

Exponential functions of finite posets and the number of extensions with a fixed set of minimal points

FRANK A CAMPO and MARCEL ERNÉ

Seilerwall 33, D 41747 Viersen, Germany
acampo.frank@gmail.com

Faculty for Mathematics and Physics, Leibniz University, Welfengarten 1,
D 30167 Hannover, Germany
erne@math.uni-hannover.de

Abstract

We establish formulas for the number of all downsets (or equivalently, of all antichains) of a finite poset P . Then, using these numbers, we determine recursively and explicitly the number of all posets having a fixed set of minimal points and inducing the poset P on the non-minimal points. It turns out that these counting functions are closely related to a collection of downset numbers of certain subposets. Since any function of that kind is an exponential sum (with the number of minimal points as exponent), we call it the exponential function of the poset. Some linear equations, divisibility relations, upper and lower bounds, and asymptotical equalities for the counting functions are deduced. A list of all such exponential functions for posets with up to five points concludes the paper.

Mathematics Subject Classification:

Primary: 06A07. Secondary: 05C30, 06D05, 054D10.

Key words: poset, extension, upset, downset, antichain, topology.

0 Introduction

The determination of $p(k)$, the number of all labeled partially ordered sets (posets), and $\bar{p}(k)$, the number of all unlabeled (i.e. isomorphism types of) posets with k points are two old order-theoretical problems that are still far from being completely solved. For early attempts see, e.g., Sharp [30], Butler and Markowsky [8, 9], Ern e [14, 15]. Numerical and structural

improvements until the last decade of the past century are reported by Erné and Stege [23]. The explicit computation of the numbers $p(k)$ was pushed forward by Heitzig and Reinhold [25] until $k=17$, and of $\bar{p}(k)$ until $k=14$. Slightly later Brinkmann and McKay [8, 9] calculated $p(k)$ for $k \leq 19$, $\bar{p}(k)$ for $k \leq 16$, and many tables of related numbers. The reason why in the labeled case one can go three steps further than in the unlabeled case is a result due to Erné [14, 16], saying that $p(k)$ equals the number of posets on $k+1$ points with a fixed set of exactly two minimal (or maximal) elements.

It seems that a way to a general solution (if any) can only be successful if one considers posets with further additional parameters, like the number of related pairs (Culberson and Rawlins [12], Erné and Stege [19, 24]), or of downsets, upsets or antichains (Erné [16], Stanley [33]). *Downsets* (alias *lower sets* or *decreasing sets*) of a poset P contain with any element all elements below it, and *upsets* (alias *upper sets* or *increasing sets*) are defined dually. The downsets and the upsets of P form two dually isomorphic superalgebraic, hence completely distributive lattices $\mathcal{D}(P)$ and $\mathcal{U}(P)$. Functorial properties of \mathcal{D} and \mathcal{U} with some combinatorial consequences will be discussed briefly in Sections 1 and 2. For example, the tensor product of $\mathcal{U}(O)$ and $\mathcal{D}(P)$ turns out to be isomorphic to the lattice of all generalized vertical sums of O and P . The numbers $d(P) = \#\mathcal{D}(P) = \#\mathcal{U}(P)$ of all upresp. downsets of P will play a crucial role in the present study.

Given natural numbers j and k , let $p(j, k)$ denote the number of all labeled posets and $\bar{p}(j, k)$ that of all unlabeled posets on k points having exactly j downsets (or upsets, or antichains). By the familiar one-to-one correspondence between finite posets and finite T_0 spaces (Alexandroff [2, 3], Birkhoff [5]), $p(j, k)$ resp. $\bar{p}(j, k)$ is also the number of all T_0 topologies on k points with exactly j open (or closed) sets. Some first progress towards the determination of these numbers (at least at the lower or upper end of the j -scale for fixed k) was made by Sharp [31], Erné [14, 16] and Stanley [33]. More comprehensive results were obtained by Erné and Stege [23, 24], and later by Benoumhani and Kolli [6, 7, 26, 27]. Note that the number $\bar{p}(j, k)$ is, by a theorem due to Birkhoff [5], equal to the number of all unlabeled distributive lattices with j elements of which k are join (or meet) irreducible; these numbers have been calculated by Erné, Heitzig and Reinhold [22] for all $j < 50$ and arbitrary k .

Many of the (often tedious and time-consuming) calculations in that area are simplified essentially by the following systematical approach, which we shall pursue in the present paper. Given a poset or partial order P on a finite set K and a finite set M disjoint from K , let $\mathcal{E}(M, P)$ denote the set of all posets with ground set $M \cup K$, inducing P on K and having M as the set of all minimal elements. The cardinality $e(m, P)$ of $\mathcal{E}(M, P)$ depends only on the cardinality $m = \#M$ but not on the specific choice of M ; $e(m, P)$

is called the *exponential function* of P , because it is always an exponential sum with exponent m . Using some further combinatorial interpretations of the numbers $e(m, P)$ by means of certain sets of order-preserving functions or relations, we shall obtain diverse explicit and recursive formulas for $e(m, P)$, all involving the downset numbers $d(U)$ of certain subposets U as the relevant ingredients. Horizontal and vertical decompositions of posets are of particular use for that purpose. A few number-theoretical divisibility properties of the numbers $e(m, P)$ are easy consequences of their exponential sum representations. We present exact upper bounds for the downset numbers $d(P)$ where P belongs to specific classes of posets with a given height or number of minimal elements. These bounds are helpful for a more precise analysis of the exponential functions $e(m, P)$.

Given a suitable enumeration of unlabeled finite posets, one may express the relevant formulas in terms of certain succinct matrix equations. Furthermore, the knowledge of the leading terms $d(P)^m$ of the functions $e(m, P)$ provides good numerical estimates and information about their asymptotical behavior. The paper concludes with a list of all unlabeled posets with up to five points and their exponential functions.

1 Basic definitions and concepts

By \mathbb{N} we denote the set of positive integers, and \underline{k} will stand for the set of all $n \in \mathbb{N}$ with $n \leq k$; in particular, $\underline{0}$ is the empty set \emptyset . While \subseteq denotes set inclusion, \subset denotes *proper* inclusion.

A *relation* between (elements of) two sets X and Y is merely a subset R of the cartesian product $X \times Y$. Instead of $(x, y) \in R$ one writes $x R y$. The *dual* or *converse* relation of R is $R^d = R^{-1} = \{(y, x) \mid x R y\}$. For $a \in X$ and $b \in Y$, we put

$$aR = \{y \in Y \mid a R y\}, \quad Rb = \{x \in X \mid x R b\},$$

and for $A \subseteq X$, $B \subseteq Y$, we define

$$AR = \bigcup \{aR \mid a \in A\}, \quad RB = \bigcup \{Rb \mid b \in B\}.$$

In case $X = Y$, a subset R of $X \times X$ is a *relation on X* , and for $A \subseteq X$, the *relation induced on A* is $R|_A = R \cap (A \times A)$. Instead of $R|_{X \setminus A}$ we simply write $R - A$. The *identity relation* on X or *diagonal* is $\Delta_X = \{(x, x) \mid x \in X\}$.

Given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, the *composite relation* $RS = S \circ R$ consists of all pairs (x, z) for which there exists a y with $x R y S z$. In particular, this applies to functions from X to Y , regarded as special relations $F \subseteq X \times Y$ such that each $x F$ is a singleton, consisting of the single element $F(x)$; in that case one writes $F[A]$ instead of AF .

A *quasiorder* (or *preorder*) on X is a reflexive and transitive relation on X , in other terms, a relation Q on X with $\Delta_X \cup QQ \subseteq Q$; a quasiorder Q that is antisymmetric (i.e. $Q \cap Q^d \subseteq \Delta_X$) is referred to as a (*partial order*) on X . The pair (X, Q) is then a *quasiordered set* (*qoset*) or a *partially ordered set* (*poset*), respectively.

For any quasiorder Q on X and arbitrary subsets $A \subseteq X$, the *down-closure* is QA , the downset generated by A (that is, the least downset containing A). *Upsets* or *upper sets* and the *up-closure* AQ are defined dually. The collection of all downsets of Q is denoted by $\mathcal{D}(Q)$, and that of all upsets by $\mathcal{U}(Q)$. It is known and easy to see that $\mathcal{D}(Q)$ as well as $\mathcal{U}(Q)$ is always an *Alexandroff topology* (closed under arbitrary unions and intersections), called the *lower* and the *upper Alexandroff topology* associated with Q (see [2, 3]). Of course, on a finite set, all topologies are Alexandroff topologies. Notice that sending each downset to its complement gives a dual isomorphism between the complete lattices $\mathcal{D}(Q)$ and $\mathcal{U}(Q)$. By passing from a qoset to the associated (upper or lower) Alexandroff space, one obtains a concrete isomorphism between the category of qosets with isotone, i.e. order-preserving maps as morphisms and the category of Alexandroff spaces with continuous maps as morphisms. Under that isomorphism, the posets correspond to the T_0 Alexandroff spaces (in which distinct points have different closures). In particular, finite qosets (resp. posets) are in one-to-one correspondence with finite topological spaces (resp. T_0 spaces).

Two fundamental constructions allow us to build from given posets larger ones (cf. Stanley [34]). Suppose O and P are partial orders on disjoint ground sets X and Y , respectively (of course, the disjointness condition is not an essential restriction, because one always may pass to isomorphic copies having disjoint ground sets). The *cardinal sum* or *disjoint sum* is

$$O + P = O \cup P,$$

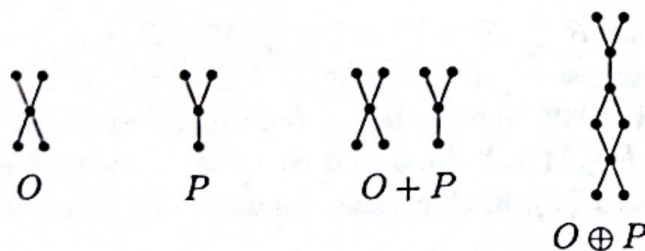
and the *ordinal sum* is

$$O \oplus P = O \cup P \cup (X \times Y).$$

Accordingly, one writes for the corresponding posets

$$(X, O) + (Y, P) = (X \cup Y, O + P),$$

$$(X, O) \oplus (Y, P) = (X \cup Y, O \oplus P).$$



Up to isomorphism, there is only one *antichain* (totally unordered set) with k elements, namely $A_k = (k, =)$, and only one *chain* (totally ordered set) with k elements, namely $C_k = (k, \leq)$. Such extremal posets are obtained by iterated formation of cardinal resp. ordinal sums, starting with singletons. Allowing a mixed iterated construction involving both kinds of sums, again starting with singletons, yields exactly all *N-free posets*, i.e., all finite posets having no subposet with diagram \mathbb{N} . By obvious reasons, such order relations have also been baptized *series-parallel orders* (see, e.g., Möhring [28]).

Sometimes orders are identified with the corresponding ordered sets; set-theoretically, this convention is not entirely consistent, but there is no danger of confusion since the ground set is determined by the order relation. If P is a poset with underlying set X , the order relation is denoted by \leq_P . For subsets A of X , we denote by $P|_A$ the induced poset on A and by $P-A$ the induced poset on $X \setminus A$; and we say P is an *extension* of a poset O if $O = P|_A$ for some A . It will then be convenient to write xP for $x \leq_P$, AP for $A \leq_P$ etc.; hence, the symbols $x \leq_P y$, $x \in Py$, and $y \in xP$ all have the same meaning, namely $(x, y) \in \leq_P$. For any poset (or qoset) P , the downset lattice $\mathcal{D}(P) = \mathcal{D}(\leq_P)$ and the upset lattice $\mathcal{U}(P) = \mathcal{U}(\leq_P)$ are *superalgebraic*, i.e. completely distributive algebraic lattices. In fact, $\mathcal{D}(P)$ is the free completely distributive lattice over P : every isotone (i.e. order-preserving) map f from P to a completely distributive lattice L uniquely extends to a join-preserving map f^\vee from $\mathcal{D}(P)$ to L , given by $f^\vee(D) = \bigvee f[D]$. Moreover, \mathcal{D} gives rise to a functorial dual equivalence between the category of posets with isotone maps and the category of superalgebraic lattices with complete homomorphisms, i.e. maps preserving arbitrary joins and meets (see, e.g., Ern  [17, 20]). As nonempty finite distributive lattices are already superalgebraic, the category of finite posets resp. T_0 spaces is equivalent to the category of finite distributive lattices (see Birkhoff [5]).

