Energy and the Zagreb index conditions for nearly balanced bipartite graphs to be traceable

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Abstract: The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. The first Zagreb index of a graph is defined as the sum of squares of the degrees of the vertices of the graph. The second Zagreb index of a graph is defined as the sum of products of the degrees of a pairs of the adjacent vertices of the graph. In this paper, we establish some sufficient conditions for a nearly balanced bipartite graph with large minimum degree to be traceable in terms of the energy, the first Zagreb index and the second Zagreb index of the quasi-complement of the graph, respectively.

Keywords: Nearly balanced bipartite graph; Traceable; Energy; The first Zagreb index; The second Zagreb index

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $e(G) = |E(G)|$ the number of edges of the graph G, $N_G(v)$ the set of vertices which are adjacent to v in

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G. The degree of v is denoted by $d_G(v) = |N_G(v)|$ (or simply $d(v)$), the minimum degree of G is denoted by $\delta(G)$, the maximum degree of G is denoted by $\Delta(G)$. Let $G = (X, Y; E)$ be a bipartite graph with two part sets X, Y. If $|X| = |Y|$, $G = (X, Y; E)$ is called a *balance bipartite graph.* If $|X| = |Y| + 1$, $G = (X, Y; E)$ is called a *nearly balance bipartite graph.* For a bipartite graph $G = (X, Y; E)$, the *quasi-complement* of G is denoted by $G^* = (X, Y; E^*)$, where $E^* = \{xy : x \in X, y \in Y, xy \notin E\}$.

The *adjacency matrix* of G is defined to be a matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The degree matrix of G is denoted by $D(G) = \text{diag}\,(d_G(v_1), d_G(v_2), \dots, d_G(v_n))$. The matrix $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* (or *Q-matrix*) of G. Obviously, $A(G)$ and $Q(G)$ are real symmetric matrix. So their eigenvalues are real number and can be ordered. Let the eigenvalues of $A(G)$ be arranged as $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. The largest eigenvalue of $A(G)$, $\lambda_n(G)$, denoted by $\lambda(G)$, is called the *spectral radius* of G. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the *signless Laplacian spectral radius* of G. The energy of G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix, i.e. $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i(G)|$.

A graph invariant is a function on a graph that does not depend on a labeling of its vertices. Such quantities are also called topological indices. Among more useful of them appear three that are known under various names, but mostly as energy and Zagreb indices. The first Zagreb index $Z_1(G)$ of G is defined as the sum of squares of the degrees of the vertices of the graph G, i.e. $Z_1(G) = \sum_{u \in V(G)} d_G^2(u)$. The second Zagreb index $Z_2(G)$ of G is defined as the sum of products of the degrees of a pairs of the adjacent vertices of the graph G, i.e. $Z_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$.

A *Hamiltonian cycle* of the graph G is a cycle which contains all vertices in G, and a *Hamiltonian path* of G is a path which contains all vertices in G. The graph G is said to be *Hamiltonian* if it contain a Hamiltonian cycle, and is said to be *traceable* if it contain a Hamiltonian path. The problem of deciding whether a graph is Hamiltonian or traceable is one of the most difficult classical problems in graph theory.

Recently, some topological indices have been applied to this problem. We refer readers to see [4, 5, 6, 7, 9, 10, 11, 12, 13, 19, 23, 24, 36, 37]. Particularly, Li [23] presents energy sufficient conditions for a graph to be traceable, Hamiltonian, respectively; Li [24] obtains sufficient conditions for some stable properties of the graphs using energy and the first Zagreb index of the complement of a graph; Yu et al. [19] give sufficient conditions for a graph with large maximum degree to be traceable, Hamiltonian, Hamilton-connected in terms of the energy of the complement of the graph, respectively. Inspired by these studies, in this paper, we study the sufficient conditions for a nearly balanced bipartite graph $G = (X, Y; E)$ with large minimum degree to be traceable in terms of the energy, the first Zagreb index and the second Zagreb index of $G^* = (X, Y; E^*)$, respectively.

2 Preliminaries

The definition of the closure of a balanced bipartite graph can be found in [3]. For an integer $k \geq 0$, the k-closure of a balanced bipartite graph $G = (X, Y; E)$, denoted by $cl_k(G)$, is a graph obtained from $G = (X, Y; E)$ by successively joining pairs of nonadjacent vertices $x \in X$and $y \in Y$, whose degree sum is at least k until no such pairs remains. We note that $d_{cl_k(G)}(x) + d_{cl_k(G)}(y) \leq k-1$ for any pair of nonadjacent vertices $x \in X$, and $y \in Y$ of $cl_k(G)$.

