$GDD(n_1, n, n+1, 4; \lambda_1, \lambda_2)$: $n_1 = 1$ or 2

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ABSTRACT. The subject matter for this paper is GDDs with three groups of sizes n_1, n $(n \geq n_1)$ and $n+1$, for $n_1 = 1$ or 2 and block size four. A block having Configuration $(1, 1, 2)$ means that the block contains 1 point from two different groups and 2 points from the remaining group. A block having Configuration $(2, 2)$ means that the block has exactly two points from two of the three groups. First, we prove that a $GDD(n_1, n, n+1, 4; \lambda_1, \lambda_2)$ for $n_1 = 1$ or 2 does not exist if we require that exactly half of the blocks have the Configuration $(1, 1, 2)$ and the other half of the blocks have the Configuration $(2, 2)$ except possibly for $n = 7$ when $n_1 = 2$. Then we provide necessary conditions for the existence of a GDD$(n_1, n, n+1, 4; \lambda_1, \lambda_2)$ for $n_1 = 1$ and 2, and prove that these conditions are sufficient for several families of GDDs. For $n_1 = 2$, these usual necessary conditions are not sufficient in general as we provide specific examples of existence and non-existence of GDDs, which also provide a glimpse of challenges which exist even for the case of $n_1 = 2$. A general result that a GDD$(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$ exists if $n_1 + n_2 + n_3 \equiv 0, 4 \pmod{12}$ is also given.

1. Introduction

Group divisible designs (GDDs) have been studied for their usefulness in statistics and for their universal application to constructions of new designs [10, 17, 18]. Certain difficulties are present especially when the number of groups is smaller than the block size. In [2, 3], the question of existence of GDDs for block size three was settled. There is a more technical proof given in the book "Triple Systems" [1]. Similar results were

K. Namyalo and D. G. Sarvate thank the Council for International Exchange of Scholars for a Fulbright Specialist grant which made this collaboration possible. Dinesh Sarvate wish to thank Dr. Bob Mignone for continued support. L. Zhang thanks The Citadel Foundation for its support.

established for GDDs with block size four in [4, 6, 8, 13, 19]. Also, results about GDDs with two groups and block size four with equal number of even and odd blocks were given in [12]. In [15], results on GDDs with number of groups 2 or 3 and block size four were established. Furthermore, in [5, 7], results about GDDs with two groups and block size five with fixed block configuration were presented. In [16], results about GDDs with three groups and block size five with fixed block configuration were addressed. In [11], results about GDDs with fours groups and block size five with fixed block configuration were established. In [14], generalizations of designs for block size five from Clatworthy's Table are given. In [9], results about GDDs with block size six with fixed block configuration were studied.

A group divisible design, GDD$(n, m, k; \lambda_1, \lambda_2)$, is a collection of k-element subsets of a v-set V called *blocks* which satisfies the following properties: the $v = nm$ elements of V are partitioned into m subsets (called *groups*) of size n each; each point of V appears in $r = \frac{\lambda_1(n-1)+\lambda_2 n(m-1)}{k-1}$ (called *replication number*) of the $b = \frac{nmr}{k}$ blocks; points within the same group are called *first associates* of each other and appear together in λ_1 blocks; any two points not in the same group are called *second associates* of each other and appear together in λ_2 blocks. In the literature this definition has been generalized to include situations where the groups are not of the same sizes.

DEFINITION 1. *A group divisible design GDD$(n_1, n_2, ..., n_m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets of a v-set V called* blocks *which satisfies the following properties:*

- *the elements of V are partitioned into m subsets (called* groups*) of sizes $n_1, n_2, \cdots, n_m$, respectively;*
- *points within the same group are called* first associates *of each other and appear together in λ_1 blocks;*
- *any two points not in the same group are called* second associates *of each other and appear together in λ_2 blocks.*

EXAMPLE 1. *The blocks of a GDD$(2, 3, 3; 3, 1)$ with $G_1 = \{a, b\}$, $G_2 = \{1, 2, 3\}$ are $\{1, a, b\}, \{2, a, b\}, \{3, a, b\}, \{1, 2, 3\}, \{1, 2, 3\}$ and $\{1, 2, 3\}$.*

In [4, 19], the necessary conditions are proved to be sufficient for the existence of a GDD$(n, 3, 4; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 2)$ (note that three groups have the same size n in this problem). The main purpose of this paper is to establish results for GDDs with three groups of different sizes $n_1, n (n \geq 2)$ and $n + 1$, respectively, and block size four. There are, as mentioned earlier, several papers where GDDs with equal number of blocks of different configuration are studied. Interestingly here we prove that such GDDs in which exactly half of the number of blocks have the Configuration

$(1,1,2)$ and the other half of the blocks have the Configuration $(2,2)$, do not exists for $n_1 = 1$ in general and for $n_1 = 2$ except for possibly $n = 7$. Unless otherwise stated, we say that the number of blocks having the Configuration $(1,1,2)$ equals the number of the blocks having the Configuration $(2,2)$ in the remaining of this paper to mean that exactly half of the number of blocks have the Configuration $(1,1,2)$ and the other half of the blocks have the Configuration $(2,2)$.

2. GDD$(1,n,n+1,4;\lambda_1,\lambda_2)$

2.1. Non-existence result with configuration restriction.

THEOREM 1. *The necessary conditions for the existence of a GDD $(1,n,n+1,4;\lambda_1,\lambda_2)$ with equal number of blocks for each Configuration $(1,1,2)$ and $(2,2)$ are $\lambda_2 = \frac{3n^2\lambda_1}{n^2+3n+1}$, $\lambda_1 \equiv 0 \pmod{3(n^2+3n+1)}$, $n \geq 3$ and $\frac{b}{2} = r_1$.*

PROOF. Suppose a GDD$(1,n,n+1,4;\lambda_1,\lambda_2)$ exists and has b blocks. The number of the first associate pairs equals $\frac{\lambda_1(n(n-1)+n(n+1))}{2}$, and it also equals $\frac{2b}{2}+\frac{b}{2}$ since half of the blocks have Configuration $(1,1,2)$ and half of the blocks have Configuration $(2,2)$, we have $\frac{2b}{2}+\frac{b}{2} = \frac{\lambda_1(n(n-1)+n(n+1))}{2}$. That is, $b = \frac{\lambda_1(n(n-1)+n(n+1))}{3}$. In addition, the number of second associate pairs equals $\lambda_2(n+(n+1)+n(n+1)) = \frac{4b}{2}+\frac{5b}{2}$, we have $b = \frac{2\lambda_2(n+(n+1)+n(n+1))}{9}$. From $\frac{2\lambda_2(n+(n+1)+n(n+1))}{9} = \frac{\lambda_1(n(n-1)+n(n+1))}{3} = b$, $\lambda_2 = \frac{3n^2\lambda_1}{n^2+3n+1}$. This implies $\lambda_2 \equiv 0 \pmod 3$.