These remarks enable us to translate all results about (finite) ordered sets into the language of topology (or, if desired, of lattice theory). For instance, the down-closure PA of a subset A of a poset or qoset P is the closure w.r.t. the associated *upper* Alexandroff topology $\mathcal{U}(P)$, while the up-closure AP is the closure w.r.t. the associated *lower* Alexandroff topology $\mathcal{D}(P)$. And, consequently, $X \setminus (X \setminus A)P$ is the topological interior A° in $\mathcal{D}(P)$, whereas $X \setminus P(X \setminus A)$ is the interior in $\mathcal{U}(P)$. See [15, 18, 21, 23] for more background.

Although for concrete computations in combinatorial order theory incidence matrices provide a very useful tool (see, e.g., [10, 11, 14, 21, 29, 30, 31] for details), we shall not need them in the subsequent considerations.

2 Generalized vertical sums of posets

In this section it will be convenient to denote by O and P two *partial orders* on sets X and Y , respectively (and not the corresponding posets). Recall that a function f between (X, O) and (Y, P) is said to be *isotone* (or *order-preserving*) if $x O x'$ implies $f(x) P f(x')$, and *antitone* (or *order-inverting*) if $x O x'$ implies $f(x') P f(x)$. Let $\mathcal{Q}(X)$ denote the set of all quasiorders and $\mathcal{Q}_0(X)$ that of all partial orders on X .

Specifically, given partial orders O and P on *disjoint* ground sets X and Y , respectively, consider the following sets:

$$\begin{aligned}\mathcal{Q}(O, P) &= \{Q \in \mathcal{Q}(X \cup Y) \mid Q|_X = O, Q|_Y = P, Q \cap (Y \times X) = \emptyset\}, \\ \mathcal{D}(O, P) &= \{f : Y \rightarrow \mathcal{D}(O) \mid f \text{ is isotone from } (Y, P) \text{ to } (\mathcal{D}(O), \subseteq)\}, \\ \mathcal{U}(O, P) &= \{g : X \rightarrow \mathcal{U}(P) \mid g \text{ is antitone from } (X, O) \text{ to } (\mathcal{U}(P), \subseteq)\}, \\ \mathcal{J}(O, P) &= \{h : \mathcal{D}(P) \rightarrow \mathcal{D}(O) \mid h \text{ preserves arbitrary joins (unions)}\}, \\ \mathcal{R}(O, P) &= \{R \subseteq X \times Y \mid OR \cup RP \subseteq R\}.\end{aligned}$$

Note that $\mathcal{Q}(O, P)$ is the collection of all *partial orders* extending $O + P$ and contained in $O \oplus P$. The posets with order relations in $\mathcal{Q}(O, P)$ are called *generalized vertical sums* of (X, O) and (Y, P) .

Being closed under arbitrary unions and intersections, the sets $\mathcal{R}(O, P)$ are Alexandroff topologies on $X \times Y$, hence superalgebraic lattices.

The *tensor product* $C \otimes D$ of two complete lattices C and D (Shmuely [32]) may be represented as the complete lattice of all join-preserving maps from C to the dual of D . Notice that $C \otimes D$ is always isomorphic to $D \otimes C$. As usual, we order sets of functions into posets pointwise.

Theorem 2.1 *The following six superalgebraic lattices are isomorphic:*

$$\mathcal{Q}(O, P), \mathcal{D}(O, P), \mathcal{U}(O, P), \mathcal{J}(O, P), \mathcal{R}(O, P), \mathcal{U}(O) \otimes \mathcal{D}(P).$$

Proof. Define $\iota : \mathcal{R}(O, P) \rightarrow \mathcal{Q}(O, P)$ by $\iota(R) = O \cup P \cup R$. This is a well-defined map: for $R \in \mathcal{R}(O, P)$ and $Q = \iota(R)$, we have $Q|_X = O$, $Q|_Y = P$, and $R \subseteq X \times Y$, hence $Q \cap (Y \times X) = \emptyset$; furthermore, Q is reflexive since $\Delta_{X \cup Y} = \Delta_X \cup \Delta_Y \subseteq Q$, and transitive since $QQ = OO \cup PP \cup OR \cup RP$ is contained in $O \cup P \cup R = Q$; this proves $Q \in \mathcal{Q}(X \cup Y)$. By the equations $\iota(R)|_X = O$ and $\iota(R)|_Y = P$, the map ι is one-to-one.

Now, let Q be an arbitrary partial order in $\mathcal{Q}(O, P)$, and put $R = Q \cap (X \times Y)$. Then $RP \subseteq QQ \cap (X \times Y) = Q \cap (X \times Y) = R$, and analogously $OR \subseteq R$. Thus, $R \in \mathcal{R}(O, P)$. In order to show that ι is onto, it remains to verify that $Q = \iota(R)$. In fact, $Q = Q|_X \cup Q|_Y \cup (Q \cap (X \times Y)) = O \cup P \cup R = \iota(R)$, because $Q \cap (Y \times X)$ is empty. By definition, $R \subseteq S$ is

equivalent to $\iota(R) \subseteq \iota(S)$, and consequently, ι is an isomorphism between the superalgebraic lattices $\mathcal{R}(O, P)$ and $\mathcal{Q}(O, P)$.

The map

$$\alpha : \mathcal{U}(O, P) \rightarrow \mathcal{R}(O, P), g \mapsto \{(x, y) \in X \times Y \mid y \in g(x)\}$$

is a well-defined isomorphism, too: for $R = \alpha(g)$ and $(x, z) \in RP$ there is a $y \in Y$ with $xRyPz$. Thus $y \in g(x) \in \mathcal{U}(P)$ and so $z \in g(x)$, i.e. $(x, z) \in R$. Hence, we have $RP \subseteq R$. For $OR \subseteq R$, note that $xOyRz$ implies $z \in g(y) \subseteq g(x)$, hence xRz . Surjectivity is straightforward: given $R \in \mathcal{R}(O, P)$, define $g : X \rightarrow \mathcal{U}(P)$ by $g(x) = xR \in \mathcal{U}(P)$ (since xRP is contained in xR by definition of $\mathcal{R}(O, P)$) and verify that g is an antitone map from (X, O) to $(\mathcal{U}(P), \subseteq)$ with $\alpha(g) = R$. The proof of the isomorphism properties is completed by the equivalences

$$g \leq h \Leftrightarrow \forall x \in X (g(x) \subseteq h(x)) \Leftrightarrow \alpha(g) \subseteq \alpha(h).$$

Similarly, one sees that the map

$$\beta : \mathcal{D}(O, P) \rightarrow \mathcal{R}(O, P), f \mapsto \{(x, y) \in X \times Y \mid x \in f(y)\}$$

is an isomorphism.

By the universal property of \mathcal{D} (see Section 1), the lattice $\mathcal{D}(O, P)$ is isomorphic to the lattice $\mathcal{J}(O, P)$ of all join-preserving maps from $\mathcal{D}(P)$ to $\mathcal{D}(O)$, which in turn is isomorphic to the tensor product $\mathcal{D}(P) \otimes \mathcal{U}(O)$ (because $\mathcal{D}(O)$ and $\mathcal{U}(O)$ are duals of each other), hence to $\mathcal{U}(O) \otimes \mathcal{D}(P)$. \square

In the special case that O is the antichain on X , we immediately deduce from Theorem 2.1 (denoting, as usual, by $\mathcal{P}(X)$ the power set of X):

Corollary 2.1 *For disjoint sets X, Y and any partial order P on Y , the following five lattices are isomorphic to $\mathcal{P}(X) \otimes \mathcal{D}(P)$:*

$$\mathcal{Q}(X, P) = \{Q \in \mathcal{Q}(X \cup Y) \mid Q|_X = \Delta_X, Q|_Y = P, Q \cap (Y \times X) = \emptyset\},$$

$$\mathcal{D}(X, P) = \{f : Y \rightarrow \mathcal{P}(X) \mid xPy \Rightarrow f(x) \subseteq f(y)\},$$

$$\mathcal{U}(X, P) = \{g \mid g : X \rightarrow \mathcal{U}(P)\} = \mathcal{U}(P)^X,$$

$$\mathcal{J}(X, P) = \{h : \mathcal{D}(P) \rightarrow \mathcal{P}(X) \mid h \text{ preserves arbitrary unions}\},$$

$$\mathcal{R}(X, P) = \{R \subseteq X \times Y \mid RP \subseteq R\}.$$

Observe that $\mathcal{Q}(X, P)$ consists of certain partial orders on $X \cup Y$ whose set of minimal elements contains but is not necessarily equal to the set X . By definition, $\mathcal{D}(X, P)$ consists of all *isotone* maps from (Y, P) to $(\mathcal{P}(X), \subseteq)$, while $\mathcal{U}(X, P)$ consists of *all* maps from X to $\mathcal{U}(P)$.

Although we shall not need the following categorical aspect in our combinatorial computations, it is worthwhile to observe that one may form a new category \mathcal{R} of posets by taking the sets $\mathcal{R}(O, P)$ as hom-sets, with the relation product \circ as composition (this works because $OR \cup RP \subseteq R$ and $PS \cup SQ \subseteq S$ imply $ORS \cup RSQ \subseteq RS$, and $O\Delta_X \cup \Delta_X O = O$). In view of Theorem 2.1, one might take alternatively the sets $\mathcal{Q}(O, P)$, $\mathcal{U}(O, P)$, $\mathcal{D}(O, P)$, or $\mathcal{J}(O, P)$ as hom-sets, with suitable compositions.

It is well-known that the superalgebraic lattices are, up to isomorphism, exactly the downset lattices of arbitrary posets, and that the category of posets and isotone maps is dual to the category of superalgebraic lattices and maps preserving arbitrary joins and meets (see Ern e [18, 20]). Now, let \mathcal{S}_\vee denote the category of superalgebraic lattices with maps preserving arbitrary joins (but not necessarily meets) as morphisms. Within the extended framework of relations instead of functions, one shows by a series of straightforward verifications, involving Theorem 2.1:

Theorem 2.2 *The following facts hold without any finiteness restrictions:*

(1) *The category \mathcal{R} is self-dual under the contravariant functor sending any poset to its dual and each relation $R \in \mathcal{R}(O, P)$ to the dual R^d .*

(2) *The category \mathcal{S}_\vee is self-dual under the contravariant functor sending any superalgebraic lattice to its dual and any join-preserving map $h : B \rightarrow C$ between superalgebraic lattices to its upper adjoint $h^* : C \rightarrow B$, given by*

$$h^*(c) = \bigvee \{b \in B \mid h(b) \leq c\}.$$

(3) *The upset operator \mathcal{U} , defined on morphisms $R \in \mathcal{R}(O, P)$ by*

$$\mathcal{U}(R) : \mathcal{U}(O) \rightarrow \mathcal{U}(P), \quad \mathcal{U}(R)(U) = UR$$

gives rise to a functorial equivalence between the categories \mathcal{R} and \mathcal{S}_\vee .

(4) *Similarly, the downset operator \mathcal{D} gives rise to a functorial duality between the categories \mathcal{R} and \mathcal{S}_\vee . On morphisms $R \in \mathcal{R}(O, P)$, the contravariant functor \mathcal{D} is defined by*

$$\mathcal{D}(R) : \mathcal{D}(P) \rightarrow \mathcal{D}(O), \quad \mathcal{D}(R)(D) = RD.$$

Of course, the category of posets and isotone maps may be regarded as a subcategory of \mathcal{R} (viewing functions as relations), and the category of superalgebraic lattices and complete homomorphisms as a subcategory of \mathcal{S}_\vee . However, these two dually equivalent subcategories fail to be self-dual!

In the finite case, Theorems 2.1 and 2.2 give rise to several combinatorial 'bijection equalities'. We shall return to that topic in Section 4.

3 Counting antichains, upsets and downsets

In what follows, we deal with *finite posets*, denoted by O, P etc. and represented in the usual way by diagrams (see Section 1 and the Appendix).

For a finite poset P , we denote by $d(P)$ the cardinality of $\mathcal{D}(P)$, the downset lattice or free distributive lattice over P . Observe that $d(P)$ is not only the number of all downsets and the number of all upsets, but also the number of all *antichains*, i.e., subsets A of pairwise incomparable elements (such that the induced order on A is the identity relation). This coincidence is caused by the bijection between antichains and downsets (or upsets) sending A to PA (or to AP).

A very flexible formula for the recursive enumeration of all downsets (hence of all antichains) of a single finite poset is given by

Theorem 3.1 *For any subset A of a finite poset P , the number $d(P)$ of downsets satisfies the inequality*

$$d(P) \leq \sum_{B \subseteq A} d(P - A \uparrow_P B) \quad \text{with } A \uparrow_P B = (A \setminus B)P \cup PB.$$

Equality holds whenever A is an antichain.