We need the following results as lemmas to prove our theorems.

LEMMA 2.1 [1] *A balanced bipartite graph $G = (X, Y; E)$, where $|X| = |Y| = n$, is Hamiltonian if and only if $cl_{n+1}(G)$ is so.*

LEMMA 2.2 [2] *Let $G = (X, Y; E)$ be a bipartite graph, where $|X| = |Y| = n \geq 2$, with degree sequence $(d_1, d_2, \cdots, d_{2n})$, where $d_1 \leq d_2 \leq \cdots \leq d_{2n}$. If there is no integer $k \leq \frac{n}{2}$ such that $d_k \leq k$ and $d_n \leq n-k$. Then G is Hamiltonian.*

LEMMA 2.3 [14] *Let e be an edge in a graph G. Then*

$$\varepsilon(G) - 2 \leq \varepsilon(G - \{e\}) \leq \varepsilon(G) + 2,$$

where $G - \{e\}$ be obtained G by deleting the edge $e \in E(G)$, the left equality holds if and only if e is an isolated edge of G and equality on the right-hand side never holds.

LEMMA 2.4 [17] *Let G be a bipartite graph. Then*

$$\lambda(G) \leq \sqrt{e(G)}.$$

LEMMA 2.5 [18] *Let G be a graph of order n with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then*

$$\lambda^2(G) \geq \frac{1}{n} \sum_{i=1}^{n} d_i^2,$$

where equality holds if and only if G is regular graph or semi-regular bipartite.

Lemma 2.6 [28] *Let G be a graph with non-empty edge set. Then*

$$\lambda(G) \geq min\{\sqrt{d(u)d(v)} : uv \in E(G)\}.$$

Moreover, if G is connected, then equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.7 [32] *Let G be a graph with at least one edge. Then*

$$q(G) \geq \frac{Z_1(G)}{e(G)},$$

where equality holds if and only if G is regular or semi-regular bipartite.

Let M be a Hermitian matrix of order n, and let $\lambda_i(M)$ be the $i-$th largest eigenvalue of M, $1 \leq i \leq n$.

Lemma 2.8 [34] *Let B and C be Hermitian matrices of order n, and let $1 \leq i, j \leq n$. If $i + j \leq n + 1$, then*

$$\lambda_i(B) + \lambda_j(C) \geq \lambda_{i+j-1}(B + C).$$

Moreover, equality holds if and only if there exists a unit vector $\mathbf{x}$ *such that* $B\mathbf{x} = \lambda_i(B)\mathbf{x}, C\mathbf{x} = \lambda_j(C)\mathbf{x}$, *and* $(B + C)\mathbf{x} = \lambda_{i+j-1}(B + C)\mathbf{x}$.

Lemma 2.9 [35]*Let $G = (V, E)$ be a graph with n vertices. Denote by $d(v)$ the degree of $v \in V$ and by $m(v)$ the average of the degrees of the vertices of G adjacent to v. Then*

$$q(G) \leq max\{m(v) + d(v) : v \in V\}.$$

Lemma 2.10 *Let G be a graph with at least one edge. Then*

$$\lambda(G) \geq \frac{Z_1(G)}{e(G)} - \Delta(G),$$

where equality holds if and only if G is regular.

Proof. Since $Q(G) = A(G) + D(G)$, by Lemma 2.8, let $i = j = 1$, we have

$$\lambda_1(A(G)) + \lambda_1(D(G)) \geq \lambda_1(Q(G)).$$

Recalling that $\lambda_1(A(G)) = \lambda(G), \lambda_1(D(G)) = \Delta(G)$ and $\lambda_1(Q(G)) = q(G)$, then

$$\lambda(G) \geq q(G) - \Delta(G).$$

By Lemma 2.7,

$$\lambda(G) \geq \frac{Z_1(G)}{e(G)} - \Delta(G).$$

Moreover, equality holds if and only if G is regular or semi-regular bipartite and there exists a unit vector $\mathbf{x}$ such that $A(G)\mathbf{x} = \lambda(G)\mathbf{x}, D(G)\mathbf{x} = \Delta(G)\mathbf{x}$, and $Q(G)\mathbf{x} = q(G)\mathbf{x}$. This implies that $\lambda(G) + \Delta(G) = q(G)$.