Note that the replication number r_1 for an arbitrary point in group G_1 is $\frac{\lambda_2(2n+1)}{3}$. Similarly, $r_2 = \frac{\lambda_1(n-1)+\lambda_2(n+2)}{3}$ and $r_3 = \frac{\lambda_1 n+\lambda_2(n+1)}{3}$. Also, we have $b = \frac{r_1+nr_2+(n+1)r_3}{4} = \frac{2n^2\lambda_1}{3}$ (use $\lambda_2 = \frac{3n^2\lambda_1}{n^2+3n+1}$ and simplify). This implies that b is even and satisfies the requirement for having equal number of blocks having Configurations $(1,1,2)$ and $(2,2)$, respectively. Since $b = \frac{2n^2\lambda_1}{3}$, either $\lambda_1 \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$. Also, since $r_2 = \frac{\lambda_1(n-1)+\lambda_2(n+2)}{3} = \frac{\lambda_1(4n^3+8n^2-2n-1)}{3(n^2+3n+1)}$, either $\lambda_1 \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$. Combining these two conditions, we have $\lambda_1 \equiv 0 \pmod 3$. Furthermore, since $\lambda_2 = \frac{3n^2\lambda_1}{n^2+3n+1}$, $\lambda_1 \equiv 0 \pmod{(n^2+3n+1)}$.

Since there are equal number of blocks having Configurations $(1,1,2)$ and $(2,2)$, respectively, the number of blocks $b \geq \lambda_1\binom{n+1}{2} = \frac{\lambda_1 n(n+1)}{2}$ (the number of first associate pairs from group 3). Note that the equal sign holds when each block has two points from group 3. Also, since $b = \frac{\lambda_1(n(n-1)+n(n+1))}{3} = \frac{2n^2\lambda_1}{3}$, we have $\frac{\lambda_1 n(n+1)}{2} \leq \frac{\lambda_1(n(n-1)+n(n+1))}{3}$, that is, $n(n+1) \leq 2n(n-1)$, or $n \geq 3$.

Lastly, since the point in G_1 must appear in $\frac{b}{2}$ blocks having Configuration $(1,1,2)$ and not appear in any blocks having Configuration $(2,2)$, we must have $r_1 = \frac{b}{2}$. ■

COROLLARY 1. *A GDD* $(1, n, n+1, 4; \lambda_1, \lambda_2)$ *with equal number of blocks for each Configuration* $(1,1,2)$ *and* $(2,2)$ *does not exist.*

PROOF. One of the necessary conditions of the problem in Theorem 1 is $r_1 = \frac{b}{2}$. That is, $\frac{\lambda_2(2n+1)}{3} = \frac{n^2(2n+1)\lambda_1}{n^2+3n+1} = \frac{n^2\lambda_1}{3}$, we have $n^2 - 3n - 2 = 0$, and there's no integer solution for n. Therefore, for any positive integer $n \geq 2$ a GDD $(1, n, n+1, 4; \lambda_1, \lambda_2)$ with equal number of blocks for each Configuration $(1,1,2)$ and $(2,2)$ does not exist. ■

Next we study GDD $(1, n, n+1, 4; \lambda_1, \lambda_2)$ without configuration restrictions.

2.2. The necessary conditions for GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$.

As a GDD $(1, n, n+1, 4; \lambda_1, \lambda_2)$ with equal number of blocks of Configuration $(1,1,2)$ and $(2,2)$ does not exist, our next step is to study the existence of these designs in general without configurations. We let G_1, G_2 and G_3 denote the three groups. We can assume that $n > 1$ as we have the following.

LEMMA 1. *A GDD*$(1,1,2,4;\lambda_1,\lambda_2)$ *exist only when* $\lambda_1 = \lambda_2$ *and the blocks of the trivial GDD are* λ_1 *copies of* V *(i.e.* $G_1 \cup G_2 \cup G_3$*).*

In this section we will find necessary conditions for the existence of GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$. By simple counting, the replication numbers r_i for the elements of i^{th} group are : $r_1 = \frac{(2n+1)\lambda_2}{3}$, $r_2 = \frac{(n+2)\lambda_2+(n-1)\lambda_1}{3}$ and $r_3 = \frac{n\lambda_1+(n+1)\lambda_2}{3}$.

Since $4b = 1 \times r_1 + n \times r_2 + (n+1) \times r_3$, we have $b = \frac{n^2\lambda_1+(n^2+3n+1)\lambda_2}{6}$.

Case i n **even.** Let $n = 2t$. Then $b = \frac{4t^2\lambda_1+(4t^2+6t+1)\lambda_2}{6}$ which implies that λ_2 must be even.

Case ii n **odd.** Let $n = 2t+1$. Then $b = \frac{(4t^2+4t+1)\lambda_1+(4t^2+10t+5)\lambda_2}{6}$ which implies that λ_1 and λ_2 are of the same parity.

The parameters r_1, r_2, r_3 and b must be integers, hence we get the following restrictions on n in Table 1 where "None" means the design does not exists for any n.

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	any n	None	None	None	None	None
1	None	$n \equiv 1 \pmod 6$	None	None	$n \equiv 4 \pmod 6$	None
2	None	None	$n \equiv 1 \pmod 3$	None	None	None
3	even n	None	None	$n \equiv 1,3,5 \pmod 6$	None	None
4	None	None	None	None	$n \equiv 1 \pmod 3$	None
5	None	None	$n \equiv 4 \pmod 6$	None	None	$n \equiv 1 \pmod 6$

TABLE 1. The necessary conditions for GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$

The following lemma gives another necessary condition.

LEMMA 2. *A necessary condition for the existence of a GDD*$(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$ *is* $b \geq max(2r_i - \lambda_1)$, $i = 1, 2, 3$. *Note that when* $n_1 = 1$, $b \geq max(2r_i - \lambda_1)$, $i = 2, 3$.