Proof. Clearly, $\mathcal{D}(P)$ is the disjoint union of the sets

$$\mathcal{D}_{AB}(P) = \{D \in \mathcal{D}(P) \mid D \cap A = B\},$$

where B runs through the subsets of A . Hence it suffices to show that

$$\varphi_{AB} : \mathcal{D}_{AB}(P) \rightarrow \mathcal{D}(P - A \uparrow_P B), \quad D \mapsto D \setminus PB$$

is a well-defined injection, and a bijection in case A is an antichain.

Any downset $D \in \mathcal{D}(P)$ satisfies the implications

$$D \cap A = B \Rightarrow D \cap (A \setminus B)P = \emptyset \Rightarrow (D \setminus PB) \cap (A \uparrow_P B) = \emptyset.$$

Consequently, $D \in \mathcal{D}_{AB}(P)$ implies $\varphi_{AB}(D) \in \mathcal{D}(P - A \uparrow_P B)$, so that φ_{AB} is well-defined, and $D = \varphi_{AB}(D) \cup (PB \setminus (A \setminus B)P)$; indeed, from $B \subseteq D$ it follows that $PB \subseteq D$, and assuming $y \in D \cap (A \setminus B)P$ we would find an $x \in (A \setminus B) \cap Py$, which would lead to the contradiction $x \in D \cap A = B$. This shows that φ_{AB} is one-to-one. (In fact, it is an embedding of the distributive lattice $\mathcal{D}_{AB}(P)$ in the distributive lattice $\mathcal{D}(P - A \uparrow_P B)$.)

Now suppose that A is an antichain of P . Given $D' \in \mathcal{D}(P - A \uparrow_P B)$, put $D = D' \cup PB$. Then D is a downset of P ; indeed, for $y \in D$ and $x \in Py$, we either have $y \in PB$ and so $x \in PB \subseteq D$, or we have the implications

$$y \in D \setminus PB \Rightarrow y \in D' \Rightarrow y \notin A \downarrow_P B \Rightarrow y \notin (A \setminus B)P \Rightarrow x \notin (A \setminus B)P \\ \Rightarrow x \notin A \downarrow_P B \text{ or } x \in PB \Rightarrow x \in D' \cup PB = D,$$

since $D' \in \mathcal{D}(P - A \downarrow_P B)$. Furthermore, D cannot meet $A \setminus B$:

$$D \cap (A \setminus B) = (D' \cup PB) \cap (A \setminus B) = PB \cap (A \setminus B) = \emptyset,$$

as A is an antichain and $B \subseteq A$. On the other hand, $B \subseteq PB \cap A \subseteq D \cap A$. This proves the equation $B = D \cap A$. By definition, $\varphi_{AB}(D) = D \setminus PB = D'$ (because D' is disjoint from PB). Thus, in that case, φ_{AB} is onto, hence bijective (and an isomorphism between $\mathcal{D}_{AB}(P)$ and $\mathcal{D}(P - A \downarrow_P B)$). \square

Since the minimal elements of a poset always form an antichain, we have the following consequence:

Corollary 3.1 For the set A of all minimal elements of a finite poset P ,

$$d(P) = \sum_{B \subseteq A} d(P - (B \cup (A \setminus B)P)) = \sum_{C \subseteq A} d(P - (A \cup CP)).$$

Already the case of a singleton A is useful (and well-known, see [14, 21]):

Corollary 3.2 For any element x of a finite poset P ,

$$d(P) = d(P - P\{x\}) + d(P - \{x\}P).$$

In particular, if x is a minimal element of P then

$$d(P) = d(P - \{x\}) + d(P - \{x\}P).$$

A further counting formula for antichains is obtained by applying a result on clique numbers in graphs (Erdős and Erné [13], Vollert [35]) to the incomparability graphs of posets. Let $\mathcal{A}(P)$ denote the set of antichains of the poset P .

Theorem 3.2 For any antichain A of a finite poset P ,

$$d(P) = \sum_{B \in \mathcal{A}(P-A)} 2^{\#(A \setminus (PB \cup BP))}.$$

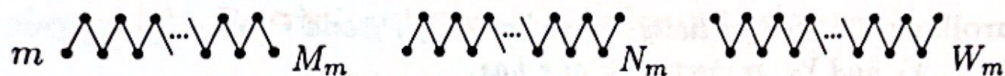
The formulas for downsets in sums of posets are almost obvious:

Lemma 3.1 For any two disjoint finite posets O and P ,

$$d(O + P) = d(O) \cdot d(P), \quad d(O \oplus P) = d(O) + d(P) - 1.$$

These equations allow to determine recursively in a straightforward manner the number of downsets in N -free posets. In particular, $d(A_k) = 2^k$ and $d(C_k) = k+1$.

Example 3.1 Typical orders that are not N -free are the zigzag orders: a zigzag poset with m minimal elements has one of the following diagrams:



For a zigzag poset with k points, the number of downsets is the k -th Fibonacci number f_k with $f_0 = 1$, $f_1 = 2$ and $f_{k+2} = f_{k+1} + f_k$.

The formulas in Lemma 3.1 are extremal instances of one formula for the number of downsets in generalized vertical sums, as introduced in Section 2. In slightly changed notation, such a vertical sum of two posets $O = (X, \leq_O)$ and $P = (Y, \leq_P)$ is of the form

$$O \oplus_R P = (X \cup Y, Q) \text{ with } Q = \leq_O \cup R \cup \leq_P$$

and $R \in \mathcal{R}(O, P)$, i.e., $R \subseteq X \times Y$ and $\leq_O R \cup R \leq_P \subseteq R$.

Theorem 3.3 The collection of all downsets in a generalized vertical sum is

$$\mathcal{D}(O \oplus_R P) = \bigcup_{D \in \mathcal{D}(P)} \{D \cup RD \cup E \mid E \in \mathcal{D}(O - RD)\}$$

and this is a disjoint union. Hence, in the finite case,

$$d(O \oplus_R P) = \sum_{D \in \mathcal{D}(P)} d(O - RD).$$

Proof. By the definition of the quasiorder Q , it follows from $D \in \mathcal{D}(P)$ that $QD = D \cup RD$, hence $X \cap QD = RD$ and $X \setminus QD = X \setminus RD$. For any D' in $\mathcal{D}(O \oplus_R P)$, we have $D = D' \cap Y \in \mathcal{D}(P)$, $E = D' \cap (X \setminus RD) \in \mathcal{D}(O - RD)$, and $QD \cup E = D'$, since D' is a downset containing D .

On the other hand, given any $D \in \mathcal{D}(P)$ and any $E \in \mathcal{D}(O - RD)$, we obtain $Q(QD \cup E) = QD \cup QE = QD \cup E$, since $QE = OE$ is contained in $RD \cup E \subseteq QD \cup E$. Thus, $D \cup RD \cup E = QD \cup E \in \mathcal{D}(O \oplus_R P)$.

The formula for $d(O \oplus_R P)$ follows from that for $\mathcal{D}(O \oplus_R P)$, because the union is disjoint (as $D = (QD \cup E) \cap Y$ for $D \in \mathcal{D}(P)$), and the map $E \mapsto QD \cup E$ is one-to-one (as $E = (QD \cup E) \setminus QD$ for $E \in \mathcal{D}(O - QD)$). \square

Corollary 3.3 For any finite poset P disjoint from an antichain A_m ,

$$d(A_m \oplus_R P) = \sum_{D \in \mathcal{D}(P)} 2^{m - \#RD}.$$

This may also be regarded as a special instance of Theorem 3.2. Another application of Theorem 3.3 is the following 'associative law':

Corollary 3.4 For partially ordered sets O , P_1 and P_2 on disjoint ground sets X , Y_1 and Y_2 , respectively, one has

$$R \in \mathcal{R}(O, P_1 + P_2) \Leftrightarrow R_i \in \mathcal{R}(O, P_i) \text{ for } R_i = R \cap (X \times Y_i), i = 1, 2,$$

and

$$O \oplus_R (P_1 + P_2) = (O \oplus_{R_1} P_1) \oplus_{R_2} P_2.$$

Hence, in the finite case,

$$d(O \oplus_R (P_1 + P_2)) = \sum_{D \in \mathcal{D}(P_2)} d((O \oplus_{R_1} P_1) - R_2 D).$$

Finally, a few additional arguments, similar to those for Corollary 4.2 in the next section, lead to the following extremality results (cf. Beinling [4]):

Corollary 3.5 The maximal downset number of posets P with k points of which exactly m are minimal (or maximal) is

$$d(P) = 2^{m-1} + 2^{k-1}.$$

All posets satisfying that equation are isomorphic to $A_{m-1} + (A_1 \oplus A_{k-m})$. There are exactly $\binom{k}{m} m$ such posets on \underline{k} if $m < k$, and only one if $m = k$.

Corollary 3.6 The maximal downset number of posets with k points and height h (the maximal size of chains in the poset) is

$$d(P) = 2^{k-h}(h+1).$$

All posets satisfying that equation are isomorphic to $A_{k-h} + C_h$. There are exactly $(k)_h = \prod_{i=0}^{h-1} (k-i)$ such posets on \underline{k} if $h > 1$, and only one if $h = 1$.

4 The exponential function of a finite poset

From now on, let M and K always be disjoint finite sets of cardinality m and k , respectively. For example, one may take $M = \underline{m}$, $K = \underline{m+k} \setminus \underline{m}$. If P is a poset with underlying set K , we write $\mathcal{Q}(M, P)$ for $\mathcal{Q}(M, \leq_P)$ etc. From Corollary 2.1, we immediately deduce a few combinatorial identities:

Corollary 4.1 For a finite poset P on K , the following six distributive lattices have all the same cardinality $d(P)^m$:

$$\mathcal{Q}(M, P), \mathcal{D}(M, P), \mathcal{U}(M, P), \mathcal{R}(M, P), \mathcal{J}(M, P), \mathcal{P}(M) \otimes \mathcal{D}(P).$$

We denote the set of minimal elements of a poset P by $\text{Min } P$ and its cardinality by $m(P)$. Furthermore, from now on, $\mathcal{Q}_0(K)$ will denote the set of all posets on K , while $\mathcal{P}_0(M)$ will denote the collection of all nonempty subsets of M . Restricting the sets in Corollary 4.1, we define:

$$\begin{aligned}\mathcal{E}(M, P) &= \{Q \in \mathcal{Q}_0(M \cup K) \mid \text{Min } Q = M, Q|_K = P\}, \\ \mathcal{F}(M, P) &= \{f : K \rightarrow \mathcal{P}_0(M) \mid x \leq_P y \Rightarrow f(x) \subseteq f(y)\}, \\ \mathcal{G}(M, P) &= \{g : M \rightarrow \mathcal{U}(P) \mid \bigcup g[M] = K\}, \\ \mathcal{H}(M, P) &= \{h : \mathcal{D}(P) \rightarrow \mathcal{P}(M) \mid h \text{ preserves unions, } h(D) = \emptyset \Rightarrow D = \emptyset\}, \\ \mathcal{I}(M, P) &= \{R \subseteq M \times K \mid R \leq_P \subseteq R, MR = K\}.\end{aligned}$$

Thus, $\mathcal{E}(M, P)$ is the set of all posets on $M \cup K$ extending P such that M is exactly the set of all minimal elements. The numbers

$$e(m, P) = \#\mathcal{E}(M, P)$$

do not depend on the special choice of the set M of cardinality m with $M \cap K = \emptyset$. Now, analogous reasonings as for Theorem 2.1 and Corollary 2.1 lead to the next crucial result.

Theorem 4.1 *For finite disjoint sets M, K and any poset P on K ,*

$$e(m, P) = \#\mathcal{E}(M, P) = \#\mathcal{F}(M, P) = \#\mathcal{G}(M, P) = \#\mathcal{H}(M, P) = \#\mathcal{I}(M, P).$$

Moreover, $e(m, P)$ is also the number of all T_0 topologies on $M \cup K$ inducing the topology $\mathcal{D}(P)$ on K and making exactly the points in M isolated.

Corollary 4.2 *Regarded as functions, \mathcal{G} and e are strictly antitone in the second argument: for $M \neq \emptyset$ and posets O, P on K ,*

$$\leq_O \subset \leq_P \Rightarrow \mathcal{U}(O) \supset \mathcal{U}(P) \Rightarrow \mathcal{G}(M, O) \supset \mathcal{G}(M, P) \Rightarrow e(m, O) > e(m, P).$$

Proof. For posets O, P on K such that \leq_O is properly contained in \leq_P , it follows from $U \in \mathcal{U}(P)$ that $UO \subseteq UP \subseteq U$, hence $U \in \mathcal{U}(O)$; and if $x \leq_P y$ but not $x \leq_O y$ then xO lies in $\mathcal{U}(O)$ but not in $\mathcal{U}(P)$.

