)^e$. Many known results on the exact values and bounds for small Ramsey numbers can be found in [7].

In 1970, Folkman [2] proved that for positive integers k and $a_1, \cdots, a_r$, $F_v(a_1, \cdots, a_r; k)$ ($F_e(a_1, a_2; k)$) exists if and only if $k > \max\{a_1, \cdots, a_r\}$ ($k > \max\{a_1, a_2\}$). For edge Folkman numbers, Folkman's method works only for two colors. The existence of $F_e(a_1, \cdots, a_r; k)$ was proved in [5] (also see [3]). The vertex Folkman numbers were studied in particular by Dudek and Rödl [1] and Hàn, Rödl and Szabó [4]. The latter work contains $F_v(s, s; s+1) \leq Cs^2 \log^2 s$ as a special case.

In this paper we study the lower and upper bounds for some generalized vertex Folkman numbers of form $F_v(K_{a_1}, \cdots, K_{a_r}; K_4 - e)$, where $r \in \{2, 3, 4\}$ and $a_i \in \{2, 3\}$ for any $i \in \{1, \cdots, r\}$.

The remaining parts of this paper are organized as follows. In Section 2, some inequalities on $F_v(K_3, K_3; K_4 - e)$ are proved. $F_v(K_2, K_2, K_2; K_4 - e) = 10$ is proved in Section 3. In Section 4, we obtain exact values and bounds for some vertex Folkman numbers, including $F_v(K_2, K_3; K_4 - e) = 19$, and $F_v(K_3, K_3; K_4 - e) \geq 25$.

2 Some inequalities on $F_v(K_3, K_3; K_4 - e)$

In [8] it was prove that $F_e(K_{t+1}, K_{t+1}; K_{t+2} - e)$ and $F_v(K_t, K_t; K_{t+1} - e)$ are finite for any integer $t \geq 3$, based a theorem in [6]. In fact, the theorem in [6] implies the following theorem on vertex Folkman numbers.

Theorem 2.1 *Suppose that integer $t \geq 3$. There is a $K_{t+1} - e$-free graph G such that $G \rightarrow (t, t)^v$, where for any pair t-cliques U_1 and U_2 in $V(G)$, U_1 and U_2 share at most one common vertex.*

When $t = 3$, Theorem 2.1 is same to that $F_v(K_3, K_3; K_4 - e)$ is finite. But for $t > 3$, Theorem 2.1 is stronger than that $F_v(K_t, K_t; K_{t+1} - e)$ is finite.

Let $\mathcal{F}_v(K_{a_1}, \cdots, K_{a_r}; H; m)$ denote the set of all H-free graphs of order m that arrow $(a_1, \cdots, a_r)^v$.

We can see that $F_v(K_2, K_2, K_2; K_4 - e) \leq F_v(K_2, K_3; K_4 - e)$ and

$$F_v(K_2, K_2, K_2, K_2; K_4-e) \leq F_v(K_2, K_2, K_3; K_4-e) \leq F_v(K_3, K_3; K_4-e).$$

We know that $R(K_4-e, K_6) = 21$. Let us prove the following theorem.

Theorem 2.2 *If $F_v(K_2, K_2, K_3; K_4 - e) \geq 21$, then*

$$F_v(K_3, K_3; K_4 - e) \geq F_v(K_2, K_2, K_3; K_4 - e) \geq F_v(K_2, K_3; K_4 - e) + 6.$$

Proof. Let $n = F_v(K_2, K_2, K_3; K_4 - e)$ and $G \in \mathcal{F}_v(K_2, K_2, K_3; K_4 - e; n)$. If $n \geq 21$, then by $R(K_4 - e, K_6) = 21$ we know that $\alpha(G) \geq 6$. Suppose that V_1 is a 6-independent set in $V(G)$. Therefore $G - V_1 \to (K_2, K_3)^v$ and $|V(G) - V_1| \geq F_v(K_2, K_3; K_4 - e)$ because that $G - V_1$ is $K_4 - e$-free. So $F_v(K_3, K_3; K_4 - e) \geq F_v(K_2, K_2, K_3; K_4 - e) \geq F_v(K_2, K_3; K_4 - e) + 6$. Hence we have proved the theorem. □

Although we know that $F_v(K_3, K_3; K_4 - e)$ is finite, no interesting bounds for it are known. It is obvious that $F_v(K_3, K_3; K_4 - e) = n \geq F_v(3, 3; 4) = 14$. We can only obtain an upper bound for n based on [6] as discussed in [8]. The upper bound obtained this way is very large, because that we have to consider a $K_5 - e$-free graph that arrows $(K_4, K_4)^e$, and consider the subgraph induced by the neighbors of a vertex in this graph. Although n may be much larger than 14, it may be much smaller than the upper bound obtained by this method.