PROOF. Consider a first associate pair, say x and y from a group G_i, then as both of them come together λ_1 times, the number of blocks must be at least $2r_i - \lambda_1$ to accommodate x and y r_i times in the design. ■

An example where $b \not\geq 2r_1 - \lambda_1$ is given below.

EXAMPLE 2. *A GDD*$(1, 2, 3, 4; 3, 6)$ *does not exist because for this design to exist* $b = 13$, $r_1 = 10, r_2 = 9$ *and* $r_3 = 8$ *and clearly* b *is less than* $2r_1 - \lambda_1$.

The following is an example of a GDD that exists when $b \geq 2r_1 - \lambda_1$.

EXAMPLE 3. *A GDD*$(1, 2, 3, 4; 9, 6)$, *here* $b = 17$ *and if* $\lambda_1 = 9, \lambda_2 = 6$ *then* $r_1 = 10, r_2 = 11$ *and* $r_3 = 12$. *Let* $G_1 = \{x\}, G_2 = \{a, b\}$ *and* $G_3 = \{1, 2, 3\}$. *The blocks of the GDD are given below in columns.*

x	x	x	x	x	x	x	x	x	x	a	a	a	1	1	1	1
a	a	a	a	a	a	1	1	1	1	b	b	b	2	2	2	2
b	b	b	b	b	b	2	2	2	2	1	2	3	3	3	3	3
1	1	2	2	3	3	3	3	3	3	2	3	1	a	b	a	b

Observe that every block contains at least two first associate pairs except for the blocks which contain the element from the first group of size 1. These r_1 blocks which contain group 1 element contain at least 1 first associate pair. So we must have at least $r_1 + 2(b - r_1) = 2b - r_1$ first associate pairs. That is we have $[\binom{n}{2} + \binom{n+1}{2}]\lambda_1 \geq 2b - r_1$. Hence, we have

LEMMA 3. *A necessary condition for the existence of a GDD*$(1, n, n+1, 4; \lambda_1, \lambda_2)$ *is* $\lambda_1 n^2 \geq 2b - r_1$.

In several cases it is easy to show that a specific design does not exist from the necessary conditions. For a GDD$(1, 4, 5, 4; 3, 6)$ to exist $b = 37, r_2 = 15$ the number of first associate pairs in the design $[\binom{4}{2} + \binom{5}{2}]3 = 48$ but as $2b - r_1 = 59$, the design does not exist. Another example, a GDD$(1, 2, 3, 4; 12, 18)$ does not exist as $b = 41$, $r_2 = 28$, $2b - r_2 = 54$ and $n^2\lambda_1 = 48$.

Though general results are hard to obtain, we have an interesting bound for how far away λ_2 can go from λ_1.

THEOREM 2. *A necessary condition for the existence of a GDD*$(1, n, n+1, 4; \lambda_1, \lambda_2)$ *is* $\lambda_2 < 2\lambda_1$.

PROOF. Substitute the values of b and r_1 in $\lambda_1 n^2 \geq 2b - r_1$, we have $\lambda_2 \leq \frac{2n\lambda_1}{n+1} < 2\lambda_1$ for $n \geq 2$. ■

2.3. GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$ when $\lambda_1 \leq 3$.

A *balanced incomplete block design* $BIBD(v, k, \lambda)$ $(\lambda \geq 1)$ is a pair (V, B) where B is a collection of blocks of V such that every block contains exactly $k < v$ points and every pair of distinct elements is contained in exactly λ blocks. It is known that a $BIBD(v, 4, \lambda)$ exists if $\lambda \equiv 1, 5 \pmod 6$ and $v \equiv 1, 4 \pmod{12}$, or if $\lambda \equiv 2, 4 \pmod 6$ and $v \equiv 1 \pmod 3$, or if $\lambda \equiv 3 \pmod 6$ and $v \equiv 0, 1 \pmod 4$, or if $\lambda \equiv 0 \pmod 6$.

We will use BIBDs extensively in the remaining of the paper. We start with $\lambda_1 = 3$.

2.3.1. *GDD*$(1, n, n+1, 4; 3, \lambda_2)$.

For $\lambda_1 = 3$, $\lambda_2 < 6$ from Theorem 2.

A GDD$(1, n, n+1, 4; 3, 3)$ is just a BIBD$(2n+2, 4, 3)$. For n odd, a BIBD$(2n+2, 4, 3)$ exists, and for n even, it does not exist as evident also from Table 1. Also, from Table 1, GDD$(1, n, n+1, 4; 3, \lambda_2)$ for $\lambda_2 = 1, 2, 4, 5$ does not exist. For $\lambda_2 = 0$, in general, the blocks of a GDD$((1, n, n+1, 4; \lambda_1, 0)$ are the collection of blocks of BIBD$(n, 4, \lambda_1)$ and BIBD$(n+1, 4, \lambda_1)$. Therefore, we have the following result.

LEMMA 4. *GDD$(1, n, n+1, 4; 3, \lambda_2)$ does not exist except when $\lambda_2 = 3$ and in this case it exists only for odd values of n and it is a BIBD$(2n+2, 4, 3)$.*

We note that when $\lambda_1 = 3$, a GDD$((1, n, n+1, 4; \lambda_1, 0)$ exists only when $n \equiv 0 \pmod 4$.

2.3.2. *GDD$(1, n, n+1, 4; 1, \lambda_2)$.*

For $\lambda_1 = 1$, $\lambda_2 \leq 1$ from Theorem 2. Therefore, we have:

LEMMA 5. *GDD$(1, n, n+1, 4; 1, \lambda_2)$ does not exist for any λ_2 except when $\lambda_2 = 1$ and in this case it is just a BIBD$(2n+2, 4, 1)$.*

2.3.3. *GDD$(1, n, n+1, 4; 2, \lambda_2)$.*

For a GDD$(1, n, n+1, 4; 2, \lambda_2)$, $\lambda_2 \leq 3$ from Theorem 2. A GDD$(1, n, n+1, 4; 2, \lambda_2)$ does not exist for $\lambda_2 = 0, 1$ or 3 from Table 1, and a GDD$(1, n, n+1, 4; 2, 2)$ is just a BIBD $(2n+2, 4, 2)$ which exist when $n \equiv 1 \pmod 3$.

LEMMA 6. *A GDD$(1, n, n+1, 4; 2, \lambda_2)$ does not exist except when $\lambda_2 = 2$ and in this case it is a BIBD$(2n+2, 4, 2)$ and $n \equiv 1 \pmod 3$.*

In summary, a *GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$* for $\lambda_1 \leq 3$ exists only when $\lambda_1 = \lambda_2$ and in that case it exists if and only if a BIBD$(2n+2, 4, \lambda)$ exists.