The second implication is clear from the definition, and the third from the proven equation $e(m, P) = \#\mathcal{G}(M, P)$ in Theorem 4.1. \square

We call $e(m, P)$ the *exponential function* of the poset P . This terminology is justified by the subsequent results, which provide some economic representations of such functions.

Theorem 4.2 *For the antichain A of minimal elements of a finite poset P ,*

$$e(m, P) = \sum_{B \subseteq A} (-1)^{\#B} d(P-B)^m \quad (1)$$

Proof. For the ground set K of P and any $g \in \mathcal{U}(P)^M$, the equation $K = \bigcup g[M]$ is tantamount to the inclusion $A \subseteq \bigcup g[M]$, since K is the upset generated by A and $\bigcup g[M]$ is an upset of P . For each subset B of A , the set

$$\mathcal{G}_B = \{g \in \mathcal{U}(P)^M \mid \bigcup g[M] \cap B = \emptyset\} = \mathcal{U}(P-B)^M$$

has cardinality $d(P-B)^m$. Using the equations

$$\mathcal{G}_B = \bigcap \{\mathcal{G}_{\{a\}} \mid a \in B\}, \quad \mathcal{G}(M, P) = \mathcal{U}(P)^M \setminus \bigcup \{\mathcal{G}_{\{a\}} \mid a \in A\},$$

and a well-known combinatorial sieve inversion formula (see, e.g., [1, II.1.17]), one obtains from Theorem 4.1:

$$e(m, P) = \#\mathcal{G}(M, P) = \sum_{B \subseteq A} (-1)^{\#B} d(P-B)^m. \quad \square$$

An informal algorithmic description of (1) was mentioned by Beinling [4].

Now, defining

$$b(i, j, P) = \#\{B \subseteq \text{Min } P \mid \#B = i, d(P-B) = j\},$$

$$c(j, P) = \sum_{i=0}^{m(P)} (-1)^i b(i, j, P),$$

we arrive at a crucial formula:

Theorem 4.3 For any finite poset P and all natural numbers m ,

$$e(m, P) = \sum_{j=1}^{d(P)} c(j, P) \cdot j^m \quad \text{with} \quad c(d(P), P) = 1 \quad (2)$$

In particular,

$$\sum_{j=1}^{d(P)} c(j, P) = e(0, P) = 0 \quad \text{for } P \neq \emptyset, \quad \sum_{j=1}^{d(P)} c(j, P) \cdot j = e(1, P) = 1.$$

A list of all exponential functions for posets with at most five points, correcting a few misprints in [4], is presented in the Appendix of this paper. We pick two posets from that list and form their ordinal sum:

Example 4.1 $P_4 = \bullet \bullet$, $P_7 = \begin{array}{c} \bullet \\ \wedge \\ \bullet \end{array}$, $P_{12} = \begin{array}{c} \bullet \\ \wedge \\ \bullet \end{array}$, $P_{47} = \begin{array}{c} \bullet \\ \wedge \\ \bullet \\ \wedge \\ \bullet \end{array} = P_4 \oplus P_7$,

$$e(m, P_{47}) = d(P_{47})^m - 2 \cdot d(P_{12})^m + d(P_7)^m = 8^m - 2 \cdot 6^m + 5^m.$$

More involved is the computation of exponential functions for so-called fences:

Example 4.2 The fence on $\underline{2t} = \{1, \dots, 2t\}$ is defined by the special zigzag order

$$x N_t y \Leftrightarrow x = y \text{ or } (x \text{ odd and } |x - y| = 1).$$

A calculation based on Theorem 4.2 yields the following explicit formula:

$$\begin{aligned} e(m, N_t) &= \sum_{s=t}^{2t} (-1)^s \sum_{i_1 + \dots + i_{2t-s+1} = s, 2 | i_r \Leftrightarrow r=1} \left(\prod_{r=1}^{2t-s+1} f_{i_r} \right)^m \\ &= \sum_{s=0}^t (-1)^s \sum_{j_0 = \frac{1}{2}, j_1 < \dots < j_{s+1} = t+1, j_r \in \mathbb{N}} \left(\prod_{r=1}^{s+1} f_{2(j_r - j_{r-1}) - 1} \right)^m, \end{aligned}$$

where f_i is the i -th Fibonacci number:

i	0	1	2	3	4	5	6	7
f_i	1	2	3	5	8	13	21	34

$$N_1 = \updownarrow, \quad e(m, N_1) = 3^m - 2^m = f_2^m - (f_0 f_1)^m,$$

$$\begin{aligned} N_2 = \updownarrow\updownarrow, \quad e(m, N_2) &= 8^m - 6^m - 5^m + 4^m \\ &= f_4^m - (f_2 f_1)^m - (f_0 f_3)^m + (f_0 f_1 f_1)^m, \end{aligned}$$

$$\begin{aligned} N_3 = \updownarrow\updownarrow\updownarrow, \quad e(m, N_3) &= 21^m - 16^m - 15^m - 13^m + 12^m + 2 \cdot 10^m - 8^m \\ &= f_6^m - (f_4 f_1)^m - (f_2 f_3)^m - (f_0 f_5)^m + (f_2 f_1 f_1)^m + 2(f_0 f_1 f_3)^m - (f_0 f_1 f_1 f_1)^m. \end{aligned}$$

The exponential sum representation entails some divisibility properties of the numbers $e(m, P) - 1$.

Corollary 4.3 For any finite poset P , natural number n , prime number p , and $m = n(p-1) + 1$, the number $e(m, P) - 1$ is divisible by p . In particular, $e(m, P) - 1$ is always even, divisible by 6 if m is odd, by 30 if $m \equiv 1 \pmod{4}$, by 42 if $m \equiv 1 \pmod{6}$, and by 210 if $m \equiv 1 \pmod{12}$.

Proof. Theorem 4.3 and Fermat's Little Theorem give for $m = n(p-1) + 1$:

$$e(m, P) = \sum_{j=1}^{d(P)} c(j, P) \cdot j \cdot (j^n)^{p-1} \equiv \sum_{j=1}^{d(P)} c(j, P) \cdot j = 1 \pmod{p}.$$

The remaining claims are obtained for the special cases $p = 2, 3, 5, 7$. \square

For cardinal and ordinal sums of posets, the exponential functions are easily calculated from those of the summands, and the exponential behavior is also observed in the second argument:

Theorem 4.4 For disjoint finite posets O and P ,

$$e(m, O + P) = e(m, O) \cdot e(m, P) \quad (3)$$

$$e(m, O \oplus P) = \sum_{j=1}^{d(O)} c(j, O) \cdot (j + d(P) - 1)^m \quad (4)$$

provided O is nonempty; hence, the exponential function of $O \oplus P$ is obtained from that of O simply by increasing all base numbers by $d(P) - 1$.

Proof. (3) The assignment $f \mapsto (f|_O, f|_P)$ gives a map from $\mathcal{F}(M, O + P)$ to $\mathcal{F}(M, O) \times \mathcal{F}(M, P)$ that is easily seen to be bijective. Now, apply Theorem 4.1.

(4) By the equations $\text{Min}(O \oplus P) = \text{Min } O$ and, for $B \subseteq \text{Min } O$,

$$O \oplus P - B = (O - B) \oplus P, \quad d((O - B) \oplus P) = d(O - B) + d(P) - 1,$$

we get

$$\begin{aligned} b(i, j, O \oplus P) &= \#\{B \subseteq \text{Min } O \mid \#B = i, d(O \oplus P - B) = j\} \\ &= b(i, j - d(P) + 1, O), \end{aligned}$$

and consequently $c(j, O \oplus P) = c(j - d(P) + 1, O)$.

Now, $d(O \oplus P - \text{Min } O) \geq d(P)$ entails $c(j, O \oplus P) = 0$ for $j < d(P)$, and Theorem 4.3 yields

$$\begin{aligned} e(m, O \oplus P) &= \sum_{j=d(P)}^{d(O) + d(P) - 1} c(j - d(P) + 1, O) \cdot j^m \\ &= \sum_{j=1}^{d(O)} c(j, O) \cdot (j + d(P) - 1)^m. \quad \square \end{aligned}$$

Corollary 4.4 For all finite posets P disjoint from the antichain A_ℓ ,

$$e(m, A_\ell + P) = (2^m - 1)^\ell \cdot e(m, P) \quad (5)$$

$$e(m, A_\ell \oplus P) = \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} \cdot (2^i + d(P) - 1)^m \quad (6)$$

$$e(m, A_\ell \oplus A_k) = \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} \cdot (2^i + 2^k - 1)^m \quad (7)$$

In particular, for a chain C_k with k elements not in $\underline{\ell}$,

$$e(m, C_k) = (k + 1)^m - k^m \quad (8)$$

$$e(m, A_\ell + C_k) = (2^m - 1)^\ell \cdot ((k + 1)^m - k^m) \quad (9)$$

$$e(m, A_\ell \oplus C_k) = \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} \cdot (2^i + k)^m \quad (10)$$

Proof. Equation (5) is obtained by an iterated application of (3). Equation (6) results from (4), observing that $c(2^i, A_\ell) = (-1)^{\ell-i} \binom{\ell}{i}$ for $i \leq \ell$, and $c(j, A_\ell) = 0$ otherwise. The formulas (7) and (8) are special cases of (6). Equation (9) is now immediate from (5) and (8), and (10) from (6). \square

With the help of Theorem 4.4 and Corollary 4.4, one may determine recursively the exponential functions for all N -free posets. In particular, this gives all exponential functions for posets with at most four points except the fence $N = (\underline{4}, N_2)$ (see Example 4.2).


Example 4.3 For posets Q with a least element, i.e. $Q \simeq A_1 \oplus P$, (6) gives

$$e(m, Q) = d(Q)^m - (d(Q) - 1)^m,$$

and for posets of the form $Q \simeq A_2 \oplus P$, (6) amounts to

$$e(m, Q) = (d(P) + 3)^m - 2 \cdot (d(P) + 1)^m + d(P)^m.$$

For instance, this confirms the formula for $P_{47} = P_4 \oplus P_7$ in Example 4.1.

Example 4.4 For the poset $P = A_3 \oplus C_2$ with diagram , (10) gives

$$e(m, P) = \sum_{i=0}^3 \binom{3}{i} (-1)^{3-i} (2^i + 2)^m = 10^m - 3 \cdot 6^m + 3 \cdot 4^m - 3^m.$$

Some insight into the downset structure of a poset P and its extensions $Q \in \mathcal{E}(M, P)$ is encoded in a sort of 'characteristic polynomial' p_Q , defined by

$$p_Q(z) = \sum_{D \in \mathcal{D}(P)} z^{\#(M \setminus QD)} = \sum_{i=0}^m p_{Q,i} z^i,$$

where

$$p_{Q,i} = \#\{D \in \mathcal{D}(P) \mid \#(M \setminus QD) = i\},$$

in particular,

$$p_{Q,0} = p_Q(0) = \#\{D \in \mathcal{D}(P) \mid M \subseteq QD\}.$$

Lemma 4.1 For any $Q \in \mathcal{E}(M, P)$, p_Q is a normalized polynomial of degree m with $p_Q(1) = d(P)$ and $p_Q(2) = d(Q)$. Specifically,

$$\text{for } Q \in \mathcal{E}(\underline{1}, P), \quad p_Q(z) = z + d(P) - 1,$$

$$\text{for } Q \in \mathcal{E}(\underline{2}, P), \quad p_Q(z) = z^2 + (d(Q) - d(P) - 3)z - d(Q) + 2d(P) + 2,$$

$$\begin{aligned} \text{for } Q \in \mathcal{E}(\underline{3}, P), \quad p_Q(z) = & z^3 + \left(\frac{1}{2}d(Q) - d(P) + \frac{1}{2}p_{Q,0} - 3\right)z^2 \\ & - \left(\frac{1}{2}d(Q) - 2d(P) + \frac{3}{2}p_{Q,0} - 2\right)z + p_{Q,0}. \end{aligned}$$

The second part is obtained by solving an easy system of linear equations.

Corollary 4.5 For $Q, Q' \in \mathcal{E}(M, P)$,

$$p_Q = p_{Q'} \Leftrightarrow d(Q) = d(Q') \text{ in case } m \leq 2,$$

$$p_Q = p_{Q'} \Leftrightarrow (d(Q) = d(Q') \text{ and } p_{Q,0} = p_{Q',0}) \text{ in case } m = 3.$$




This is also an immediate consequence of the fact that the difference polynomial $p_Q - p_{Q'}$ has degree at most $m - 1$ and the roots 1 and 2, whence

$$p_Q(z) - p_{Q'}(z) = (z - 1)(z - 2)r(z) = (z^2 - 3z + 2) \sum_{i=0}^{m-3} r_i z^i$$

for an integer polynomial r of degree at most $m - 3$; in particular, r is an integer constant if $m \leq 3$ and equal to 0 if $m \leq 2$. The above factorization provides the difference between p_Q and $p_{Q'}$ coefficientwise:

$$p_{Q,i} - p_{Q',i} = r_{i-2} - 3r_{i-1} + 2r_i \text{ for } 0 \leq i < m, \text{ and } r_i = 0 \text{ for all other } i.$$

Example 4.5 We sketch the three posets P_n with $k_n = 5$ points, $m_n = 3$ minimal points, and $d_n = 14$ downsets (see the Appendix). One of them has height $h_n = 3$ and belongs to $\mathcal{E}(\underline{3}, C_2)$, while the other two have height $h_n = 2$ and belong to $\mathcal{E}(\underline{3}, A_2)$.

