$GDD(n, 2, 4; \lambda_1, \lambda_2)$ with Equal Number of Blocks of Three Configurations

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Abstract

Recently GDDs with two groups and block size four were studied in a paper where the authors constructed two families out of four possible cases with an equal number of even, odd, and group blocks. In this paper, we prove partial existence of one of the two remaining families, namely $GDD(11t + 1, 2, 4; 11t + 1, 7t)$, with $7 \nmid (11t + 1)$. In addition, we show a useful construction of $GDD(6t + 4, 2, 4; 2, 3)$.

1 Introduction

Group Divisible Designs (GDDs) are building blocks for constructions of many other combinatorial designs, those including Balanced Incomplete Block Designs. These designs and other combinatorial structures play an important role in the constructions of GDDs as well.

Definition 1.1. *A group divisible design $GDD(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets, called blocks, of an nm-set X where the elements of X are partitioned into m subsets (called groups) of size n each; pairs of distinct elements within the same group are called first associates of each other and appear together in λ_1 blocks while any two elements not in the same group are called second associates and appear together in λ_2 blocks.*

Definition 1.2. *A Balanced Incomplete Block Design, $BIBD(v, b, r, k, \lambda)$, is a collection of b k-subsets (called blocks) of a v-set V, such that each element appears in exactly r blocks, every pair of distinct elements of V occurs in λ blocks and $k < v$. A $BIBD(v, b, r, k, \lambda)$ can also be denoted by a $BIBD(v, k, \lambda)$.*

. A $GDD(v, k, k; 0, 1)$ is also called a transversal design $TD(k, v)$, and it has v^2 blocks. GDDs are building blocks for constructions of many other designs including balanced incomplete block designs defined below.

Another helpful tool for constructing the GDDs in question is the factorization of a complete graphs. A complete graph K_n is a graph on n vertices where each distinct pair of vertices is connected by an edge. Later, we use the edges of certain complete graphs, paired with a single element, in order to achieve required λ_2 values, whereas the Balanced Incomplete Block Designs are useful for contributing to λ_1 value of the GDDs.

2 Necessary Conditions

The replication number r of an element x, i.e. the number of blocks in which the element x appears in the GDD$(n, 2, 4; \lambda_1, \lambda_2)$ can be found by $r = \frac{\lambda_1(n-1)+\lambda_2 n}{3}$. Clearly r is independent of the element chosen. Therefore, $\lambda_1(n-1) + \lambda_2 n \equiv 0 \pmod 3$. Let b be the number of blocks of the GDD. Since each of the $2n$ elements occur in r blocks, $2nr = 4b$ and $b = \frac{n(n(\lambda_1+\lambda_2)-\lambda_1)}{6}$. Both b and r must be integers.

A block of size four with elements from either one or both of two groups can have only three intersection patterns: $(1,3)$, which is called an odd block, $(2,2)$, which is called an even block, and $(0,4)$, which is called a group block. Nanfuka and Sarvate have studied such GDDs with equal numbers of all three block types in [11]. However, only two of the four cases described in [11] have been constructed.

As seen above, $\frac{b}{3} = \frac{n(n(\lambda_1+\lambda_2)-\lambda_1)}{18}$ is the number of blocks of each type, and, from [11], $\lambda_1 = \frac{11n\lambda_2}{7(n-1)}$. The two remaining cases for which existence is not known are:

(i) GDD$(11t+1, 2, 4; 11t+1, 7t)$

(ii) GDD$(77t+56, 2, 4; 11t+8, 7t+5)$

3 GDD$(11t+1, 2, 4; 11t+1, 7t)$

In order to show that such a GDD with equal number of even, odd, and group blocks exists for all $t \geq 1$, one should look at the cases of t modulo 12.