2.4. GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2$.

In this section, we study GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2$. We proceed with the cases on the main diagonal of Table 1 first (where $\lambda_1 \equiv \lambda_2 \pmod 6$), and then study the three cases on the off-diagonal entries (where $\lambda_1 \equiv 3 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$, and $\lambda_1 \equiv 5 \pmod 6$ and $\lambda_2 \equiv 2 \pmod 6$. and $\lambda_1 \equiv 1 \pmod 6$ and $\lambda_2 \equiv 4 \pmod 6$). Note that a GDD does not exist for all other cases. Also, a brief discussion of GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$ where $\lambda_1 < \lambda_2$ is included in the summary of this paper.

2.4.1. *Case* $\lambda_1 \equiv \lambda_2 \pmod 6$.

In general, a GDD$(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$ exists when $\lambda_2 \leq \lambda_1$ and $\lambda_1 \equiv \lambda_2 \pmod 6$, if a BIBD$(n_1 + n_2 + n_3, 4, \lambda_2)$ exists as a BIBD$(v, 4, 6)$ exists for all integers $v \geq 4$. More generally,

THEOREM 3. *If a BIBD$(n_1 + n_2 + n_3, 4, \lambda_2)$, and a BIBD$(n_i, 4, \lambda_1)$ exists for $i = 1, 2, 3$, then a GDD$(n_1, n_2, n_3, 4; \lambda_1 + \lambda_2, \lambda_2)$ exists.*

For example, a GDD$(1, 4, 5, 4; 5, 2)$ exists.

COROLLARY 2. *A GDD$(n_1, n_2, n_3, 4; \lambda_1 = \lambda_2 + 6s, \lambda_2)$ exists if a BIBD $(n_1 + n_2 + n_3, 4, \lambda_2)$ exists.*

PROOF. As BIBD$(n, 4, 6)$ exist for all n. The blocks of the GDD are obtained by taking the union of the collection of blocks of BIBD$(n, 4, 6s)$ on G_i, $i = 1, 2, 3$ and BIBD$(n_1 + n_2 + n_3, 4, \lambda_2)$ on $G_1 \cup G_2 \cup G_3$ that exist. ■

COROLLARY 3. *For all values of n_1, n_2, n_3 where $n_1 + n_2 + n_3 \equiv 0, 4 \pmod{12}$, a GDD$(n_1, n_2, n_3, 4; \lambda_2 + 6s, \lambda_2)$ exists for all λ_2's.*

COROLLARY 4. *A GDD$(1, 6t+1, 6t+2, 4; \lambda_2 + 6s, \lambda_2)$ exists for all λ_2, and $t, s \geq 0$.*

PROOF. The blocks of GDD are obtained by taking the union of the collection of blocks of a BIBD$(6t + 1, 4, 6)$ on G_2 and a BIBD$(6t + 2, 4, 6)$ on G_3 and BIBD$(12t + 4, 4, \lambda_2)$ on $G_1 \cup G_2 \cup G_3$. ■

2.4.1.1. Subcase $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$.

For any value of n, a GDD$(1, n, n + 1, 4; 6s, 6t)$ is possible, but as we will see the situation is more involved.

First from Theorem 2, a GDD$(1, n, n + 1, 4; 6t, 6T)$ does not exist for $T \geq 2t$. Note for $t = 2, n = 2$, for example, a GDD$(1, 2, 3, 4; 12, 18)$ satisfies this condition but does not exist as seen in Section 2 and similarly a GDD$(1, n, n + 1, 4; 12, 18)$ does not exist for $n = 3$ though the parameters of a GDD$(1, 3, 4, 4; 12, 18)$ satisfy the condition. Here $b = 75$, $r_1 = 42$, $r_2 = 38$ and $r_3 = 36$. Now $12 \cdot 9 < 150 - 38 = 112$.

LEMMA 7. *For all values of $n \geq 4$, a GDD$(1, n, n + 1, 4; 6s, 6t)$ exists if $6t \leq 6s$.*

PROOF. The blocks of the GDD are obtained by taking union of the collection of blocks of these BIBDs: BIBD$(2n + 2, 4, 6t)$ on $G_1 \bigcup G_2 \bigcup G_3$, BIBD$(n, 4, 6(s - t))$ on G_2 and BIBD$(n + 1, 4, 6(s - t))$ on G_3. ■

2.4.1.2. Subcase $\lambda_1 \equiv 1 \pmod 6$ and $\lambda_2 \equiv 1 \pmod 6$.

From Table 1, for this case $n \equiv 1 \pmod 6$, and from Corollary 4, we proved the existence of the required GDDs.

2.4.1.3. Subcase $\lambda_1 \equiv 2 \pmod 6$ and $\lambda_2 \equiv 2 \pmod 6$.

LEMMA 8. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n+1, 4, \lambda_1 \equiv 2 \pmod 6, \lambda_2 \equiv 2 \pmod 6)$ *for* $\lambda_2 \leq \lambda_1$.

PROOF. From the necessary conditions $n \equiv 1 \pmod 3$. Also note that λ_2 is even. As the cardinality of the union of the three groups is $6t+4$ for some integer t, a BIBD$(6t+4, 4, \lambda_2)$ exists. The blocks of the BIBD along with the blocks of $\frac{\lambda_1 - \lambda_2}{6}$ copies of BIBD$(3t+1, 4, 6)$ and BIBD$(3t+2, 4, 6)$ provide the blocks of the required GDD. ■

2.4.1.4. Subcase $\lambda_1 \equiv 3 \pmod 6$ and $\lambda_2 \equiv 3 \pmod 6$.

Here from Table 1, n is odd, say $n = 2t+1$ for some positive integer t. Notice that a BIBD$(4t+4, 4, \lambda_2)$ exists. The blocks of the BIBD along with the blocks of $\frac{\lambda_1 - \lambda_2}{6}$ copies of BIBD$(2t+1, 4, 6)$ and BIBD$(2t+2, 4, 6)$ provide the blocks of a GDD$(1, 2t+1, 2t+2, 4; 6a+3, 6b+3)$, when $a \geq b$. Hence we have

LEMMA 9. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n+1, 4, \lambda_1 \equiv 3 \pmod 6, \lambda_2 \equiv 3 \pmod 6)$ *for* $\lambda_2 \leq \lambda_1$.