	$P = C_2$ $Q = P_{73}$		$P = A_2$ $Q = P_{78}$		$P = A_2$ $Q = P_{79}$
$p_Q(z) = z^3 + z^2 + z$		$p_Q(z) = z^3 + 3z$		$p_Q(z) = z^3 + z^2 + 2$	

The difference between the last two polynomials is just $z^2 - 3z + 2$.

5 Recursions and matrix equations

Henceforth, given a subset U of a poset P , we write $e(m, U)$ instead of $e(m, P|_U)$, regarding U as a subposet of P . We now are going to establish two recursion formulas for the exponential functions of posets P by representing $e(m, P)$ (resp. $e(m+1, P)$) as integer linear combinations of the functions $e(m, U)$, where U runs through all proper (resp. all) upsets of P .

Theorem 5.1 For all finite posets P and natural numbers m ,

$$d(P)^m = \sum_{D \in \mathcal{D}(P)} e(m, P-D) = \sum_{U \in \mathcal{U}(P)} e(m, U); \text{ hence}$$

$$r(m, P) = d(P)^m - e(m, P) = \sum_{U \in \mathcal{U}(P) \setminus \{P\}} e(m, U) \quad (11)$$

Proof. For each $U \in \mathcal{U}(P)$ let ι_U be the inclusion map from $\mathcal{U}(P|_U)$ into $\mathcal{U}(P)$. Since for $g \in \mathcal{U}(P)^M$ the union $\bigcup g[M]$ is an upset of P , we have

$$\mathcal{U}(P)^M \subseteq \bigcup_{U \in \mathcal{U}(P)} \{\iota_U \circ g \mid g \in \mathcal{G}(M, P|_U)\}.$$

Since the reverse inclusion is trivial and the union on the right is disjoint, we get the result by invoking Theorem 4.1 once more. \square

Example 5.1 For the fence $N = P_{19}$ and its eight upsets, isomorphic to P_1, P_2 (twice), P_3, P_4, P_6, P_8 , and P_{19} (see the Appendix), one obtains:

$$\begin{aligned} 8^m &= e(m, P_1) + 2e(m, P_2) + e(m, P_3) + e(m, P_4) + e(m, P_6) \\ &\quad + e(m, P_8) + e(m, P_{19}) \\ &= 1 + (2 \cdot 2^m - 2) + (3^m - 2^m) + (4^m - 2 \cdot 2^m + 1) + (5^m - 4^m) \\ &\quad + (6^m - 4^m - 3^m + 2^m) + (8^m - 6^m - 5^m + 4^m). \end{aligned}$$

In order to proceed from m to $m+1$, consider the sets

$$\mathcal{F}'(M, P) = \{(D, U, f) \mid D \in \mathcal{D}(P), U \in \mathcal{U}(P - DP), f \in \mathcal{F}(M, P - D)\}.$$

A careful distinction of cases (nine in all) shows:

Lemma 5.1 For all $(D, U, f) \in \mathcal{F}'(M, P)$ and elements i , the function

$$f_i : P \rightarrow \mathcal{P}_0(M \cup \{i\}) \text{ with}$$

$$f_i(x) = \begin{cases} \{i\} & \text{if } x \in D \\ f(x) \cup \{i\} & \text{if } x \in (U \cup DP) \setminus D \\ f(x) & \text{if } x \in K \setminus (U \cup DP) \end{cases}$$

is isotone, hence a member of $\mathcal{F}(M \cup \{i\}, P)$ if $i \notin K$.

Recall that for every subset A of a poset P on the ground set K , the set $A^\circ = K \setminus (K \setminus A)P$ is the interior in the lower Alexandroff topology $\mathcal{D}(P)$.

Theorem 5.2 For every finite poset P disjoint from the set $\underline{m+1}$, the assignment $(D, U, f) \mapsto f_{m+1}$ gives a bijection φ between $\mathcal{F}'(\underline{m}, P)$ and $\mathcal{F}(\underline{m+1}, P)$. Hence

$$e(m+1, P) = \sum_{D \in \mathcal{D}(P)} d(P - DP) e(m, P - D) = \sum_{U \in \mathcal{U}(P)} d(U^\circ) e(m, U) \quad (12)$$

Proof. Let us verify that the function below is well-defined and inverse to φ :

$$\psi : \mathcal{F}(\underline{m+1}, P) \rightarrow \mathcal{F}'(\underline{m}, P), f \mapsto (D_f, U_f, f') \text{ with}$$

$$D_f = \{x \in K \mid f(x) = \{m+1\}\},$$

$$U_f = \{x \in K \mid m+1 \in f(x)\} \setminus D_f P,$$

$$f' : K \setminus D_f \rightarrow \mathcal{P}_0(\underline{m}), x \mapsto f(x) \setminus \{m+1\}.$$

(1) ψ is well-defined: as f is isotone and each $f(x)$ is nonempty, D_f lies in $\mathcal{D}(P)$, U_f in $\mathcal{U}(P - D_f P)$, and f' is isotone, too. Furthermore, $f'(x)$ belongs to $\mathcal{P}_0(\underline{m})$ for each $x \in K \setminus D_f$, since the set $f(x)$ contains at least one element distinct from $m+1$.

(2) $(\psi \circ \varphi)(D, U, f) = (D, U, f)$: in fact, $\{x \in K \mid f_{m+1}(x) = \{m+1\}\} = D$, $\{x \in K \mid m+1 \in f_{m+1}(x)\} \setminus DP = U$, and $f_{m+1}(x) \setminus \{m+1\} = f(x)$.

(3) $(\varphi \circ \psi)(f) = f$ for $f \in \mathcal{F}(\underline{m+1}, P)$: in fact, $\psi(f) = (D_f, U_f, f')$, and by definition, $f'_{m+1} = f$.

The recursion formula for $e(m+1, P)$ is now clear, again by Theorem 4.1. \square

We find it instructive to express the explicit and recursive formulas found for $e(m, P)$ in terms of certain matrix equations. Let $(P_n \mid n \in \mathbb{N})$ be an enumeration of all unlabeled (i.e. isomorphism types of) finite posets such that $m \leq n$ implies $k_m \leq k_n$, where k_n is the ground set of P_n (see the Appendix). Then we have the following (infinite) matrices:

$$A = (a_{mn}), a_{mn} = \sum_{U \in \mathcal{U}(P_n), U \simeq P_m} d(U^\circ),$$

$$B = (b_{mn}), b_{mn} = \#\{U \in \mathcal{U}(P_n) \mid U \simeq P_m\},$$

$$C = (c_{mn}), c_{mn} = \delta_{mn} - \sum_{j=m+1}^n b_{mj} c_{jn},$$

$$D = (d_{mn}), d_{mn} = d(P_n)^m,$$

$$E = (e_{mn}), e_{mn} = e(m, P_n).$$

The columns of the matrix E represent the exponential functions $e(m, P_n)$ for fixed n , while the rows E_m list the values $e(m, P_n)$ for fixed m . Using the identity matrix $I = (\delta_{mn})$ and its upper triangular shift $\vec{I} = (\delta_{m, n-1})$, we now are able to prove some nice matrix equations.

Theorem 5.3 *A, B, and C are upper triangular matrices with*

$$(1) BC = I, \text{ hence } C = B^{-1},$$

$$(2) EB = D, \text{ hence } DC = E,$$

$$(3) EA = \vec{I}E, \text{ hence } E_m A = E_{m+1}.$$

Proof. For $m > n$, no $U \in \mathcal{U}(P_n)$ can be isomorphic to P_m , because that would imply that P_m is embeddable in P_n , and then either $k_m < k_n$ (hence $m < n$) or $\#U = k_m = k_n$ (hence $P_m \simeq U = P_n$, and so $m = n$). This shows that A and B are upper triangular, and from the recursive definition of the coefficients of C it follows that C is upper triangular, too, and inverse to B , proving (1).

(2) By Theorem 5.1, $d_{mn} = d(P_n)^m = \sum_{U \in \mathcal{U}(P_n)} e(m, U) = \sum e_{mj} b_{jn}$, whence $D = EB$, and by (1), $DC = E$.

(3) By Theorem 5.2, $(E_m A)_n$ is equal to

$$\sum_{j: U \in \mathcal{U}(P_j), U \simeq P_m} e(m, P_j) d(P_m - (P_m - U)P_m) = e(m+1, P_n) = (E_{m+1})_n. \quad \square$$

6 Estimates and asymptotical results

As expected, the largest exponential functions are obtained for antichains.

Theorem 6.1 For all posets P with k points, except the antichain A_k , and for all natural numbers $m > 1$, one has the inequality

$$e(m, P) < \left(\frac{3}{4}\right)^m (2^m - 1)^k = \left(\frac{3}{4}\right)^m e(m, A_k).$$

Proof. P contains two comparable elements. Hence, by Corollary 4.2, we obtain $e(m, P) \leq e(m, A_{k-2} + C_2)$, and so by (9),

$$e(m, P) \leq \frac{12^m - 8^m}{4^m} (2^m - 1)^{k-2} < \frac{12^m - 2 \cdot 6^m + 3^m}{4^m} (2^m - 1)^{k-2} = \left(\frac{3}{4}\right)^m (2^m - 1)^k$$

since $2 \cdot 6^2 - 3^2 = 63 < 64 = 8^2$ and $2 \cdot 6^m \leq 6^m + 2m \cdot 6^{m-1} < (6+2)^m = 8^m$ for $m > 2$. □

Compare Theorem 6.1 with the well-known inequality

$$d(P) \leq \frac{3}{4} 2^k = \frac{3}{4} d(A_k) \text{ for } P \neq A_k.$$

The leading term in an exponential sum is very dominant and makes asymptotical estimates rather easy.

Theorem 6.2 For any poset P on K , the function $e(m, P)$ is asymptotically equal to $d(P)^m$. More precisely, for

$$d'(P) = \max\{d(P|_A) \mid A \subset K\}, \text{ one has } d'(P) \leq 2^{k-1} \text{ and}$$

$$0 \leq r(m, P) = d(P)^m - e(m, P) \leq 2^{m(P)-1} d'(P)^m \leq 2^{(k-1)(m+1)} \quad (13)$$

$$e(m, P) = d(P)^m (1 - O(\varepsilon(P)^m)) \text{ with } \varepsilon(P) = \frac{d'(P)}{d(P)} < 1 \quad (14)$$

Proof. By (11), we have $0 \leq r(m, P) = d(P)^m - e(m, P)$, and by (1),

$$r(m, P) \leq \sum_{B \subseteq \text{Min } P, \#B \text{ odd}} d(P - B)^m \leq 2^{m(P)-1} d'(P)^m.$$

The asymptotical equation (14) is an immediate consequence of (13). \square

The definition of the coefficients used in Theorem 4.3 gives the inequality

$$\sum_{j=1}^{d(P)} |c(j, P)| \leq 2^{m(P)}$$

and one might conjecture that even equality holds; indeed, that is true very frequently, but not always (because certain summands $(-1)^i b(i, j, P)$ may absorb each other). For posets with up to five points, there are only two exceptions (see the Appendix):

Example 6.1 For the posets with the numbers $n=71$ and $n=81$ one has

$$P_{71} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \quad e(m, P_{71}) = 12^m - 2 \cdot 8^m + 2 \cdot 4^m - 3^m,$$

$$P_{81} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \quad e(m, P_{81}) = 16^m - 12^m - 10^m + 6^m + 5^m - 4^m,$$

$$m(P_n) = 3, \quad \sum_{j=1}^{d(P_n)} |c(j, P_n)| = 6 < 2^3.$$

Theorem 6.3 $e(m+1, P)$ is asymptotically equal to $d(P)e(m, P)$.