Let $G \to (K_3, K_3; K_4 - e)^v$, and v be any vertex in $V(G)$. Suppose that $|V(G)| = n = F_v(K_3, K_3; K_4 - e)$. We may suppose that the subgraph of G induced by the neighbors of v is a perfect match. In fact, if G is not such a graph, we can obtain such a graph by deleting some edges from G if necessary. For any edge deleted, say uv, v is a neighbor of u, and u and v have no common neighbors.

If $G \to (K_3, K_3; K_4 - e)^v$, then we can prove that the subgraph of G induced by all the non-neighbors of v, say H, arrows $(K_2, K_3)^v$.

Theorem 2.3 *Let integer $t \geq 3$ and G be $K_{t+1} - e$-free. Suppose that $G \to (K_t, K_t)^v$. If v is any vertex in $V(G)$, and H is the subgraph of G induced by all the non-neighbors of v, then $H \to (K_2, K_t)^v$.*

Proof. Let V_1 be any independent set in the $V(H)$. The subgraph of G induced by the joint of V_1 and the set of all neighbors of v is K_t-free. In fact, if there is a K_t in such an induced subgraph of G, it must contain one non-neighbor of v and $t-1$ adjacent neighbors of v. But this can not be true because that G is $K_{t+1}-e$-free and such a K_t together with v is isomorphic to $K_{t+1}-e$. Therefore by $G \to (K_t, K_t)^v$ we know that the subgraph of H obtained by deleting any independent set must contain K_t. Hence $H \to (K_2, K_t)^v$. □

The case $t = 3$ in Theorem 2.3 is useful in this paper. If $|V(G)| = F_v(K_3, K_3; K_4 - e)$, then we may suppose that $\delta(G) \geq 4$. Therefore by Theorem 2.3 we have $n \geq 5 + F_v(K_2, K_3; K_4 - e)$. If n is not small, then we can obtain better lower bound by $n \geq \alpha(G) + F_v(K_2, K_3; K_4 - e)$. Similarly, we have the following theorem.

Theorem 2.4 *Suppose that G is $K_4 - e$-free and $G \to (K_2, K_3)^v$. If v is any vertex in $V(G)$, and H is the subgraph of G induced by all the non-neighbors of v, then $H \to (K_2, K_2)^v$.*

We can also generalize Theorem 2.3 to multicolor cases. The following theorem is an interesting special case.

Theorem 2.5 *Suppose that G is $K_4 - e$-free and $G \to (K_2, K_2, K_3)^v$. If v is any vertex in $V(G)$, and H is the subgraph of G induced by all the non-neighbors of v, then $H \to (K_2, K_2, K_2)^v$.*

3 The exact value of $F_v(K_2, K_2, K_2; K_4 - e)$

By $R(K_4 - e, K_4) = 11$, there is a $K_4 - e$-free graph G of order 10, of which the independence number $\alpha(G) \leq 3$. Hence $G \to (K_2, K_2, K_2)^v$, and $F_v(K_2, K_2, K_2; K_4 - e) \leq 10$. On the other hand, by computing we have proved that for any $K_4 - e$-free graph G of order 9, the chromatic number of G is no larger than 3. Therefore $F_v(K_2, K_2, K_2; K_4 - e) \geq 10$. So we have the following result.

Theorem 3.1 $F_v(K_2, K_2, K_2; K_4 - e) = 10$.

Similar to Theorem 2.2, we can see that $F_v(K_2, K_2, K_2, K_2; K_4 - e) \geq F_v(K_2, K_2, K_2; K_4 - e) + 4$, because that $R(K_4 - e, K_4) = 11$. Therefore $F_v(K_2, K_2, K_2, K_2; K_4 - e) \geq 14$. Note that $R(K_4 - e, K_5) = 16$. If there is a $K_4 - e$-free graph G that arrows $(2, 2, 2, 2)^v$, and $|V(G)| = 15$, then $\alpha(G) \leq 5$. On the other hand, $F_v(K_2, K_2, K_2, K_2; K_4 - e) \leq 22 = F_v(2, 2, 2, 2; 3)$.