2.4.1.5. Subcase $\lambda_1 \equiv 4 \pmod 6$ and $\lambda_2 \equiv 4 \pmod 6$.

This case is identical to the case when $\lambda_1 \equiv 2 \pmod 6$ and $\lambda_2 \equiv 2 \pmod 6$ and hence we have

LEMMA 10. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n+1, 4, \lambda_1 \equiv 4 \pmod 6, \lambda_2 \equiv 4 \pmod 6)$ *for* $\lambda_2 \leq \lambda_1$.

2.4.1.6. Subcase $\lambda_1 \equiv 5 \pmod 6$ and $\lambda_2 \equiv 5 \pmod 6$.

From Table 1, $n \equiv 1 \pmod 6$, and hence from Corollary 4, we have

LEMMA 11. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n+1, 4, \lambda_1 \equiv 5 \pmod 6, \lambda_2 \equiv 5 \pmod 6)$ *for* $\lambda_2 \leq \lambda_1$.

In summary, the above subsections together prove

THEOREM 4. *Necessary conditions are sufficient for the existence of* $GDD(1, n, n+1, 4; \lambda_1, \lambda_2)$ *when* $\lambda_1 \geq \lambda_2$ *and* $\lambda_1 \equiv \lambda_2 \pmod 6$.

2.4.2. *Case* $\lambda_1 \equiv 5 \pmod 6$ *and* $\lambda_2 \equiv 2 \pmod 6$.

The necessary condition for the existence of a GDD$(1, n, n+1, 4; 5, 2)$ is $n \equiv 4 \pmod 6$ (i.e., $n \equiv 4, 10 \pmod{12}$) from Table 1. The following result gives half of these designs (where $n \equiv 4 \pmod{12}$).

THEOREM 5. *If* $n \equiv 4 \pmod{12}$ *then a* $GDD(1, n, n+1, 4; 5, 2)$ *exists and hence a* $GDD(1, n, n+1, 4; 5t, 2t)$ *exists for all* $t \geq 1$.

PROOF. The blocks of the GDD with groups G_1, G_2 and G_3 are blocks of a BIBD$(2n+2, 4, 2)$ on $G_1 \bigcup G_2 \bigcup G_3$, the blocks of a BIBD$(n, 4, 3)$ on G_2 and the blocks of BIBD $(n+1, 4, 3)$ on G_3.

Note when $n \equiv 4 \pmod{12}$, $n \equiv 0 \pmod 4$ and $n+1 \equiv 1 \pmod 4$ and $2n+2 \equiv 1 \pmod 3$ conditions required for the existence of the BIBDs used in the construction are satisfied. ■

2.4.3. *Case* $\lambda_1 \equiv 3 \pmod 6$ *and* $\lambda_2 \equiv 0 \pmod 6$.

In this case, the smallest λ_1 we need to consider is 9 as $\lambda_1 = 3$ case has been discussed in Section 2.3. From Table 1, n is even, i.e., $n \equiv 0, 2 \pmod 4$.

If $n \equiv 0 \pmod 4$, for any $\lambda_1 = 6s_1 + 3$ and $\lambda_2 = 6s_2$ where $\lambda_1 \geq \lambda_2$, a GDD$(1, n, n+1, 4; 6s_1+3, 6s_2)$ exists by using the blocks of BIBD$(2n+2, 4, 6s_2)$ and BIBD$(n, 4, 6(s_1 - s_2) + 3)$.

If $n \equiv 2 \pmod 4$, a GDD$(1, 4t+2, 4t+3, 4; 6s+3, 0)$ does not exist since a BIBD$(4t+2, 4, 6s+3)$ does not exist. For $n = 2$, a GDD$(1, 2, 3, 4; 9, 6)$ exists from Example 3, also it is known that a BIBD$(6, 4, 6)$ exists. So the blocks of the copies of $s - 1$ BIBD$(6, 4, 6)$ together with blocks of a GDD$(1, 2, 3, 4; 9, 6)$ give a GDD$(1, 2, 3, 4; 6t+3, 6t)$. For $n = 6$, a GDD$(1, 6, 7, 4; 9, 6)$ exists. Suppose x is the single element from G_1. The blocks of a GDD$(1, 6, 7, 4; 9, 6)$ are the union of the following collection of blocks: (1) $\{x\} \cup B$ where B is a block of a BIBD$(13, 3, 1)$ on elements from $G_2 \cup G_3$; (2) $\{i\} \cup B$ where $i \in G_2$ and B is a block of a BIBD$(7, 3, 1)$ on elements from G_3; (3) blocks from a BIBD$(6, 4, 6)$ on elements from G_2; (4) blocks from a BIBD$(13, 4, 2)$ on elements from $G_2 \cup G_3$.

2.4.4. *Case* $\lambda_1 \equiv 1 \pmod 6$ *and* $\lambda_2 \equiv 4 \pmod 6$.

The necessary condition for the existence of a GDD$(1, n, n+1, 4; 7, 4)$ is $n \equiv 4 \pmod 6$ (i.e., $n \equiv 4, 10 \pmod{12}$) from Table 1. The following result gives half of these designs (where $n \equiv 4 \pmod{12}$).

THEOREM 6. *If $n \equiv 4 \pmod{12}$ then a GDD$(1, n, n+1, 4; 7, 4)$ exists and hence a GDD$(1, n, n+1, 4; 7t, 4t)$ exists for all $t \geq 1$.*

PROOF. The blocks of the GDD with groups G_1, G_2 and G_3 are blocks of a BIBD$(2n+2, 4, 4)$ on $G_1 \bigcup G_2 \bigcup G_3$, the blocks of a BIBD$(n, 4, 3)$ on G_2 and the blocks of BIBD $(n+1, 4, 3)$ on G_3. ■

In summary of the GDDs where $n_1 = 1$, we have the following theorem.

THEOREM 7. *Necessary conditions are sufficient for a GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2$, except possibly for a GDD$(1, n, n+1, 4; 5t, 2t)$ or a GDD$(1, n, n+1, 4; 7t, 4t)$ $(t \geq 1)$ when $n \equiv 10 \pmod{12}$.*

In the next section, we study GDDs where $n_1 = 2$, and give some examples and find necessary conditions for their existence, and prove that even for this case GDDs with equal number of blocks of Configuration $(1, 1, 2)$ and $(2, 2)$ do not exist except possibly for $n = 7$.

3. GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$

3.1. Non-existence result with configuration restriction.

In this subsection, we establish that if we require equal number of blocks of Configurations $(1, 1, 2)$ and $(2, 2)$, respectively, a GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ does not exist except for possibly $n = 7$.

LEMMA 12. *A GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ with equal number of blocks with Configuration $(1, 1, 2)$ and $(2, 2)$ does not exist for $n \geq 8$.*

PROOF. There are $\lambda_1(n^2+1)$ first associate pairs, and there are equal number of blocks of Configuration $(1, 1, 2)$ and $(2, 2)$. Therefore, $\frac{3b}{2} = \lambda_1(n^2+1)$ implies $b = \frac{2\lambda_1(n^2+1)}{3}$.

There are four second associate pairs from the blocks with Configuration $(2, 2)$ and five from the blocks with Configuration $(1, 1, 2)$ and there are $(n^2+5n+2)\lambda_2$ second associate pairs. Hence, $\frac{9b}{2} = (n^2+5n+2)\lambda_2$. Therefore, we have $\lambda_1 = \frac{(n^2+5n+2)}{3(n^2+1)}\lambda_2$. From Lemma 16, $b \leq \frac{\lambda_1}{6}[(3n^2+2) + \frac{2(2n+1)3(n^2+1)}{n^2+5n+2}]$. Substituting b and simplify, $n^4 \leq 7n^3 + 2n^2 + 2n + 2$, which is true only for $n \leq 7$. ■

LEMMA 13. *A GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ with equal number of blocks with Configuration $(1, 1, 2)$ and $(2, 2)$ does not exist for $2 \leq n \leq 6$.*

PROOF. We observe that the blocks with Configuration $(1,1,2)$ contain at least one element from G_1. Since there are exactly $2r_1 - 2\lambda_1$ blocks containing exactly one element from G_1, we must have $(2r_1 - \lambda_1) - \frac{b}{2} \le \lambda_1$. Using $\frac{b}{2} = \frac{\lambda_1(n^2+1)}{3}$ and $r_1 = \frac{\lambda_1+(2n+1)\lambda_2}{3}$, we get $7n^3 - n^2 - 13n - 4 \le n^4$, which is true only when $n = 1$ or $n \ge 7$ ■

Combining the above two lemmas we have

THEOREM 8. *A GDD*$(2, n, n+1, 4; \lambda_1, \lambda_2)$ *with equal number of blocks with Configuration* $(1,1,2)$ *and* $(2,2)$ *does not exist except possibly for* $n = 7$.

3.2. Necessary conditions for GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$.

As a GDD $(2, n, n+1, 4; \lambda_1, \lambda_2)$ with equal number of blocks with Configuration $(1,1,2)$ and $(2,2)$ does not exist except possibly for $n = 7$, our next step is to find the necessary conditions of the existence of these designs in general without configurations. A GDD$(2, n, n+1, 4; \lambda_1, 0)$ does not exist as group size 2 is less than the block size 4. A GDD$(2, n, n+1, 4; 0, \lambda_2)$ does not exist because there are only three groups but the block size is four. Unless otherwise stated, we assume $\lambda_1 > 0$ and $\lambda_2 > 0$ in this section.

The parameters for GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ are given by:

$b = \frac{\lambda_1(n^2+1)+\lambda_2(n^2+5n+2)}{6}$, $r_1 = \frac{\lambda_1+(2n+1)\lambda_2}{3}$, $r_2 = \frac{(n-1)\lambda_1+(n+3)\lambda_2}{3}$, and $r_3 = \frac{n\lambda_1+(n+2)\lambda_2}{3}$.

There are $2(r_1 - \lambda_1)$ blocks containing exactly one element of G_1. These blocks contain at least one first associate pair from G_2 or G_3. Therefore, a necessary condition for existence of these GDDs is $[\binom{n}{2} + \binom{n+1}{2}]\lambda_1 \ge 2(r_1 - \lambda_1)$, i.e., $(n^2+2)\lambda_1 \ge 2r_1$. Substituting the expression for r_1, we have $\lambda_2 \le \frac{3n^2+4}{4n+2}\lambda_1$. For example, a GDD$(2,5,6,4;1,4)$ does not exist.

LEMMA 14. *If a GDD*$(2, n, n+1, 4; \lambda_1, \lambda_2)$ *exists, then* $\lambda_1 \le \frac{(2n+1)\lambda_2}{2}$.

PROOF. Let $G_1 = \{a, b\}$. The blocks that contain both a and b have $2\lambda_1$ second associate pairs with a and $2\lambda_1$ second associate pairs with b giving a total of $4\lambda_1$ second associate pairs with the first group. Hence, as there are a total of $2(2n+1)\lambda_2$ second associate pairs with a and b, $4\lambda_1 \le 2(2n+1)\lambda_2$. Thus $\lambda_1 \le \frac{(2n+1)\lambda_2}{2}$. ■

LEMMA 15. *If a GDD*$(2, n, n+1, 4; \lambda_1, \lambda_2)$ *exists, then* $b \le (n^2+1)\lambda_1$.

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	Any n	None	None	Any n	None	None
1	None	$n \equiv 5 \pmod 6$	None	None	$n \equiv 5 \pmod 6$	None
2	None	None	$n \equiv 2,5 \pmod 6$	None	None	$n \equiv 2,5 \pmod 6$
3	Odd n	None	None	Odd n	None	None
4	None	$n \equiv 2,5 \pmod 6$	None	None	$n \equiv 2,5 \pmod 6$	None
5	None	None	$n \equiv 5 \pmod 6$	None	None	$n \equiv 5 \pmod 6$

TABLE 2. Necessary conditions for GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ subject to the bound $\lambda_1 \leq \frac{(2n+1)\lambda_2}{2}$

PROOF. Every block has at least one first associate pair, and there are $(n^2+1)\lambda_1$ first associate pairs. For example, a GDD$(2,3,4,4;3,6)$ does not exist as $b = 31$ and there are 30 first associate pairs. ■

LEMMA 16. *If a GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$ exists, then* $b \leq \frac{1}{6}[(3n^2+2)\lambda_1 + (2(2n+1))\lambda_2]$ *and* $\lambda_2 \leq \frac{(2n^2+1)}{n^2+n}\lambda_1$.