More precisely, for $\varepsilon(P) = \frac{d'(P)}{d(P)} \leq 1 - \frac{1}{d(P)}$ and $m \geq (m(P)+1)d(P) \ln 2$:

$$1 \leq \frac{e(m+1, P)}{d(P)e(m, P)} \leq 1 + 2^{m(P)} \varepsilon(P)^m, \quad \frac{e(m+1, P)}{d(P)e(m, P)} = 1 + O(\varepsilon(P)^m).$$

Proof. For $m \geq -\frac{(m(P)+1) \ln 2}{\ln \varepsilon(P)}$, that is, $2^{m(P)} \varepsilon(P)^m \leq 2^{-1}$, we get

$$\begin{aligned} 0 &\stackrel{(12)}{\leq} \frac{e(m+1, P)}{d(P)e(m, P)} - 1 = \frac{d(P)^{m+1} - r(m+1, P) - d(P)^{m+1} + d(P)r(m, P)}{d(P)e(m, P)} \\ &\leq \frac{r(m, P)}{e(m, P)} = \frac{r(m, P)}{d(P)^m - r(m, P)} \stackrel{(13)}{\leq} \frac{2^{m(P)-1} \varepsilon(P)^m}{1 - 2^{m(P)-1} \varepsilon(P)^m} \leq 2^{m(P)} \varepsilon(P)^m, \end{aligned}$$

and using the straightforward inequality

$$d(P) \geq -\frac{1}{\ln(1 - 1/d(P))} \geq -\frac{1}{\ln \varepsilon(P)}$$

we arrive at the claimed estimate in Theorem 6.3. \square

This estimate is quite good for large m , but not for small m , because then $\epsilon(P)^m$ may be close to 1. For posets of size k much smaller than m , the following estimates may sometimes be more effective:

Lemma 6.1 For any nonempty downset D of a finite poset P ,

$$m \cdot e(m, P-D) \leq 2^{\#(DP \setminus D)} e(m, P).$$

Proof. For each $f \in \mathcal{F}(M, P-D)$ and each $i \in M$, the map $f_i : K \rightarrow \mathcal{P}_0(M)$ with

$$f_i(x) = \begin{cases} \{i\} & \text{if } x \in D \\ f(x) \cup \{i\} & \text{if } x \in DP \setminus D \\ f(x) & \text{if } x \in K \setminus DP \end{cases}$$

is a member of $\mathcal{F}(M, P)$ (Lemma 5.1 with $U = \emptyset$). For any $g \in \mathcal{F}(M, P-D)$, the equation $f_i = g_j$ forces $i = j$ (because the set D is nonempty) and then $f(x) \subseteq f_i(x) = f(x) \cup \{i\} = g_i(x) = g(x) \cup \{i\}$ for all $x \in DP \setminus D$.

In all, there are at most $2^{\#(DP \setminus D)}$ possibilities for g so that $f_i = g_j$. Now, Theorem 4.1 yields $e(m, P-D) = \#\mathcal{F}(M, P-D)$ and so

$$m \cdot e(m, P-D) \leq 2^{\#(DP \setminus D)} \#\mathcal{F}(M, P) = 2^{\#(DP \setminus D)} e(m, P). \quad \square$$

Theorem 6.4 For all finite posets P on k points and natural numbers m ,

$$1 \leq \frac{e(m+1, P)}{d(P) e(m, P)} < 1 + m^{-1} 2^k d(P) \leq 1 + m^{-1} 4^k.$$

Proof. By Theorem 5.2, we have

$$\frac{e(m+1, P)}{d(P) e(m, P)} - 1 = \sum_{\emptyset \neq D \in \mathcal{D}(P)} \frac{d(P-D) e(m, P-D)}{d(P) e(m, P)} < d(P) \max_{\emptyset \neq D \in \mathcal{D}(P)} \frac{e(m, P-D)}{e(m, P)}.$$

Now, Lemma 6.1 and the inequality $\#(DP \setminus D) < k$ conclude the proof. \square

As in Section 5, let $(P_n : n \in \mathbb{N})$ be an enumeration of all unlabeled finite posets, arranged in such a manner that P_n is a poset on k_n and $m \leq n$ implies $k_m \leq k_n$. For any finite poset P with k points, let $a(P)$ denote the number of all automorphisms (symmetries) of P . Then $i(P) = k!/a(P)$ is the number of isomorphic copies of P on the same ground set, and

$$e_k(m) = \sum_{P \in \mathcal{Q}_0(k)} e(m, P) = \sum_{n \text{ with } k_n = k} i(P_n) e(m, P_n) \quad (15)$$

is the number of all (labeled) partial orders or posets on $\underline{m+k}$ such that \underline{m} is the set of all minimal elements. From this description it is clear that

$$e_k(1) = p(k) = \#\mathcal{Q}_0(k)$$

but it is a surprising and non-trivial fact, established in [16], that also

$$e_{k-1}(2) = p(k).$$

The asymptotical behavior of the exponential sums $e_k(m)$ for fixed k is easily determined with the help of an old result due to Stanley [33]: putting

$$z_2 = 2, z_3 = 6, z_4 = 20, \text{ and } z_i = 2i \text{ for } i \geq 5,$$

there are $z_i \binom{k}{i}$ T_0 topologies (or partial orders) on k points with $2^{k-1} + 2^{k-i}$ open sets (resp. downsets), and no others with more than 2^{k-1} open sets. Now, Theorems 6.1 and 6.2 together with $(2^m - 1)^k = 2^{km} + O(2^{(k-1)m})$ lead to the following result:

Theorem 6.5 $e_k(m)$ is asymptotically equal to 2^{km} . More precisely,

$$e_k(m) = 2^{km} + \sum_{i=2}^k z_i \binom{k}{i} (2^{k-1} + 2^{k-i})^m + O(2^{(k-1)m}) = 2^{km} (1 + O((\frac{3}{4})^m)).$$

The leading difference has an interesting number-theoretical property:

Theorem 6.6 $e_k(m) - 2^{mk} - (-1)^k$ is divisible by k if k is prime.

Proof. By (5) and (15), $e_k(m) - (2^m - 1)^k = \sum_{P_n \in \mathcal{Q}_0(k) \setminus \{A_k\}} \frac{k!}{a(P_n)} e(m, P_n)$. But if $k_n = k$ is prime, it cannot divide $a(P_n)$ unless $P_n = A_k$; indeed, if k divides $a(P_n)$ then, by Cauchy's Theorem, $\text{Aut } P_n$ must have a cyclic subgroup of order k , and consequently all points must have the same height, i.e., P_n must be an antichain. Hence, k must divide the sum on the right. The congruence $(2^m - 1)^k \equiv 2^{mk} + (-1)^k \pmod{k}$ completes the proof. \square

Finally, note a generalization of Corollary 3.5, based on Corollary 3.3:

Theorem 6.7 For any $Q \in \mathcal{E}(M, P)$,

$$d(Q) = \sum_{D \in \mathcal{D}(P)} 2^{\#(M \setminus QD)} \leq 2^{m-1} (d(P) + 1),$$

and the upper bound is attained exactly for those posets which are isomorphic to

$$A_{m-1} + (A_1 \oplus P).$$

Hence, for disjoint finite sets K, M, N with $\#M = m$ and $\#N = n$, and for any poset P on K , the number of posets $Q \in \mathcal{Q}_0(K \cup M \cup N)$ with $Q|_K = P$, $\text{Min } Q|_{K \cup M} = M$ and $\text{Min } Q = N$ is asymptotically equal to

$$2^{(m-1)n} (d(P) + 1)^n.$$

Appendix: posets with at most five points

	n	1	2	3	4	5	6	7	8	9
poset	P_n	\emptyset	\bullet	\downarrow	$\bullet \bullet$	\downarrow	\vee	\wedge	\downarrow	\dots
number of points	k_n	0	1	2		3				
number of minimal points	m_n	0	1	1	2	1	1	2	2	3
height (maximal size of chains)	h_n	0	1	2	1	3	2	2	2	1
number of automorphisms	a_n	1	1	1	2	1	2	2	1	6
number of isomorphic copies	i_n	1	1	2	1	6	3	3	6	1
number of downsets	d_n	1	2	3	4	4	5	5	6	8

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
P_n	\downarrow	\vee	\wedge	\vee	\vee	\wedge	\wedge	\downarrow	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\dots
k_n	4															
m_n	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	4
h_n	4	3	3	3	2	3	3	3	2	2	2	2	2	2	2	1
a_n	1	2	2	1	6	2	1	1	4	1	2	2	6	2	2	24
i_n	24	12	12	24	4	12	24	24	6	24	12	12	4	12	12	1
d_n	5	6	6	7	9	6	7	8	7	8	9	10	9	10	12	16

n	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
P_n	\downarrow	\vee	\wedge	\vee	\vee	\wedge	\wedge	\downarrow	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee
k_n	5																				
m_n	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
h_n	5	4	4	4	3	4	4	4	3	3	3	3	3	3	3	3	2	4	4	4	4
a_n	1	2	2	1	6	2	1	1	4	1	2	2	6	2	2	24	2	1	1	1	4
i_n	120	60	60	120	20	60	120	120	30	120	60	60	20	60	60	5	60	120	120	120	30
d_n	6	7	7	8	10	7	8	9	8	9	10	11	10	11	13	17	7	8	9	10	8

n	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67
P_n	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee
k_n	5																				
m_n	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
h_n	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2
a_n	4	2	1	2	2	2	1	1	1	1	1	1	2	2	1	1	12	2	2	2	2
i_n	30	60	120	60	60	60	120	120	120	120	120	120	60	60	120	120	10	60	60	60	60
d_n	8	9	9	10	11	9	10	10	10	11	11	12	12	12	12	14	11	12	13	14	15

n	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88
P_n	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee	\vee
k_n	5																				
m_n	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	5
h_n	2	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	1
a_n	6	6	2	2	2	1	2	12	2	2	4	2	2	1	2	4	24	6	4	6	120
i_n	20	20	60	60	60	120	60	10	60	60	30	60	60	120	60	30	5	20	30	20	1
d_n	18	10	11	12	13	14	16	11	12	13	14	14	15	16	18	20	17	18	20	24	32

The exponential functions

$e_k(m)$ denotes the number of those labeled posets on $m+k$ for which m is the set of minimal points. In particular, $e_k(0) = \delta_{k0}$. By Theorem 4.3,

$$e_k(m) = \sum_{j=1}^{2^k} \sum_{i=0}^k (-1)^i b_{ijk} j^m \quad \text{with}$$

$$b_{ijk} = \#\{(P, B) \mid P \in \mathcal{Q}_0(k), B \subseteq \text{Min } P, \#B = i, d(P-B) = j\}.$$

In particular, $b_{0jk} = p(j, k)$ and $b_{kjk} = \delta_{j1}$.

For $k \leq 5$, one obtains the following exponential sums (cf. Theorem 6.5):

$$e_0(m) = 1$$

$$e_1(m) = 2^m - 1 = 2^m + O(1)$$

$$e_2(m) = 4^m + 2 \cdot 3^m - 4 \cdot 2^m + 1 = 4^m + 2 \binom{2}{2} 3^m + O(2^m)$$

$$\begin{aligned} e_3(m) &= 8^m + 6 \cdot 6^m + 6 \cdot 5^m - 6 \cdot 4^m - 18 \cdot 3^m + 12 \cdot 2^m - 1 \\ &= 8^m + 2 \binom{3}{2} 6^m + 6 \binom{3}{3} 5^m + O(4^m) \end{aligned}$$

$$\begin{aligned} e_4(m) &= 16^m + 12 \cdot 12^m + 24 \cdot 10^m + 20 \cdot 9^m + 16 \cdot 8^m + 54 \cdot 7^m - 108 \cdot 6^m \\ &\quad - 96 \cdot 5^m + 108 \cdot 3^m - 32 \cdot 2^m + 1 \\ &= 16^m + 2 \binom{4}{2} 12^m + 6 \binom{4}{3} 10^m + 20 \binom{4}{4} 9^m + O(8^m) \end{aligned}$$

$$\begin{aligned} e_5(m) &= 32^m + 20 \cdot 24^m + 60 \cdot 20^m + 100 \cdot 18^m + 10 \cdot 17^m + 100 \cdot 16^m \\ &\quad + 120 \cdot 15^m + 390 \cdot 14^m + 240 \cdot 13^m - 180 \cdot 12^m + 500 \cdot 11^m - 540 \cdot 10^m \\ &\quad - 300 \cdot 9^m - 830 \cdot 8^m - 1650 \cdot 7^m + 1200 \cdot 6^m + 900 \cdot 5^m + 320 \cdot 4^m \\ &\quad - 540 \cdot 3^m + 80 \cdot 2^m - 1 \\ &= 32^m + 2 \binom{5}{2} 24^m + 6 \binom{5}{3} 20^m + 20 \binom{5}{4} 18^m + 10 \binom{5}{5} 17^m + O(16^m) \end{aligned}$$