(i) If G is regular with n vertices and m edges. Let any vertex $u \in V(G)$, and $d_G(u) = d$. By Lemma 2.5, $\lambda(G) = d$. Note that $\Delta(G) = d$, $Z_1(G) = nd^2$, $e = \frac{nd}{2}$, then we have $\lambda(G) = \frac{Z_1(G)}{e(G)} - \Delta(G)$. Namely, the equality can be hold.

(ii) If $G = (X, Y)$ is a semi-regular bipartite graph. Let any vertex $u \in X, d_G(u) = d_1$, any vertex $v \in Y, d_G(v) = d_2$, where $d_1 \neq d_2$.

Suppose that $F_1 = (X_1, Y_1)$ is a non-trivial connected component of G, which has largest eigenvalue of $A(G)$, then by Lemma 2.6, $\lambda(G) = \lambda(F_1) = \sqrt{d_1 d_2}$. Obvious, $\Delta(G) = max\{d_1, d_2\}$. Next we will consider $q(G)$.

Suppose that $F_2 = (X_2, Y_2)$ is a non-trivial connected component which has largest eigenvalue of $Q(G)$, and $d_1 > d_2$. By Lemma 2.9

$$q(G) \leq max\{m(v) + d(v) : v \in V(G)\} = d_1 + d_2.$$

Then by $\lambda(G) + \Delta(G) = q(G)$, we get

$$\sqrt{d_1 d_2} + max\{d_1, d_2\} = q(G),$$

thus

$$\sqrt{d_1 d_2} + d_1 = q(G) \leq d_1 + d_2,$$

hence

$$d_1 \leq d_2,$$

a contradiction.

So the result follows. ■

3 The main results

Let $G = (X, Y; E)$ be a nearly balanced bipartite graph with $|X| = |Y| + 1$, G' be obtained from G by adding a vertex v to Y which is adjacent to every vertices in X, then G' be a balanced bipartite graph. We note that if G' is Hamiltonian then G is traceable.

Denote $O_{m,n} = K^*_{m,n}$ a bipartite graph without edges. In this section, $O_{n,m}$ and $O_{m,n}$ are considered as different bipartite graphs, unless $m = n$. Let G_1, G_2 be two bipartite graphs, with the bipartitions $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between X_1 and Y_2. We use $G_1 \cup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between X_1 and Y_2, and all possible edges between X_2 and Y_1.

THEOREM 3.1 *Let $G = (X, Y; E)$, where $|X| = |Y| + 1 = r \geq k + 1$, be a nearly balanced bipartite graph of order $n = 2r - 1 \geq 3$ and $\delta(G) \geq k \geq 1$. G^* is the quasi-complement of G. Denote $\varepsilon(G^*)$ the energy of G^*. Then G is traceable,*

(i) if

$$\varepsilon(G^*) \geq \sqrt{e(G^*)}(\sqrt{2r-2} + \sqrt{2}) + 2e(G^*) - 2kr + 2k^2.$$

(ii) if

$$\varepsilon(G^*) \geq 2\sqrt{(r-1)(e(G^*) - (r - \Delta(G^*))^2)} + 2e(G^*) - 2\Delta(G^*) - 2(k-1)r + 2k^2$$

and $\Delta(G^*) \leq r - \sqrt{r}$.

(iii) if

$$\varepsilon(G^*) \geq 2\sqrt{(r-1)(e(G^*) - kr + k^2)} + 2e(G^*) - 2kr + 2k^2 + 2(\sqrt{k(r-k)}),$$

and $G \neq K_{r-k-1,r-k} \sqcup K_{k,k}$ $(r \geq 2k + 1)$.