PROOF. The blocks containing the elements from first group have at least one first associate pair and the remaining $b-(2r_1-\lambda_1)$ blocks have at least two first associate pairs. The total number of first associate pairs is $\lambda_1(n^2+1)$. Therefore, $2(b-2r_1+\lambda_1)+2r_1-\lambda_1 \leq \lambda_1(n^2+1)$. Substituting r_1 and simplify, we have $b \leq \frac{1}{6}[(3n^2+2)\lambda_1 + 2(2n+1)\lambda_2]$. Substituting b, we have $\lambda_2 \leq \frac{(2n^2+1)}{n^2+n}\lambda_1$. ■

From Lemma 16, if $\lambda_1 = 1$, then $\lambda_2 \leq 1$, i.e. $\lambda_2 = 1$. That is, a GDD$(2, n, n+1, 4; 1, \lambda_2)$ only exists for $\lambda_2 = 1$. Notice that a GDD$(2, n, n+1, 4; 1, 1)$ is a BIBD$(2n+3, 4, 1)$ which exists when $n \equiv 5 \pmod 6$ as required from Table 2 as well. We summarize this in the following remark.

REMARK 1. *A GDD$(2, n, n+1, 4; \lambda, \lambda)$ is equivalent to a BIBD$(2n+3, 4, \lambda)$. For example, a GDD$(2, n, n+1, 4; \lambda, \lambda)$ exists for any λ when $n \equiv 5 \pmod 6$, as a BIBD$(2n+3, 4, \lambda)$ exists. Similarly, a GDD$(2, n, n+1, 4; 3\lambda, 3\lambda)$ = BIBD$(2n+3, 4, 3\lambda)$ exists for all odd n and any λ as BIBD$(2n+3, 4, 3)$ exists for odd n. If one construct a GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$, then we also have a GDD$(2, n, n+1, 4; \lambda_1+\lambda, \lambda_2+\lambda)$ if we have a BIBD$(2n+3, 4, \lambda)$ exists.*

In the next three subsections, we provide construction examples of a GDD $(2, n, n+1, 4; \lambda_1, \lambda_2)$ for $n = 2, 3$ and 4, respectively.

3.3. $n = 2$.

If a GDD$(2,2,3,4;\lambda_1,\lambda_2)$ exists, then $r_1 = r_2 = \frac{\lambda_1+5\lambda_2}{3}$ and $r_3 = \frac{2\lambda_1+4\lambda_2}{3}$, which imply $\lambda_1 \equiv \lambda_2 \pmod 3$. Also, since $b = \frac{4r_1+3r_3}{4}$, we have $\lambda_1 \equiv 0 \pmod 2$.

Since Every block of a GDD$(2,2,3,4;\lambda_1,\lambda_2)$ must contain at least one first associate pair, we have $b \le 5\lambda_1$, which gives a necessary condition: $\lambda_2 \le \frac{25}{16}\lambda_1$. Another bound for λ_1 is obtained by observing that the two elements in the first group, say $\{a,b\}$, occur as a pair in λ_1 blocks, then these blocks must have $2\lambda_1$ second associate pairs with a. Also, the required number of second associate pairs with a is $5\lambda_2$, and it must be greater than or equal to $2\lambda_1$. Hence we have the following lemma.

LEMMA 17. *Necessary conditions for the existence of a GDD$(2,2,3,4;\lambda_1,\lambda_2)$ include $\lambda_2 \le \frac{25}{16}\lambda_1$ and $\lambda_1 \le \frac{5}{2}\lambda_2$.*

As an immediate example of the above lemma, a GDD$(2,2,3,4;\lambda_1 = 2,\lambda_2)$ does not exist for any λ_2. Notice that for $\lambda_1 = 2$, the only possible value of λ_2 is 3, but 2 is not congruent 3 modulo 3. Similarly, a GDD$(2,2,3,4;\lambda_1 = 8,\lambda_2 = 2)$, a GDD$(2,2,3,4;\lambda_1 = 6,\lambda_2 = 12)$ and a GDD$(2,2,3,4;18,6)$ do not exist. Also, a GDD$(2,2,3,4;12,18)$ and a GDD$(2,2,3,4;18,12)$ do not exist, but it requires a different argument to prove these as follows.

A GDD$(2,2,3,4;12,18)$ does not exist: If a design exists, then the required number of blocks is 58 and there are 60 first associate pairs. As each block contains at least one first associate pairs, there can be at most one block containing G_3. If a block contains G_3, then there should be 33 more blocks containing a pair from G_3, and the replication of each element from G_3 in these 34 blocks is $1+11+11 = 23$. The required replication number r_3 is 32 if a design exists, hence we need $3 \times (32-23) = 27$ more blocks for the elements of G_3, but the required number of blocks for the design is 58.

A GDD$(2,2,3,4;18,12)$ does not exist: Since at most 18 blocks can contain G_3 and $r_3 = 28$, the minimum number of blocks needed for the elements from G_3 is $18 + 3 \times (28-18) = 48$, but the required number of blocks for a design is 47 if it exists.

On the other hand a GDD$(2,2,3,4;\lambda_1 = 12,\lambda_2 = 6)$ exists, the blocks are given below: Let $G_1 = \{a,b\}, G_2 = \{c,d\}$ and $G_3 = \{1,2,3\}$. Take six copies of $G_1 \cup G_2$, two copies of $G_i \cup e$ for $i = 1,2$ and for all edges e of K_3 on G_3, and two copies of $G_3 \cup \{x\}$ for all $x \in G_1 \cup G_2$.

3.4. $n = 3$.

In this section, we provide GDD constructions for GDD$(2,3,4,4;\lambda_1,\lambda_2)$ where $\lambda_2 = 3$ or 6. We let $G_1 = \{a,b\}, G_2 = \{x,y,z\}$ and $G_3 = \{1,2,3,4\}$.

If $\lambda_2 = 3$, then $\lambda_1 \le 10$. Since $\lambda_1 \equiv 0,3 \pmod 6$ from Table 2, the possible values of λ_1 are 3, 6 and 9.