3.1 $t \equiv 1 \pmod{12}$

Let $t = 12s+1$ for some integer $t \geq 1$. Then, the GDD to is construct GDD$(132s+12, 2, 4; 132s+12, 84s+7)$. Let $G_1 = \{x_1, \ldots, x_{132s+12}\}$ and $G_2 = \{y_1, \ldots, y_{132s+12}\}$. Split the groups into three subsets each in a natural way, e.g. split G_1 into $A_1 = \{x_1, \ldots, x_{44s+4}\}$, $B_1 = \{x_{44s+5}, \ldots, x_{88s+8}\}$, and $C_1 = \{x_{88s+9}, \ldots, x_{132s+12}\}$. Similarly, split G_2 into A_2, B_2, and C_2. It is required that there are $\frac{b}{3} = \frac{\lambda_2 n^2}{7} = (12s+1)(132s+12)^2$ blocks of each type. In order to get odd blocks, use the blocks of TD$(44s+4, 4, 4; 0, 1)$ on certain groups as described in Table 1.

These blocks contribute $3(12s+1)$ to the count of λ_2. However, this will contribute to λ_1 count in between the groups A_1B_1, A_1C_1, and B_1C_1 by $6(12s+1)$ and between the groups A_2B_2, A_2C_2, and B_2C_2 by $3(12s+1)$. In order to get even blocks, use the blocks of TD$(44s+4, 4, 4; 0, 1)$ on certain groups as described in Table 2.

Table 1:

A_1	A_2	B_2	C_2	A_2	A_1	B_1	C_1
B_1	A_2	B_2	C_2	B_2	A_1	B_1	C_1
C_1	A_2	B_2	C_2	C_2	A_1	B_1	C_1
$(12s+1)$ Copies				$2(12s+1)$ Copies			

Table 2:

A_1	B_1	A_2	B_2	A_1	B_1	A_2	C_2	A_1	B_1	B_2	C_2
A_1	C_1	A_2	B_2	A_1	C_1	A_2	C_2	A_1	C_1	B_2	C_2
B_1	C_1	A_2	B_2	B_1	C_1	A_2	C_2	B_1	C_1	B_2	C_2
$(12s+1)$ Copies				$(12s+1)$ Copies				$(12s+1)$ Copies			

These contribute to λ_2 count by $4(12s+1)$, making $\lambda_2 = 7(12s+1)$ as required by the necessary conditions. In addition, it will contribute to λ_1 count between A_1B_1, A_1C_1, B_1C_1, A_2B_2, A_2C_2, and B_2C_2 by $3(12s+1)$. From here, use the blocks of $BIBD(132s+12, 4, 24s+3)$ on G_1 and $BIBD(132s+12, 4, 60s+6)$ on G_2 to get required λ_1 between groups. To get required λ_1 within each group, we use the blocks of $BIBD(44s+4, 4, 108s+9)$ on each of A_1, B_1, and C_1 and the blocks of $BIBD(44s+4, 4, 72s+6)$ on each of A_2, B_2, and C_2. One can see that this will make $\lambda_1 = 132s+12$ within groups as well as get our required number of group blocks.

3.2 $t \equiv 4, 10 \pmod{12}$

Let $t = 6s+4$ for some integer $s \geq 1$. The aim now is to construct GDD$(66s+45, 2, 4; 66s+45, 42s+28)$. Let $G_1 = \{x_1, \ldots, x_{66s+45}\}$ and $G_2 = \{y_1, \ldots, y_{66s+45}\}$. From here, split G_1 into three subsets each:

(i) $A_1 = \{x_1, \ldots, x_{22s+15}\}$,

(ii) $B_1 = \{x_{22s+16}, \ldots, x_{44s+30}\}$,

(iii) $C_1 = \{x_{44s+31}, \ldots, x_{66s+45}\}$.

And similarly, split G_2 into A_2, B_2, and C_2. From the necessary conditions, it is required to have $\frac{b}{3} = \frac{\lambda_2 n^2}{7} = (6s+4)(66s+45)^2$ blocks of each type. In order to get the odd blocks, use the blocks of TD$(22s+15, 4, 4; 0, 1)$ on certain groups as described in Table 3.