Proof. Let $G' = (X, Y'; E')$ be obtained from G as mentioned above, then G' be a balanced bipartite graph. Suppose that G be not traceable. Then G' be not Hamiltonian. By Lemma 2.1, $H := cl_{r+1}(G')$ be not Hamiltonian too. Then we note that H is not $K_{r,r}$, and $d_H(u) + d_H(v) \leq r$ for any pair of nonadjacent vertices $u \in X$ and $v \in Y'$ (always existing) in H. Hence for any edge $uv \in E(H^*)$, we have that

$$d_{H^*}(u) + d_{H^*}(v) = r - d_H(u) + r - d_H(v) \geq 2r - r = r. \tag{3.1}$$

We notice that for any graph G of order n, $\sum_{i=1}^{n} \lambda_i^2(G)$ must be equal to the trace of $A^2(G)$, and the trace of $A^2(G)$ just equal $\sum_{i=1}^{n} d_G(v_i)$. So

$$\sum_{i=1}^{n+1} \lambda_i^2(H^*) = \sum_{i=1}^{n+1} d_{H^*}(v_i) = 2e(H^*).$$

Since H^* is a balanced bipartite graph, $\lambda(H^*) = \lambda_{n+1}(H^*) = -\lambda_1(H^*)$.

From the definition of $\varepsilon(H^*)$ and Cauchy-Schwartz inequality, we have that

$$\begin{aligned}\varepsilon(H^*)=\sum_{i=1}^{n+1}|\lambda_i(H^*)| &\leq \lambda(H^*)+|\lambda_1(H^*)|+\sqrt{(n-1)\sum_{i=2}^{n}\lambda_i^2(H^*)}\\ &= 2\lambda(H^*)+\sqrt{(n-1)(\sum_{i=1}^{n+1}\lambda_i^2(H^*)-2\lambda^2(H^*))}\\ &= 2\lambda(H^*)+\sqrt{(n-1)(2e(H^*)-2\lambda^2(H^*))} \qquad (3.2)\end{aligned}$$

the equality holds if and only if $\lambda_2(H^*)=\cdots=\lambda_n(H^*)$.

Let $f(x)=2x+\sqrt{(n-1)(2e(H^*)-2x^2)}$. If $\sqrt{\frac{2e(H^*)}{n+1}}\leq x\leq\sqrt{e(H^*)}$, we have

$$f'(x)=2+\frac{-2(n-1)x}{\sqrt{(n-1)(2e(H^*)-2x^2)}}\leq 0,$$

then $f(x)$ is monotonously decreasing when $\sqrt{\frac{2e(H^*)}{n+1}}\leq x\leq\sqrt{e(H^*)}$.

We consider converse-negative proposition of Lemma 2.2. Suppose H has the degree sequence $(d_1,d_2,\cdots,d_{2r})$, since H be not Hamiltoninan, there is a integer $p\leq\frac{r}{2}$, such that $d_p\leq p$ and $d_r\leq r-p$. So

$$\begin{aligned}e(H)=\frac{1}{2}\sum_{i=1}^{2r}d_i &\leq \frac{p^2+(r-p)^2+r^2}{2}\\ &= r^2-pr+p^2\\ &= r^2-kr+k^2-(p-k)(r-p-k)\end{aligned}$$

Since $p\geq d_p\geq\delta(H)\geq\delta(G)\geq k$ and $r-p\geq d_r\geq\delta(H)\geq\delta(G)\geq k$. Hence $(p-k)(r-p-k)\geq 0$. Thus

$$e(H)\leq r^2-kr+k^2. \qquad (3.3)$$

The equality of (3.3) holds if and only if $e(H)=r^2-kr+k^2$ and $(p-k)(r-p-k)=0$. Then $p-k=0$ or $r-p-k=0$.

If $p=k$, then H is a bipartite graph with r^2-kr+k^2 edges and $d_1=\cdots=d_k=k$, $d_{k+1}=\cdots=d_r=r-k$ and $d_{r+1}=\cdots=d_{2r}=r$. This implies that $H=K_{r-k,r-k}\sqcup K_{k,k}$ $(r\geq 2k+1)$. If $p=r-k$, then H is a bipartite graph with r^2-kr+k^2 edges and $d_1=\cdots=d_{r-k}=r-k$, $d_{r-k+1}=\cdots=d_r=k$ and $d_{r+1}=\cdots=d_{2r}=r$. This also implies that $H=K_{r-k,r-k}\sqcup K_{k,k}$ $(r\geq 2k+1)$.

Then $H^*=K_{r-k,k}+O_{k,r-k}$, and $\lambda(H^*)=\sqrt{k(r-k)}$, $e(H^*)=k(r-k)$.