4 Computing $F_v(K_2, K_3; K_4 - e)$

Let us consider the upper bound for $F_v(K_2, K_3; K_4 - e)$. We know that Exoo proved $R(K_4 - e, K_{10} - e) \geq 41$ based on a $K_4 - e$-free graph of order 40. We prove $F_v(K_2, K_3; K_4 - e) \leq 19$ based on an induced subgraph of this graph of order 40. The adjacent matrix of the graph of order 19 used is shown in Fig. 1.

0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	1	0	1	1
0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1
0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	1	1
0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0
1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0
0	1	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0
0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0
1	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0
1	1	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0
0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0
1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0
0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	1	0
0	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	1
0	0	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0
1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	1	1	0
0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0	1
1	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0

Figure 1: Adjacency matrix of a $(K_2, K_3; K_4 - e; 19)^v$ graph

For the lower bound on $F_v(K_2, K_3; K_4{-}e)$, we know that $F_v(K_2, K_3; K_4 - e) \geq F_v(K_2, K_2, K_2; K_4 - e) = 10$. This lower bound is weak.

Suppose that graph G is a $K_4 - e$-free graph such that $G \rightarrow (K_2, K_3)^v$. It is not difficult to see that for any vertex $v \in V(G)$, the subgraph induced by the non-neighbors of v can not be an independent set. Otherwise, if we color v and all its non-neighbors in red, and color all neighbors of v in blue, then we obtain a $(K_2, K_3)^v$-coloring of $V(G)$, which contradicts with $G \rightarrow (K_2, K_3)^v$. Hence for any vertex $v \in V(G)$, v has at least two non-neighbors.

Furthermore, if $G - v \nrightarrow (K_2, K_3)^v$ for any $v \in V(G)$, then $\delta(G) \geq 3$. Otherwise, suppose that $f_1 : V(G) - \{v\} \rightarrow \{red, blue\}$ is a $(K_2, K_3)^v$-coloring of $V(G) - \{v\}$ for a vertex $v \in V(G)$, and $d(v) \leq 2$, then we can obtain a $(K_2, K_3)^v$-coloring of $V(G)$ based on f_1.

Let G be a $K_4 - e$-free graph of order n, and $V(G) = \{v_1, \cdots, v_n\}$. Suppose that $V(G) = \bigcup_{i=0}^{4} V_i$, where $V_0 = \{v_1, v_2, v_3\}$, and all vertices in V_i are neighbors of v_i for any $i \in \{1, 2, 3\}$, and any vertex in V_4 has no neighbors in V_0. If $n = F_v(K_2, K_3; K_4 - e)$ and $G \to (K_2, K_3)^v$, then by the discussion above we know that $\delta(G) \geq 3$. We can see there is 3-clique in $V(G)$, otherwise we can color all vertices in $V(G)$ with color blue, which contradicts with $G \to (K_2, K_3)^v$.

We may study the lower bound for $F_v(K_2, K_3; K_4 - e)$ based on $R(K_4 - e, K_5) = 16$. Let G be a $K_4 - e$-free graph on 18 vertices such that $G \to (K_2, K_3)^v$. By $R(K_4 - e, K_5) = 16$ we know there must be a 5-independent set V_1 in $V(G)$. We can see that there is a 3-clique in $V(G) - V_1$. It is to say, if G is a $K_4 - e$-free graph on 18 vertices such that $G \to (K_2, K_3)^v$, we may suppose that U_1 is a 3-clique in $V(G)$, and V_1 is a 5-independent set $V(G) - U_1$. Hence we can generate all possible $K_4 - e$-free graphs on 15 vertices that contain 5-independent set, and generate possible G of these properties as discussed above.

We have checked all such $K_4 - e$-free graph of order 18, and found that none of them arrows $(K_2, K_3)^v$. We use Theorem 2.4 when necessary. Hence $F_v(K_2, K_3; K_4 - e) > 18$. By $F_v(K_2, K_3; K_4 - e) \leq 19$ we have $F_v(K_2, K_3; K_4 - e) = 19$.

Theorem 4.1 $F_v(K_2, K_3; K_4 - e) = 19$.

We know that $R(K_4 - e, K_6) = 21$. So by Theorem 4.1 we have

$$F_v(K_2, K_2, K_3; K_4 - e) \geq 6 + 19 = 25.$$

Hence

$$F_v(K_3, K_3; K_4 - e) \geq 25.$$

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