These sums are obtained by summation of the exponential functions of all posets P_n with $k_n = k$, taken with multiplicity $i_n = i(P_n)$, as listed on the following two pages. The total number of posets with k points is then

$$p(k) = e_k(1) = e_{k-1}(2) = \sum_{m=1}^k \binom{k}{m} e_{k-m}(m)$$

$$p(1) = 1$$

$$p(2) = 3$$

$$p(3) = 19$$

$$p(4) = 219$$

$$p(5) = 4231$$

$$p(6) = 130023$$

n	i_n	$e(m, P_n)$
1	1	1^m
2	1	$2^m - 1^m$
3	2	$3^m - 2^m$
4	1	$4^m - 2 \cdot 2^m + 1^m$
5	6	$4^m - 3^m$
6	3	$5^m - 4^m$
7	3	$5^m - 2 \cdot 3^m + 2^m$
8	6	$6^m - 4^m - 3^m + 2^m$
9	1	$8^m - 3 \cdot 4^m + 3 \cdot 2^m - 1^m$
10	24	$5^m - 4^m$
11	12	$6^m - 5^m$
12	12	$6^m - 5^m$
13	24	$7^m - 6^m$
14	4	$9^m - 8^m$
15	12	$6^m - 2 \cdot 4^m + 3^m$
16	24	$7^m - 5^m - 4^m + 3^m$
17	24	$8^m - 6^m - 4^m + 3^m$
18	6	$7^m - 2 \cdot 5^m + 4^m$
19	24	$8^m - 6^m - 5^m + 4^m$
20	12	$9^m - 2 \cdot 6^m + 4^m$
21	12	$10^m - 8^m - 5^m + 4^m$
22	4	$9^m - 3 \cdot 5^m + 3 \cdot 3^m - 2^m$
23	12	$10^m - 2 \cdot 6^m - 5^m + 4^m + 2 \cdot 3^m - 2^m$
24	12	$12^m - 8^m - 2 \cdot 6^m + 2 \cdot 4^m + 3^m - 2^m$
25	1	$16^m - 4 \cdot 8^m + 6 \cdot 4^m - 4 \cdot 2^m + 1^m$
26	120	$6^m - 5^m$
27	60	$7^m - 6^m$
28	60	$7^m - 6^m$
29	120	$8^m - 7^m$
30	20	$10^m - 9^m$
31	60	$7^m - 6^m$
32	120	$8^m - 7^m$
33	120	$9^m - 8^m$
34	30	$8^m - 7^m$
35	120	$9^m - 8^m$
36	60	$10^m - 9^m$
37	60	$11^m - 10^m$
38	20	$10^m - 9^m$
39	60	$11^m - 10^m$
40	60	$13^m - 12^m$
41	5	$17^m - 16^m$
42	60	$7^m - 2 \cdot 5^m + 4^m$
43	120	$8^m - 6^m - 5^m + 4^m$
44	120	$9^m - 7^m - 5^m + 4^m$

n	i_n	$e(m, P_n)$
45	120	$10^m - 8^m - 5^m + 4^m$
46	30	$8^m - 2 \cdot 6^m + 5^m$
47	30	$8^m - 2 \cdot 6^m + 5^m$
48	60	$9^m - 2 \cdot 7^m + 6^m$
49	120	$9^m - 7^m - 6^m + 5^m$
50	60	$10^m - 2 \cdot 7^m + 5^m$
51	60	$11^m - 9^m - 6^m + 5^m$
52	60	$9^m - 7^m - 6^m + 5^m$
53	120	$10^m - 8^m - 7^m + 6^m$
54	120	$10^m - 8^m - 6^m + 5^m$
55	120	$10^m - 8^m - 7^m + 6^m$
56	120	$11^m - 2 \cdot 8^m + 6^m$
57	120	$11^m - 9^m - 7^m + 6^m$
58	120	$12^m - 10^m - 7^m + 6^m$
59	60	$12^m - 10^m - 6^m + 5^m$
60	60	$12^m - 10^m - 6^m + 5^m$
61	120	$12^m - 9^m - 8^m + 6^m$
62	120	$14^m - 12^m - 7^m + 6^m$
63	10	$11^m - 2 \cdot 9^m + 8^m$
64	60	$12^m - 10^m - 9^m + 8^m$
65	60	$13^m - 2 \cdot 10^m + 8^m$
66	60	$14^m - 12^m - 9^m + 8^m$
67	60	$15^m - 12^m - 10^m + 8^m$
68	20	$18^m - 16^m - 9^m + 8^m$
69	20	$10^m - 3 \cdot 6^m + 3 \cdot 4^m - 3^m$
70	60	$11^m - 2 \cdot 7^m - 6^m + 5^m + 2 \cdot 4^m - 3^m$
71	60	$12^m - 2 \cdot 8^m + 2 \cdot 4^m - 3^m$
72	60	$13^m - 9^m - 2 \cdot 7^m + 2 \cdot 5^m + 4^m - 3^m$
73	120	$14^m - 10^m - 8^m - 7^m + 6^m + 5^m + 4^m - 3^m$
74	60	$16^m - 12^m - 2 \cdot 8^m + 2 \cdot 6^m + 4^m - 3^m$
75	10	$11^m - 3 \cdot 7^m + 3 \cdot 5^m - 4^m$
76	60	$12^m - 2 \cdot 8^m - 7^m + 6^m + 2 \cdot 5^m - 4^m$
77	60	$13^m - 9^m - 2 \cdot 8^m + 2 \cdot 6^m + 5^m - 4^m$
78	30	$14^m - 2 \cdot 10^m + 8^m - 7^m + 2 \cdot 5^m - 4^m$
79	60	$14^m - 10^m - 2 \cdot 8^m + 2 \cdot 6^m + 5^m - 4^m$
80	60	$15^m - 10^m - 2 \cdot 9^m + 3 \cdot 6^m - 4^m$
81	120	$16^m - 12^m - 10^m + 6^m + 5^m - 4^m$
82	60	$18^m - 2 \cdot 12^m - 9^m + 8^m + 2 \cdot 6^m - 4^m$
83	30	$20^m - 16^m - 2 \cdot 10^m + 2 \cdot 8^m + 5^m - 4^m$
84	5	$17^m - 4 \cdot 9^m + 6 \cdot 5^m - 4 \cdot 3^m + 2^m$
85	20	$18^m - 3 \cdot 10^m - 9^m + 3 \cdot 6^m + 3 \cdot 5^m - 4^m - 3 \cdot 3^m + 2^m$
86	30	$20^m - 2 \cdot 12^m - 2 \cdot 10^m + 8^m + 4 \cdot 6^m + 5^m - 2 \cdot 4^m - 2 \cdot 3^m + 2^m$
87	20	$24^m - 16^m - 3 \cdot 12^m + 3 \cdot 8^m + 3 \cdot 6^m - 3 \cdot 4^m - 3^m + 2^m$
88	1	$32^m - 5 \cdot 16^m + 10 \cdot 8^m - 10 \cdot 4^m + 5 \cdot 2^m - 1$

In the same way one calculates the following two types of exponential sums. Let $e_{kn}(m)$ denote the exponential sum counting all posets Q on $M \cup K$ such that $\# \text{Min}(Q|_K) = n$ and $\text{Min} Q = M$. Then, by Corollary 3.5 (with n for m) and Theorem 4.2, the leading term of $e_{kn}(m)$ is $\binom{k}{n} \cdot n \cdot (2^{k-1} + 2^{n-1})^m$ if $n < k$, and 2^{km} if $n = k$.

$$e_{11}(m) = 2^m - 1$$

$$e_{21}(m) = 2 \cdot 3^m - 2 \cdot 2^m$$

$$e_{22}(m) = 4^m - 2 \cdot 2^m + 1$$

$$e_{31}(m) = 3 \cdot 5^m + 3 \cdot 4^m - 6 \cdot 3^m$$

$$e_{32}(m) = 6 \cdot 6^m + 3 \cdot 5^m - 6 \cdot 4^m - 12 \cdot 3^m + 9 \cdot 2^m$$

$$e_{33}(m) = 8^m - 3 \cdot 4^m + 3 \cdot 2^m - 1$$

$$e_{41}(m) = 4 \cdot 9^m - 4 \cdot 8^m + 24 \cdot 7^m - 24 \cdot 4^m$$

$$e_{42}(m) = 12 \cdot 10^m + 12 \cdot 9^m + 36 \cdot 8^m + 30 \cdot 7^m - 60 \cdot 6^m - 72 \cdot 5^m - 18 \cdot 4^m + 60 \cdot 3^m$$

$$e_{43}(m) = 12 \cdot 12^m + 12 \cdot 10^m + 4 \cdot 9^m - 12 \cdot 8^m - 48 \cdot 6^m - 24 \cdot 5^m + 36 \cdot 4^m + 48 \cdot 3^m - 28 \cdot 2^m$$

$$e_{44}(m) = 16^m - 4 \cdot 8^m + 6 \cdot 4^m - 4 \cdot 2^m + 1$$

$$e_{51}(m) = 5 \cdot 17^m - 5 \cdot 16^m + 60 \cdot 13^m - 60 \cdot 12^m + 120 \cdot 11^m - 20 \cdot 10^m + 140 \cdot 9^m + 30 \cdot 8^m - 90 \cdot 7^m - 60 \cdot 6^m - 120 \cdot 5^m$$

$$e_{52}(m) = 20 \cdot 18^m - 20 \cdot 16^m + 60 \cdot 15^m + 180 \cdot 14^m + 60 \cdot 13^m + 180 \cdot 12^m + 310 \cdot 11^m + 60 \cdot 10^m - 100 \cdot 9^m - 390 \cdot 8^m - 1080 \cdot 7^m + 180 \cdot 6^m + 120 \cdot 5^m + 420 \cdot 4^m$$

$$e_{53}(m) = 30 \cdot 20^m + 60 \cdot 18^m + 150 \cdot 16^m + 60 \cdot 15^m + 210 \cdot 14^m + 120 \cdot 13^m - 180 \cdot 12^m + 70 \cdot 11^m - 460 \cdot 10^m - 300 \cdot 9^m - 570 \cdot 8^m - 480 \cdot 7^m + 840 \cdot 6^m + 780 \cdot 5^m + 50 \cdot 4^m - 380 \cdot 3^m$$

$$e_{54}(m) = 20 \cdot 24^m + 30 \cdot 20^m + 20 \cdot 18^m + 5 \cdot 17^m - 20 \cdot 16^m - 120 \cdot 12^m - 120 \cdot 10^m - 40 \cdot 9^m + 90 \cdot 8^m + 240 \cdot 6^m + 120 \cdot 5^m - 140 \cdot 4^m - 160 \cdot 3^m + 75 \cdot 2^m$$

$$e_{55}(m) = 32^m - 5 \cdot 16^m + 10 \cdot 8^m - 10 \cdot 4^m + 5 \cdot 2^m - 1$$

Let $e_k^h(m)$ denote the exponential sum counting all posets Q on $M \cup K$ of height $h+1$ with $\text{Min} Q = M$. By Corollary 3.6 (with n for m) and Corollary 4.4, we have

$$e_k^1(m) = (2^m - 1)^k, \quad e_k^h(m) = (k)_h (2^{k-h}(h+1))^m + \dots \quad (1 < h < k), \quad e_k^k(m) = k!((k+1)^m - k^m).$$

$$e_3^1(m) = 8^m - 3 \cdot 4^m + 3 \cdot 2^m - 1$$

$$e_3^2(m) = 6 \cdot 6^m + 6 \cdot 5^m - 9 \cdot 4^m - 12 \cdot 3^m + 9 \cdot 2^m$$

$$e_3^3(m) = 6 \cdot 4^m - 6 \cdot 3^m$$

$$e_4^1(m) = 16^m - 4 \cdot 8^m + 6 \cdot 4^m - 4 \cdot 2^m + 1$$

$$e_4^2(m) = 12 \cdot 12^m + 24 \cdot 10^m + 20 \cdot 9^m - 4 \cdot 8^m + 6 \cdot 7^m - 96 \cdot 6^m - 72 \cdot 5^m + 90 \cdot 4^m + 48 \cdot 3^m - 28 \cdot 2^m$$

$$e_4^3(m) = 24 \cdot 8^m + 48 \cdot 7^m - 12 \cdot 6^m - 48 \cdot 5^m - 72 \cdot 4^m + 60 \cdot 3^m$$

$$e_4^4(m) = 24 \cdot 5^m - 24 \cdot 4^m$$

$$e_5^1(m) = 32^m - 5 \cdot 16^m + 10 \cdot 8^m - 10 \cdot 4^m + 5 \cdot 2^m - 1$$

$$e_5^2(m) = 20 \cdot 24^m + 60 \cdot 20^m + 100 \cdot 18^m + 10 \cdot 17^m + 45 \cdot 16^m + 120 \cdot 15^m + 150 \cdot 14^m + 120 \cdot 13^m - 360 \cdot 12^m + 20 \cdot 11^m - 720 \cdot 10^m - 440 \cdot 9^m + 150 \cdot 8^m - 120 \cdot 7^m + 960 \cdot 6^m + 600 \cdot 5^m - 630 \cdot 4^m - 160 \cdot 3^m + 75 \cdot 2^m$$

$$e_5^3(m) = 60 \cdot 16^m + 240 \cdot 14^m + 120 \cdot 13^m + 180 \cdot 12^m + 480 \cdot 11^m + 60 \cdot 10^m - 100 \cdot 9^m - 1110 \cdot 8^m - 1410 \cdot 7^m + 420 \cdot 6^m + 900 \cdot 5^m + 540 \cdot 4^m - 380 \cdot 3^m$$

$$e_5^4(m) = 120 \cdot 10^m + 240 \cdot 9^m + 120 \cdot 8^m - 120 \cdot 7^m - 300 \cdot 6^m - 480 \cdot 5^m + 420 \cdot 4^m$$

$$e_5^5(m) = 120 \cdot 6^m - 120 \cdot 5^m$$

The representing matrices

\emptyset	\bullet	\vdots	\dots	\vdots	\vee	\wedge	\cdot	\dots	\vdots	\vee	\vee	\vee	\vee	\wedge	\wedge	\cdot	\dots	\vdots	\vee	\wedge	\wedge	\cdot	\dots	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