This will contribute $3(6s+4)$ towards λ_2. In addition, λ_1 increases by $9(3s+2)$ between the elements of the subsets of the groups. From here, in order to get our even blocks, use the blocks of TD$(22s+15, 4, 4; 0, 1)$ on certain groups as described in Table 4.

Table 3:

A_1	A_2	B_2	C_2	A_2	A_1	B_1	C_1
B_1	A_2	B_2	C_2	B_2	A_1	B_1	C_1
C_1	A_2	B_2	C_2	C_2	A_1	B_1	C_1
$3(3s+2)$ Copies				$3(3s+2)$ Copies			

Table 4:

A_1	B_1	A_2	B_2	A_1	B_1	A_2	C_2	A_1	B_1	B_2	C_2
A_1	C_1	A_2	B_2	A_1	C_1	A_2	C_2	A_1	C_1	B_2	C_2
B_1	C_1	A_2	B_2	B_1	C_1	A_2	C_2	B_1	C_1	B_2	C_2
$(6s+4)$ Copies				$(6s+4)$ Copies				$(6s+4)$ Copies			

This contributes $4(6s+4)$ to λ_2 count, making $\lambda_2 = 7(6s+4)$ as required. In addition, λ_1 increases by $6(3s+2)$ between the elements of the groups. Now, use the blocks of $BIBD(66s+45, 4, 21s+15)$ on both G_1 and G_2 to get required λ_1 between subsets. Finally, use the blocks of $BIBD(22s+15, 4, 45s+30)$ on each of the subsets of G_1 and G_2 to get required λ_1 within subsets. One can also see that this will give the required number of group blocks.

3.3 $t \equiv 5, 9 \pmod{12}$

Let $t = 4s+1$ for some integer $s \geq 1$. The aim now is to construct GDD$(44s+12, 2, 4; 44s+12, 28s+7)$. Let $G_1 = \{x_1, \ldots, x_{44s+12}\}$ and $G_2 = \{y_1, \ldots, y_{44s+12}\}$. From here, split G_1 into four subsets:

(i) $A_1 = \{x_1, \ldots, x_{11s+3}\}$,

(ii) $B_1 = \{x_{11s+4}, \ldots, x_{22s+6}\}$,

(iii) $C_1 = \{x_{22s+7}, \ldots, x_{33s+9}\}$,

(iv) $D_1 = \{x_{33s+10}, \ldots, x_{44s+12}\}$.

Similarly, split G_2 into A_2, B_2, C_2, and D_2. It is necessary to have $\frac{b}{3} = \frac{\lambda_2 n^2}{7} = (4s+1)(44s+12)^2$ blocks of each type. In order to get the odd blocks, use the blocks of TD$(11s+3, 4, 4; 0, 1)$ on certain groups as described in Table 5:

This will increase λ_2 by $3(4s+1)$ and λ_1 in between the elements of the subsets of G_1 and G_2 by $4(4s+1)$. In order to get the even blocks, use the blocks of $2(4s+1)$ copies of $TD(22s+6, 4, 4; 0, 2)$ on the following sets:

Table 5:

A_1	A_2	B_2	C_2	A_2	A_1	C_1	D_1
A_1	A_2	B_2	D_2	A_2	B_1	C_1	D_1
B_1	A_2	B_2	C_2	B_2	A_1	C_1	D_1
B_1	A_2	B_2	D_2	B_2	B_1	C_1	D_1
C_1	A_2	C_2	D_2	C_2	A_1	B_1	C_1
C_1	B_2	C_2	D_2	C_2	A_1	B_1	D_1
D_1	A_2	C_2	D_2	D_2	A_1	B_1	C_1
D_1	B_2	C_2	D_2	D_2	A_1	B_1	D_1
$(4s+1)$ Copies				$(4s+1)$ Copies			