Let $s:=e(G^*)-e(H^*)$. Namely,

$$s=e(G^*)-[r^2-e(H)]=e(G^*)+e(H)-r^2.$$

Then by Lemma 2.3 and (3.3), we get that

$$\varepsilon(H^*) \geq \varepsilon(G^*) - 2s \geq \varepsilon(G^*) - 2e(G^*) + 2kr - 2k^2. \tag{3.4}$$

(i) By (3.1), we get

$$\sum_{uv \in E(H^*)} (d_{H^*}(u) + d_{H^*}(v)) \geq re(H^*).$$

Moreover, we note that

$$\sum_{u \in V(H^*)} d_{H^*}^2(u) = \sum_{uv \in E(H^*)} (d_{H^*}(u) + d_{H^*}(v)) \geq re(H^*).$$

By Lemma 2.5, we have that

$$2r\lambda^2(H^*) \geq \sum_{u \in V(H^*)} d_{H^*}^2(u) \geq re(H^*).$$

Thus

$$\lambda(H^*) \geq \sqrt{\frac{e(H^*)}{2}}.$$

By Lemma 2.4 and $n \geq 3$, we find that

$$\sqrt{\frac{2e(H^*)}{n+1}} \leq \sqrt{\frac{e(H^*)}{2}} \leq \lambda(H^*) \leq \sqrt{e(H^*)}.$$

Then by (3.2) and monotonicity of $f(x)$, we get

$$\varepsilon(H^*) \leq f(\lambda(H^*)) \leq f(\sqrt{\frac{e(H^*)}{2}}) = \sqrt{e(H^*)}(\sqrt{n-1} + \sqrt{2}). \tag{3.5}$$

By $e(H^*) \leq e(G^*)$, (3.4) and (3.5), we have that

$$\varepsilon(G^*) - 2e(G^*) + 2kr - 2k^2 \leq \varepsilon(H^*) \leq \sqrt{e(H^*)}(\sqrt{n-1} + \sqrt{2}) \leq \sqrt{e(G^*)}(\sqrt{n-1} + \sqrt{2})$$

So

$$\begin{aligned} \varepsilon(G^*) &\leq \sqrt{e(G^*)}(\sqrt{n-1} + \sqrt{2}) + 2e(G^*) - 2kr + 2k^2 \\ &= \sqrt{e(G^*)}(\sqrt{2r-2} + \sqrt{2}) + 2e(G^*) - 2kr + 2k^2, \end{aligned}$$

where the equality holds if and only if all inequalities in the above argument should be equalities. By above discussion, if (3.3) holds, when $r \geq k+1$ and $k \geq 1$, we get $\sqrt{k(r-k)} \neq \sqrt{\frac{k(r-k)}{2}}$, namely $\lambda(H^*) \neq \sqrt{\frac{e(H^*)}{2}}$, and then the equality of (3.5) can't hold.

Thus

$$\varepsilon(G^*) < \sqrt{e(G^*)}(\sqrt{2r-2}+\sqrt{2}) + 2e(G^*) - 2kr + 2k^2,$$

a contradiction.

(ii) By (3.1), we get

$$Z_1(H^*) = \sum_{u\in V(H^*)} d_{H^*}^2(u) = \sum_{uv\in E(H^*)} (d_{H^*}(u) + d_{H^*}(v)) \geq re(H^*).$$

By Lemma 2.10, we have that

$$\lambda(H^*) \geq \frac{Z_1(H^*)}{e(H^*)} - \Delta(H^*) \geq r - \Delta(H^*) \geq r - \Delta(G'^*) = r - \Delta(G^*). \quad (3.6)$$

By Lemma 2.4, we notice that

$$\sqrt{\frac{2e(H^*)}{n+1}} \leq \sqrt{\Delta(H^*)} \leq \sqrt{\Delta(G^*)} \leq \sqrt{r} \leq r - \Delta(G^*) \leq \lambda(H^*) \leq \sqrt{e(H^*)},$$

when $\Delta(G^*) \leq r - \sqrt{r}$.