A: $a_{mn} = \sum_{P_m \simeq U \in \mathcal{U}(P_n)} d(U^\circ) (= \delta_{mn} d(P_n) \text{ for } m \geq 10 \text{ and } n \leq 25)$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	4	1	2	1	3	6	1	2	1	2	3	1	1	3	2	2	2	4	1	3	5	8
		3	0	1	0	4	3	0	1	0	2	1	0	1	3	1	0	2	6	0	6	4	3	0
			4	0	1	0	2	12	0	1	0	1	3	0	0	2	1	1	1	5	0	2	8	24
				4	0	0	0	0	1	0	0	0	0	4	3	4	0	0	0	0	0	0	0	0
					5	0	0	0	0	1	0	0	0	0	0	0	4	3	0	5	0	0	0	0
						5	0	0	0	0	1	0	0	0	2	0	0	0	0	0	12	5	0	0
							6	0	0	0	0	1	0	0	0	2	0	2	6	0	0	8	12	0
								8	0	0	0	0	1	0	0	0	0	0	0	2	0	0	4	32

B: $b_{mn} = \#\{U \in \mathcal{U}(P_n), U \simeq P_m\} (= \delta_{mn} \text{ for } m \geq 10 \text{ and } n \leq 25)$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	2	1	2	1	2	3	1	2	1	2	3	1	1	2	2	2	2	3	1	2	3	4
		1	0	1	0	2	1	0	1	0	2	1	0	1	2	1	0	1	2	0	3	2	1	0
			1	0	1	0	1	3	0	1	0	1	3	0	0	1	1	1	1	3	0	1	3	6
				1	0	0	0	0	1	0	0	0	0	2	1	1	0	0	0	0	0	0	0	0
					1	0	0	0	0	1	0	0	0	0	0	0	2	1	0	1	0	0	0	0
						1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	3	1	0	0
							1	0	0	0	1	0	0	0	1	0	1	2	0	0	2	2	0	0
								1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	4

C: $c_{mn} = \delta_{mn} - \sum_{j < m} b_{mj} c_{jn} (= \delta_{mn} \text{ for } m \geq 10 \text{ and } n \leq 25)$

1	-1	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	1	-1	-2	0	0	1	1	3	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-4
		1	0	-1	0	-2	-1	0	0	0	0	0	1	1	1	0	0	0	0	0	3	2	1	0
			1	0	-1	0	-1	-3	0	0	0	0	0	0	0	0	1	1	1	1	0	1	2	6
				1	0	0	0	0	-1	0	0	0	0	-2	-1	-1	0	0	0	0	0	0	0	0
					1	0	0	0	0	-1	0	0	0	0	0	0	-2	-1	0	-1	0	0	0	0
						1	0	0	0	0	-1	0	0	0	-1	0	0	0	0	0	-3	-1	0	0
							1	0	0	0	0	-1	0	0	0	-1	0	-1	-2	0	0	-2	-2	0
								1	0	0	0	0	-1	0	0	0	0	0	0	-1	0	0	-1	-4

D: $d_{mn} = d(P_n)^m$

1^m	2^m	3^m	4^m	4^m	5^m	5^m	6^m	8^m	5^m	6^m	6^m	7^m	9^m	6^m	7^m	8^m	7^m	8^m	9^m	10^m	9^m	10^m	12^m	16^m
-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	--------	-------	--------	--------	--------

E^T: $e_{mn} = e(m, P_n) = \#\mathcal{E}(\underline{m}, P_n) = \#\mathcal{F}(\underline{m}, P_n) = \#\mathcal{G}(\underline{m}, P_n) = \#\mathcal{H}(\underline{m}, P_n)$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383	32767	65535	131071	262143	524287	1048575	2097151	4194303	8388607	16777215	$2^m - 1$
1	5	19	65	211	665	2059	6305	19171	58513	175535	526607	1580023	4740071	14120215	42360647	127081943	381245831	1143737495	3431212487	10293637463	30880912389	92642737167	277928211503	$3^m - 2^m$
1	9	49	225	961	3969	16129	65025	261121	1004817	3981017	15524065	60096257	231377009	885443841	3381675361	12726053441	47457792001	175002007041	630007225441	2268025611041	8164892200001	29294811920001	105043324880001	$4^m - 2 \cdot 2^m + 1$
1	7	37	175	781	3367	14197	58975	242461	977817	3911265	15645065	62580257	249921009	979684041	3878736161	15115044641	59660178561	232640714241	910552856961	3562011427841	13927645711361	53704172845441	207816651381761	$4^m - 3^m$
1	9	61	369	2101	11529	61741	325089	1690981	8913217	46141609	237169281	1211250561	6161252801	31366264001	159292320001	812171776001	4110858752001	20744293760001	105181293440001	527406471680001	2647032358400001	13235161792000001	66175808960000001	$5^m - 4^m$
1	11	79	479	2671	14231	73879	377759	1914271	9771361	49811361	254061761	1270308801	6351544001	31757720001	158788600001	793943000001	3969715000001	19848575000001	99242875000001	496214375000001	2481071875000001	12405359375000001	62026796875000001	$5^m - 2 \cdot 3^m + 2^m$
1	15	133	975	6541	41895	261493	1607775	9796381	58778281	358670081	2216688001	13856050001	86600312001	541251937601	3382824582401	21172653616001	13230158260001	82938466624001	518365416400001	3239784102400001	2024865064000001	12655406656000001	79100041600000001	$6^m - 4^m - 3^m + 2^m$
1	27	343	3375	29791	250047	2048383	16581375	133432831	1067462657	8539701247	68317610001	546540880001	4372327040001	34978616320001	283828930560001	2270631444480001	18165051555840001	145320412446720001	11625633000000001	93005064000000001	744040512000000001	5952324096000000001	47618592768000000001	$8^m - 3 \cdot 4^m + 3 \cdot 2^m - 1$

References

- [1] M. Aigner: *Kombinatorik I. Grundlagen und Zähltheorie*. Springer-Verlag, Berlin – Heidelberg – New York, 1975.
- [2] P. Alexandroff: Sur les espaces discrets. *Comptes Rendus de l'Académie des Sciences* **200** (1935), 1469-1471.
- [3] P. Alexandroff: Diskrete Räume. *Mat. Sb. (N.S.)* **2** (1937), 501-518.
- [4] M. Beinling: *Kombinatorik geordneter Mengen und ihrer Antiketten*. Diploma Thesis, University of Hannover, 2006.
- [5] G. Birkhoff: *Lattice Theory*. Amer. Math. Soc. Coll. Publ. **25**, 3rd ed., Providence, R. I., 1973.
- [6] M. Benoumhani: The number of topologies on a finite set. *J. Integer Seq.* **9** (2006), 06.2.6.
- [7] M. Benoumhani, M. Kolli: Finite topologies and partitions. *J. Integer Seq.* **13** (2010), 10.3.5.
- [8] G. Brinkmann, B.D. McKay: Posets on up to 16 Points. *Order* **19** (2002), 147–179.
- [9] G. Brinkmann, B.D. McKay: Counting unlabelled topologies and transitive relations. *J. Integer Seq.* **8** (2005), Article 05.2.1.
- [10] K.K.-H. Butler: The number of partially ordered sets. *J. Combin. Theory (B)* **13** (1972), 276–289.
- [11] K.K.-H. Butler, G. Markowsky: Enumeration of finite topologies. In: *Proc. 4th Southeastern Conf. on Combinatorics, Graph Theory and Computing (Winnipeg) 1973*, pp. 169-184.
- [12] J.C. Culberson and J.E. Rawlins: New results from an algorithm for counting posets. *Order* **7** (1991), 361–374.
- [13] P. Erdős, M. Erné: Clique numbers in graphs. *Discrete Math.* **59** (1986), 235–242.
- [14] M. Erné: *Struktur- und Anzahlformeln für Topologien auf endlichen Mengen*. PhD Thesis, Westf. Wilhelms-Universität Münster, 1972.
- [15] M. Erné: *Struktur- und Anzahlformeln für Topologien auf endlichen Mengen*. *Manuscripta Math.* **11** (1974), 221-259.
- [16] M. Erné: On the cardinalities of finite topologies and the number of antichains in partially ordered sets. *Discrete Math.* **35** (1981), 119–133.
- [17] M. Erné: Lattice representations for categories of closure spaces. In: H.L. Bentley et al. (eds.), *Categorical Topology, Proc. Conference Toledo, Ohio 1983*, Heldermann, Berlin, 1984, 197–222.
- [18] M. Erné: The ABC of order and topology. In: H. Herrlich, H.E. Porst (eds.), *Category Theory at Work*, Heldermann, Berlin, 1991, 57-83.

- [19] M. Erné: The number of partially ordered sets with more points than incomparable pairs. *Discrete Math.* **105** (1992), 49–60.
- [20] M. Erné: Algebraic ordered sets and their generalizations. In: I. Rosenberg and G. Sabidussi (eds.), *Algebras and Orders, Proc. Montreal 1991*, NATO ASI, Series C, Vol. 389, Kluwer Acad. Publ., Dordrecht-Boston-London, 1993, 113–192.
- [21] M. Erné: *Ordnungskombinatorik*. Lecture Notes, University of Hannover, 2005. <http://www.iazd.uni-hannover.de/~erne/>
- [22] M. Erné, J. Heitzig, J. Reinhold: On the number of distributive lattices. *Electronic J. Comb.* **9** (2002), R24.
- [23] M. Erné, K. Stege: Counting finite posets and topologies. *Order* **8** (1991), 247–265.
- [24] M. Erné, K. Stege: Combinatorial applications of ordinal sum decompositions. *Ars Combinatoria* **40** (1995), 65–88.
- [25] J. Heitzig, J. Reinhold: The number of unlabeled orders on fourteen elements. *Order* **17** (2000), 333–341.
- [26] M. Kolli: Direct and elementary approach to enumerate topologies on a finite set. *J. Integer Seq.* **10** (2007), 07.3.1.
- [27] M. Kolli: On the cardinality of the T_0 -topologies on a finite set. *Intern. J. of Combin.* (2014), ID 798074.
- [28] R.H. Möhring: Computationally tractable classes of ordered sets. In: Rival, Ivan (ed.), *Algorithms and Order. Proceedings of the NATO Advanced Study Institute on Algorithms and Order, Ottawa, Canada, 1987*. NATO Science Series C, **255**, Springer-Verlag, Berlin-Heidelberg-New York, 105–194.
- [29] R. Parchmann: On the cardinalities of finite topologies. *Discrete Math.* **11** (1975), 161–172.
- [30] H. Sharp: Quasi-orderings and topologies on finite sets. *Proc. Amer. Math. Soc.* **17** (1966), 1344–1349.
- [31] H. Sharp: Cardinality of finite topologies. *J. Combin. Theory* **5** (1968), 82–86.
- [32] Z. Shmueli: The structure of Galois connections. *Pacific J. Math.* **54** (1974), 209–225.
- [33] R.P. Stanley: On the number of open sets of finite topologies. *J. Combin. Theory (A)* **10** (1971), 74–79.
- [34] R.P. Stanley: Enumeration of posets generated by disjoint unions and ordinal sums. *Proc. Amer. Math. Soc.* **45** (1974), 295–299.
- [35] U. Vollert: Mächtigkeiten von Topologien auf endlichen Mengen und Cliquenzahlen in endlichen Graphen. PhD Thesis, U. Hannover, 1987.