$A_1 \cup B_1$, $C_1 \cup D_1$, $A_2 \cup B_2$, and $C_2 \cup D_2$. This will increase λ_2 by $4(4s+1)$, making the total $\lambda_2 = 7(4s+1)$ as required. λ_1 will increase by $4(4s+1)$ in between the subsets of G_1 and G_2 except between the following sets: A_1 and B_1, C_1 and D_1, A_2 and B_2, and C_2 and D_2.
Take the blocks of $BIBD((11s+3)+(11s+3), 4, 4(4s+1))$ on each of the pairings of sets given above. This also serves to increase λ_1 within subsets by $4(4s+1)$. All that remains is to increase λ_1 within subsets by $(28s+8)$ and between subsets by $(12s+4)$. This can easily be done by taking the blocks of $BIBD(11s+3, 4, 28s+8)$ for each of the subsets of G_1 and G_2 as well as forming $TD(11s+3, 4, 4; 0, 12s+4)$ on the four subsets of G_1 and the four subsets of G_2. One can see that this will give the required number of group blocks. However, such $BIBD$s do not exist when $s \equiv 0 \pmod 3$.

3.4 $t \equiv 7 \pmod{12}$

Let $t = 12s + 7$ for some integer $t \geq 1$. The aim now is to construct $GDD(132s+78, 2, 4; 132s+78, 84s+49)$. Let $G_1 = \{x_1, \ldots, x_{132s+78}\}$ and $G_2 = \{y_1, \ldots, y_{132s+78}\}$. We split G_1 into six subsets each:

(i) $A_1 = \{x_1, \ldots, x_{44s+25}\}$,

(ii) $B_1 = \{x_{44s+26}, \ldots, x_{88s+50}\}$,

(iii) $C_1 = \{x_{88s+51}, \ldots, x_{132s+75}\}$

(iv) $D_1 = \{x_{132x+76}\}$,

(v) $E_1 = \{x_{132x+77}\}$,

(vi) $F_1 = \{x_{132x+78}\}$.

Similarly, we split G_2 into A_2, B_2, C_2, D_2, E_2, and F_2. We need $\frac{b}{3} = \frac{\lambda_2 n^2}{7} = (12s+7)(132s+78)^2$ blocks of each type. To get our odd blocks, we use the blocks of $(6s+4)$ copies of $TD((44s+25)+(1),4,4;0,1)$ on the groups on the left and $(6s+3)$ copies of the same design on the groups on the right:

Table 6:

$A_1 \cup D_1$	$B_1 \cup E_1$	$C_1 \cup F_1$	$A_2 \cup D_2$	$A_2 \cup p$	$B_2 \cup q$	$C_2 \cup r$	$A_1 \cup a$
$A_1 \cup E_1$	$B_1 \cup F_1$	$C_1 \cup D_1$	$A_2 \cup E_2$	$A_2 \cup q$	$B_2 \cup r$	$C_2 \cup p$	$A_1 \cup b$
$A_1 \cup F_1$	$B_1 \cup D_1$	$C_1 \cup E_1$	$A_2 \cup F_2$	$A_2 \cup r$	$B_2 \cup p$	$C_2 \cup q$	$A_1 \cup c$
$A_1 \cup D_1$	$B_1 \cup E_1$	$C_1 \cup F_1$	$B_2 \cup D_2$	$A_2 \cup p$	$B_2 \cup q$	$C_2 \cup r$	$B_1 \cup a$
$A_1 \cup E_1$	$B_1 \cup F_1$	$C_1 \cup D_1$	$B_2 \cup E_2$	$A_2 \cup q$	$B_2 \cup r$	$C_2 \cup p$	$B_1 \cup b$
$A_1 \cup F_1$	$B_1 \cup D_1$	$C_1 \cup E_1$	$B_2 \cup F_2$	$A_2 \cup r$	$B_2 \cup p$	$C_2 \cup q$	$B_1 \cup c$
$A_1 \cup D_1$	$B_1 \cup E_1$	$C_1 \cup F_1$	$C_2 \cup D_2$	$A_2 \cup p$	$B_2 \cup q$	$C_2 \cup r$	$C_1 \cup a$
$A_1 \cup E_1$	$B_1 \cup F_1$	$C_1 \cup D_1$	$C_2 \cup E_2$	$A_2 \cup q$	$B_2 \cup r$	$C_2 \cup p$	$C_1 \cup b$
$A_1 \cup F_1$	$B_1 \cup D_1$	$C_1 \cup E_1$	$C_2 \cup F_2$	$A_2 \cup r$	$B_2 \cup p$	$C_2 \cup q$	$C_1 \cup c$