Because $f(x)$ is monotonously decreasing when $\sqrt{\frac{2e(H^*)}{n+1}} \leq x \leq \sqrt{e(H^*)}$, by (3.2), we get

$$\begin{aligned}\varepsilon(H^*) &\leq f(\lambda(H^*)) \leq f(r-\Delta(G^*)) \\ &= 2(r-\Delta(G^*)) + \sqrt{(n-1)(2e(H^*)-2(r-\Delta(G^*))^2)}. \quad (3.7)\end{aligned}$$

By $e(H^*) \leq e(G^*)$, (3.4) and (3.7), we have that

$$\begin{aligned}\varepsilon(G^*) &\leq \varepsilon(H^*) + 2e(G^*) - 2kr + 2k^2 \\ &\leq 2r-\Delta(G^*)) + \sqrt{(n-1)(2e(H^*)-2(r-\Delta(G^*))^2)} + 2e(G^*) - 2kr + 2k^2 \\ &\leq 2\sqrt{(r-1)(e(G^*)-(r-\Delta(G^*))^2)} + 2e(G^*) - 2\Delta(G^*) - 2(k-1)r + 2k^2,\end{aligned}$$

where the equality holds if and only if all inequalities in the above argument should be equalities. By above discussion, if (3.3) holds, $H^* = K_{r-k,k} + O_{k,r-k}$ is not regular. If (3.6) holds, H^* be regular, a contradiction. So the equality doesn't hold. Namely,

$$\varepsilon(G^*) < 2\sqrt{(r-1)(e(G^*)-(r-\Delta(G^*))^2)} + 2e(G^*) - 2\Delta(G^*) - 2(k-1)r + 2k^2,$$

a contradiction.

(iii) By (3.1), for any edge $uv \in E(H^*)$, we get

$$d_{H^*}(u)d_{H^*}(v) \geq d_{H^*}(u)(r - d_{H^*}(u)).$$

Since $d_H(u) \geq d_{G'}(u) \geq d_G(u) \geq k$ and $d_H(v) \geq d_{G'}(v) \geq d_G(v) \geq k$, we have $d_{H^*}(u) \leq r - k$ and $d_{H^*}(v) \leq r - k$. Thus $d_{H^*}(u) \geq r - d_{H^*}(v) \geq k$ and $d_{H^*}(v) \geq k$. Let $g(x) = x(r - x)$ with $k \leq x \leq r - k$, we note $g(x)$ is convex in x. Then we have $g(x) \geq g(k)$(or $g(r-k)$), namely $g(x) \geq k(r-k)$, the equality holds if and only if $x = k$(or $x = r - k$). Hence

$$d_{H^*}(u)d_{H^*}(v) \geq d_{H^*}(u)(r - d_{H^*}(u)) \geq k(r-k), \tag{3.8}$$

where the equality holds if and only if $d_{H^*}(u) = k$ and $d_{H^*}(v) = r - k$. By lemma 2.6, we have that

$$\lambda(H^*) \geq min\sqrt{d_{H^*}(u)d_{H^*}(v)} \geq \sqrt{k(r-k)}. \tag{3.9}$$

Because $f(x)$ is monotonously decreasing when $\sqrt{\frac{2e(H^*)}{n+1}} \leq x \leq \sqrt{e(H^*)}$, $r \geq k+1$, $k \geq 1$, and $(k-1)(r-k-1) = k(r-k) - (r-1) \geq 0$, we have

$$\sqrt{\frac{2e(H^*)}{n+1}} \leq \sqrt{\frac{2e(G^*)}{n+1}} \leq \sqrt{r-1} \leq \sqrt{k(r-k)} \leq \lambda(H^*) \leq \sqrt{e(H^*)}.$$

Then by (3.2),

$$\varepsilon(H^*) \leq f(\lambda(H^*)) \leq f(\sqrt{k(r-k)}) = 2\sqrt{k(r-k)} + \sqrt{(n-1)(2e(H^*) - 2(kr-k^2))}. \tag{3.10}$$

By (3.4), (3.10) and $e(H^*) \leq e(G^*)$, we get that

$$\begin{aligned}\varepsilon(G^*) &\leq \varepsilon(H^*) + 2e(G^*) - 2kr + 2k^2 \\ &\leq 2(\sqrt{k(r-k)}) + \sqrt{(n-1)(2e(H^*) - 2(kr-k^2))} + 2e(G^*) - 2kr + 2k^2 \\ &\leq 2\sqrt{(r-1)(e(G^*) - kr + k^2)} + 2e(G^*) - 2kr + 2k^2 + 2(\sqrt{k(r-k)}), \quad (3.11)\end{aligned}$$

where the equality holds if and only if all inequalities in the above argument should be equalities. The equality of (3.3) holds if and only if $H^* = K_{r-k,k} + O_{k,r-k}$ $(r \geq 2k+1)$. The equality of (3.11) holds if and only if $e(G^*) = e(H^*)$, namely $G = K_{k,k} \sqcup K_{r-k,r-k-1}$ $(r \geq 2k+1)$, a contradiction. ■

Remark :

Now we compare (i), (ii) and (iii) of Theorem 3.1.