This will contribute $3(12s+7)$ to λ_2 count. However, the effect on λ_1 is much more complicated.
From here on, a "multi-element subset pair" is a pair X_1, Y_1 that fulfills the inequality $|X_1|, |Y_1| > 1$. A "single-element subset pair" is a pair X_1, Y_1, where $|X_1|, |Y_1| = 1$. Finally, a "multi-element / single-element subset pair" is a pair X_1, Y_1, where $|X_1| > 1$ and $|Y_1| = 1$. So, saying that a λ_1 count increases by x between one of these pairs means that each element in the first subset comes x times with each element of the second subset. Looking at our six groups in pairings like this is important for making sure our required λ_1 is met.

(i) Between each multi-element subset pair and between each single-element subset pair of G_1, λ_1 count will increase by $9(6s+4)$.

(ii) Between each multi-element / single-element subset pair of G_1, λ_1 count will increase by $6(6s+4)$.

(iii) Between each multi-element subset pair and between each single-element subset pair of G_2, λ_1 count will increase by $9(6s+3)$.

(iv) Between each multi-element / single-element subset pair of G_2, λ_1 count will increase by $6(6s+3)$.

To get even blocks, we use the blocks of $(12s+7)$ copies of each $TD((44s+25)+(1),4,4;0,1)$ on the following groups:

Table 7:

$A_1 \cup a$	$B_1 \cup b$	$A_2 \cup p$	$B_2 \cup q$
$A_1 \cup b$	$B_1 \cup c$	$A_2 \cup q$	$C_2 \cup r$
$A_1 \cup c$	$B_1 \cup a$	$B_2 \cup r$	$C_2 \cup p$
$A_1 \cup a$	$C_1 \cup b$	$A_2 \cup q$	$B_2 \cup r$
$A_1 \cup b$	$C_1 \cup c$	$A_2 \cup r$	$C_2 \cup p$
$A_1 \cup c$	$C_1 \cup a$	$B_2 \cup p$	$C_2 \cup q$
$B_1 \cup a$	$C_1 \cup b$	$A_2 \cup r$	$B_2 \cup p$
$B_1 \cup b$	$C_1 \cup c$	$A_2 \cup p$	$C_2 \cup q$
$B_1 \cup c$	$C_1 \cup a$	$B_2 \cup q$	$C_2 \cup r$

This will contribute $4(12s+7)$ to λ_2 count, making $\lambda_2 = 7(12s+7)$ as required by our necessary conditions. Again, the effect on λ_1 is more involved:

(i) Between each multi-element subset pair and between each single-element subset pair of G_1, λ_1 count will increase by $3(12s+7)$ to $(90s+57)$.

(ii) Between each multi-element / single-element subset pair of G_1, λ_1 count will increase by $2(12s+7)$ to $(60s+38)$.

(iii) Between each multi-element subset pair and between each single-element subset pair of G_2, λ_1 count will increase by $3(12s+7)$ to $(90s+48)$.

(iv) Between each multi-element / single-element subset pair of G_2, λ_1 count will increase by $2(12s+7)$ to $(60s+32)$.

Now, we need the group blocks. However, the way we get such group blocks depends on whether or not $s \equiv 2 \pmod 3$. As such, we need to look at the process through two separate constructions for the required group blocks.