We consider that $f(x) = 2x + \sqrt{(n-1)(2e(G^*) - 2x^2)}$ is monotonously decreasing when $\sqrt{\frac{2e(G^*)}{n+1}} \leq x \leq \sqrt{e(G^*)}$, and some other results in the proof of Theorem 3.1.

Assume 1: (1) Theorem 3.1 (ii) improves Theorem 3.1 (i), when $e(G^*) \leq 2(r - \Delta(G^*))^2$ and $\Delta(G^*) \leq r - \sqrt{r}$;

(2) Theorem 3.1 (i) improves Theorem 3.1 (ii), when $e(G^*) \geq 2(r-\Delta(G^*))^2$ and $\Delta(G^*) \leq r - \sqrt{r}$.

Proof. We notice that if $e(G^*) \leq 2(r - \Delta(G^*))^2$,

$$\sqrt{\frac{2e(G^*)}{n+1}} \leq \sqrt{\frac{e(G^*)}{2}} \leq r - \Delta(G^*) \leq \sqrt{e(G^*)},$$

Then by monotonicity of $f(x)$, we get

$$\begin{aligned}
&\sqrt{e(G^*)}(\sqrt{2r-2}+\sqrt{2}) + 2e(G^*) - 2kr + 2k^2 \\
&= 2e(G^*) - 2kr + 2k^2 + f(\sqrt{\frac{e(G^*)}{2}}) \\
&\geq 2e(G^*) - 2kr + 2k^2 + f(r - \Delta(G^*)) \\
&= 2\sqrt{(r-1)(e(G^*)-(r-\Delta(G^*))^2)}+2e(G^*)-2\Delta(G^*)-2(k-1)r+2k^2.
\end{aligned}$$

Thus (1) follows. By the similar discussion, we get (2). ■

Assume 2: (1) Theorem 3.1 (iii) improves Theorem 3.1 (i), when $e(G^*) \leq 2k(r-k)$;

(2) Theorem 3.1 (i) improves Theorem 3.1 (iii), when $e(G^*) \geq 2k(r-k)$.

Proof. We notice that if $e(G^*) \leq 2k(r-k)$, we have

$$\sqrt{\frac{2e(G^*)}{n+1}} \leq \sqrt{\frac{e(G^*)}{2}} \leq \sqrt{k(r-k)} \leq \sqrt{e(G^*)}.$$

Then by monotonicity of $f(x)$, we get

$$\begin{aligned}
&\sqrt{e(G^*)}(\sqrt{2r-2}+\sqrt{2}) + 2e(G^*) - 2kr + 2k^2 \\
&= 2e(G^*) - 2kr + 2k^2 + f(\sqrt{\frac{e(G^*)}{2}}) \\
&\geq 2e(G^*) - 2kr + 2k^2 + f(\sqrt{k(r-k)}) \\
&= 2\sqrt{(r-1)(e(G^*)-kr+k^2)} + 2e(G^*) - 2kr + 2k^2 + 2(\sqrt{k(r-k)}).
\end{aligned}$$

Thus (1) follows. By the similar discussion, we get (2). ■

Assume 3: (1) Theorem 3.1 (iii) improves Theorem 3.1 (ii), when $\Delta(G^*) \geq r - \sqrt{k(r-k)}$;

(2) Theorem 3.1 (ii) improves Theorem 3.1 (iii), when $\Delta(G^*) \leq r - \sqrt{k(r-k)}$, where $k = 1$ or $k \geq 2$, $r \geq \frac{k^2}{k-1}$.