3.4.1 $s \equiv 0, 1 \pmod 3$

To get the group blocks for $s \equiv 0, 1 \pmod 3$, we take the blocks of $BIBDs$ of four separate designs. The first two are $BIBD((44s+25)+(44s+25)+(1)+(1), 4, 14s+7)$ on G_1 with the upper groups and $BIBD((44s+25)+(44s+25)+(1)+(1), 4, 14s+10)$ on G_2 with the lower groups. These are presented in Table 8. One can see such $BIBD$s will always exist for $s \equiv 0, 1 \pmod 3$. Finally, we take the blocks of $BIBD((44s+25)+(1), 4, 16s+12)$ on the upper groups and $BIBD((44s+25)+(1), 4, 16s+6)$ on the lower groups. These are presented in Table 9.

Table 8:

A_1	B_1	a	b	A_1	B_1	a	c	A_1	B_1	b	c
A_1	C_1	a	b	A_1	C_1	a	c	A_1	C_1	b	c
B_1	C_1	a	b	B_1	C_1	a	c	B_1	C_1	b	c
-	-	-	-	-	-	-	-	-	-	-	-
A_2	B_2	p	q	A_2	B_2	p	r	A_2	B_2	q	r
A_2	C_2	p	q	A_2	C_2	p	r	A_2	C_2	q	r
B_2	C_2	p	q	B_2	C_2	p	r	B_2	C_2	q	r

Table 9:

A_1	a	A_1	b	A_1	c
B_1	a	B_1	b	B_1	c
C_1	a	C_1	b	C_1	c
-	-	-	-	-	-
A_2	p	A_2	q	A_2	r
B_2	p	B_2	q	B_2	r
C_2	p	C_2	q	C_2	r

Again, one can see such $BIBD$s will always exist for $s \equiv 0, 1 \pmod 3$. These $BIBD$s will give us all of the required group blocks. In addition, we can see we have the required λ_1:

(i) Between each multi-element subset pair and between each single-element subset pair of G_1, λ_1 count will increase by $3(14s + 7)$ to $(132s + 78)$.

(ii) Between each multi-element / single-element subset pair of G_1, λ_1 count will increase by $4(14s + 7) + (16s + 12)$ to $(132s + 78)$.

(iii) Within each multi-element subset of G_1, λ_1 count will increase by $6(14s + 7) + 3(16s + 12)$ to $(132s + 78)$.

(iv) Between each multi-element subset pair and between each single-element subset pair of G_2, λ_1 count will increase by $3(14s + 10)$ to $(132s + 78)$.

(v) Between each multi-element / single-element subset pair of G_2, λ_1 count will increase by $4(14s + 10) + (16s + 6)$ to $(132s + 78)$.

(vi) Within each multi-element subset of G_2, λ_1 count will increase by $6(14s + 7) + 3(16s + 12)$ to $(132s + 78)$.

4 Additional GDD Construction

In the process of constructing the above GDDs, we came across another useful construction. The GDD($6t+4, 2, 4; 2, 3$) for t even provided a second construction for one of the above cases. This GDD can easily be shown to exist.

Proof. Let $G_1 = \{a_1, a_2, \ldots, a_{6t+4}\}$, with $A_1 = \{a_1, a_2, \ldots, a_{3t+2}\}$ and $B_1 = \{a_{3t+3}, a_{3t+4}, \ldots, a_{6t+4}\}$. Let G_2, A_2, and B_2 be defined in a similar fashion. First, take the blocks of TD($3t+2, 4, 4; 0, 2$). Then, take the blocks of BIBD($6t+4, 4, 1$) on each of $A_1 \cup B_1$, $A_1 \cup B_2$, $A_2 \cup B_1$, and $A_2 \cup B_2$, all of which exist for t even. □

This construction is useful for GDD($11t+1, 2, 4; 11t+1, 7t$) for the case of $t \equiv 9 \pmod{12}$. In this case, the GDD to construct is GDD($132s + 100, 2, 4; 132s + 100, 84t + 63$) where $t = 12s + 9$ for some positive integer s. First, take the blocks of $28t + 21$ copies of GDD($132s + 100, 2, 4; 2, 3$). This fulfills λ_2 requirement and contributes $56t + 42$ to λ_1 requirement. Finally, use the blocks of BIBD($132s + 100, 4, 76t + 58$) on G_1 and on G_2.

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