Proof. If $\Delta(G^*) \geq r - \sqrt{k(r-k)}$, we can find that

$$\sqrt{\frac{2e(G^*)}{n+1}} \leq r - \Delta(G^*) \leq \sqrt{k(r-k)} \leq \sqrt{e(G^*)}.$$

Then by monotonicity of $f(x)$, we get

$$
\begin{aligned}
&2\sqrt{(r-1)(e(G^*)-(r-\Delta(G^*))^2)}+2e(G^*)-2\Delta(G^*)-2(k-1)r+2k^2\\
&=2e(G^*)-2kr+2k^2+f(r-\Delta(G^*))\\
&\geq 2e(G^*)-2kr+2k^2+f(\sqrt{k(r-k)})\\
&=2\sqrt{(r-1)(e(G^*)-kr+k^2)}+2e(G^*)-2kr+2k^2+2(\sqrt{k(r-k)}).
\end{aligned}
$$

Thus (1) follows. By the similar discussion, we get (2). ■

THEOREM 3.2 *Let $G=(X,Y;E)$, where $|X|=|Y|+1=r\geq k+1$, be a nearly balanced bipartite graph of order $n=2r-1$ and $\delta(G)\geq k\geq 1$. Then $G=(X,Y;E)(\neq K_{r-k,r-k-1}\sqcup K_{k,k}\ (r\geq 2k+1))$ is traceable, if*

(i)

$$Z_1(G^*)\leq kr(r-k).$$

(ii)

$$Z_2(G^*)\leq k^2(r-k)^4.$$

(iii)

$$4\sum_{u\in V(G^*)} d^4_{G^*}(u)\leq kr^4-k^2r^3.$$

Proof. Similar to the Proof of Theorem 3.1.

(i) By (3.1), we get

$$Z_1(H^*)=\sum_{u\in V(H^*)} d^2_{H^*}(u)=\sum_{uv\in E(H^*)}(d_{H^*}(u)+d_{H^*}(v))\geq re(H^*).$$

Note that

$$Z_1(H^*)=\sum_{u\in V(H^*)} d^2_{H^*}(u)\leq \sum_{u\in V(G^*)} d^2_{G^*}(u)=Z_1(G^*), \tag{3.12}$$

then by (3.3), we have

$$Z_1(G^*)\geq Z_1(H^*)\geq re(H^*)=r(r^2-e(H))\geq kr(r-k),$$

where the equality holds if and only if all inequalities in the above argument should be equalities. The equality of (3.3) holds if and only if $H=K_{r-k,r-k}\sqcup K_{k,k}\ (r\geq 2k+1)$. The equality of (3.12) holds if and only if $G=H$. This implies that $G=K_{r-k,r-k-1}\sqcup K_{k,k}\ (r\geq 2k+1)$, a contradiction.

(ii) By (3.8), we have that

$$Z_2(H^*) = \sum_{uv \in E(H^*)} d_{H^*}(u)d_{H^*}(v) \geq k(r-k)e(H^*).$$

Note that

$$Z_2(H^*) = \sum_{uv \in E(H^*)} d_{H^*}(u)d_{H^*}(v) \leq \sum_{uv \in E(G^*)} d_{G^*}(u)d_{G^*}(v) = Z_2(G^*).$$

Then by (3.3), we have

$$Z_2(G^*) \geq Z_2(H^*) \geq k(r-k)e(H^*) = k(r-k)(r^2 - e(H)) \geq k^2(r-k)^4.$$

By the similar analysis, the equality hold if and only if $G = K_{r-k,r-k-1} \sqcup K_{k,k}$ $(r \geq 2k+1)$, a contradiction

(iii) By (3.1) and Hölder inequality, we have that

$$\begin{aligned} re(H^*) &\leq \sum_{uv \in E(H^*)} (d_{H^*}(u) + d_{H^*}(v)) \\ &= \sum_{u \in V(H^*)} d^2_{H^*}(u) \\ &= \sum_{u \in V(H^*)} d^{\frac{2}{3}}_{H^*}(u) \sum_{u \in V(H^*)} (d^4_{H^*}(u))^{\frac{1}{3}} \\ &\leq (\sum_{u \in V(H^*)} d_{H^*}(u))^{\frac{2}{3}} (\sum_{u \in V(H^*)} d^4_{H^*}(u))^{\frac{1}{3}} \\ &= (2e(H^*))^{\frac{2}{3}} (\sum_{u \in V(H^*)} d^4_{H^*}(u))^{\frac{1}{3}}. \end{aligned}$$

Also by (3.3), we have that

$$4 \sum_{u \in V(G^*)} d^4_{G^*}(u) \geq 4 \sum_{u \in V(H^*)} d^4_{H^*}(u) \geq r^3 e(H^*) = r^3(r^2 - e(H)) \geq kr^4 - k^2r^3.$$

By the similar analysis, the equality hold if and only if $G = K_{r-k,r-k-1} \sqcup K_{k,k}$ $(r \geq 2k+1)$, a contradiction. ■

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