EXAMPLE 4. *We construct a GDD$(2,3,4,4;\lambda_1,3)$ where $\lambda_1 = 3,6$ and 9.*

(a) *A GDD$(2,3,4,4;3,3)$: It is a BIBD$(9,4,3)$ which is known to exist.*

(b) *A GDD$(2,3,4,4;6,3)$: The blocks are as follows.*

a	a	a	a	a	a	a	a	a	b	b	b
b	b	b	b	b	b	1	1	1	1	1	1
2	3	4	x	x	y	2	2	2	2	2	3
x	y	z	y	z	z	3	4	4	3	4	4

and

1	1	1	2	2	2	x	x	x	x	1
2	3	4	3	3	3	y	y	y	y	2
y	x	x	4	4	4	z	z	z	4	3
z	z	y	x	y	z	1	2	3	4	4

(c) *A GDD$(2,3,4,4;9,3)$: Seven copies of G_3 along with the following blocks written in columns give the required collection of blocks.*

x	x	x	x	x	x	x	x	x	x	y	1	1	2	1	1	2	1	1	1	1
y	y	y	y	y	y	y	y	y	1	1	2	3	3	2	3	3	2	2	2	2
1	2	3	a	a	a	b	b	c	2	2	a	a	a	c	b	b	3	3	3	3
2	3	1	b	c	d	c	d	d	3	3	b	c	d	d	d	c	a	b	c	d

If $\lambda_2 = 6$, we have $\lambda_1 \le \frac{7}{2} \times 6 = 21$. Since $\lambda_1 \equiv 0,3 \pmod 6$ from Table 2, the possible values of λ_1 are 3, 6, 9, 12, 15, 18 and 21. Notice that if $\lambda_1 = 3$, a GDD does not exist because $b = 31$, and from Lemma 15, we require $b \le 30$.

EXAMPLE 5. *We construct a GDD$(2,3,4,4;\lambda_1,6)$ where $\lambda_1 = 6,9,12,15$, 18 and 21.*

(a) *A GDD$(2,3,4,4;6,6)$: It is a BIBD$(9,4,6)$ which is known to exist.*

(b) *A GDD$(2,3,4,4;9,6)$: It can be constructed by combining a GDD $(2,3,4,4;3,3)$ with a GDD$(2,3,4,4;\ 6,3)$.*

(c) *A GDD$(2,3,4,4;12,6)$: It can be obtained by combining a GDD$(2,3,4,4;9,3)$ with a GDD$(2,3,4,4;\ 3,3)$(i.e. a BIBD$(9,4,3)$).*

(d) *A GDD(2,3,4,4;15,6): It can be constructed by combining a GDD (2,3,4,4;6,3) with a GDD(2,3,4,4; 9,3).*

(e) *A GDD(2,3,4,4;18,6): The blocks are two copies of* $\{a,b\} \cup e$ *where* $e \in K_3$ *on* G_2 *and* $e \in K_3$ *on* G_3*, respectively, two copies of* $\{x,y,z,a\}$ *and* $\{x,y,z,b\}$*, 3 copies of* $\{x,y,z,c\}$ *for all* $c \in G_3$*,* $\{1,2,3,d\}, \{1,2,4,d\}, \{1,3,4,d\}, \{2,3,4,d\}$ *for all* $d \in G_2$*, and 10 copies of* $\{1,2,3,4\}$*.*

(f) *A GDD(2,3,4,4,21,6): The blocks are* $\{a,b\} \cup e$ *for all* $e \in 2K_4$ *on* G_3*,* $\{a,b\} \cup e$ *for all* $e \in 3K_3$ *on* G_2*, 4 copies of* $G_2 \cup \{u\}$ *for all* $u \in G_3$*,* $\{x,y,1,2\}, \{x,y,3,4\}, \{x,z,1,3\}, \{x,z,2,4\}, \{y,z,1,4\}, \{y,z,2,3\}$*, and 18 copies of* G_3*.*

3.5. $n = 4$.

In general, if n is even and $\lambda_2 = 3$, then $\lambda_1 \equiv 0 \pmod 6$. For $n = 4$ and $\lambda_2 = 3$, an additional condition is $\lambda_1 \leq 13$. Therefore, only possible values for λ_1 are 6 and 12. We construct GDDs for these two values of λ_1 below: a GDD(2,4,5,4;6,3) and a GDD(2,4,5,4;12,3). As usual, let the groups be $G_1 = \{a,b\}$, $G_2 = \{x,y,z\}$ and $G_3 = \{1,2,3,4,5\}$.

If the blocks in a design can be partitioned into *resolution (or parallel) classes* such that the blocks of each class partition the set V, then the design is called *resolvable*. A complete multigraph λK_v ($\lambda \geq 1$) is a graph on v points with λ edges between every pair of distinct points. A construction of a GDD(2,4,5,4;6,3) uses the following blocks. The blocks of a BIBD(9,4,3) on $G_2 \cup G_3$, blocks obtained by taking the union of the first 3-resolvable class of a BIBD(5,3,3) on G_3 with $\{a\}$ and union of the second 3-resolvable classes with $\{b\}$, the blocks obtained by taking the union of G_1 with each edge of a K_4 labeled with the elements of G_2, and 2 copies of G_2 as two blocks.

A construction of a GDD(2,4,5,4;12,3) uses the following blocks. The blocks of a BIBD(9,4,3) on $G_2 \cup G_3$, blocks of a BIBD(5,4,3) on G_3, blocks obtained by taking the union of G_1 with each edge of a K_4 labeled with G_2, 8 copies of G_2, the blocks obtained by taking union of G_1 with $\{1,4\}, \{2,4\}, \{3,4\}, \{1,5\}, \{2,5\}$, and $\{3,5\}$, the blocks $\{a,1,2,3\}$, $\{b,1,2,3\}$, $\{4,1,2,3\}$, $\{5,1,2,3\}$ and two copies of the blocks $\{1,2,4,5\}$, $\{1,3,4,5\}$, and $\{2,3,4,5\}$.

In addition, for $\lambda_2 = 6$, we have the following three construction where $\lambda_1 = 12$. A *2-factor* of a graph G is a spanning subgraph of G which is regular of degree 2. A *2-factorization* of a graph G is an edge disjoint decomposition of G into 2 factors.

First, the blocks of a construction of a GDD(2, 4, 5, 4; 12, 6) are as below. The first set of blocks are constructed by taking union of G_1 with all edges of a K_4 on G_2 twice. In other words, the blocks $\{a, b\} \cup e$ for all $e \in 2K_4$ on G_2. The second set of blocks are obtained by taking the union $B \cup \{a\}$ and $B \cup \{b\}$ for all blocks B of a BIBD(5, 3, 2) on G_3. The next set of blocks is constructed using $3K_5$ on G_3 and $5K_4$ on G_2 as follows: a $3K_5$ has 6 2−factors, say F_1, F_2, F_3, F_4, F_5 and F_6. Let K_4 has the edges e_1, e_2, e_3, e_4, e_5 and e_6. The blocks are constructed by taking the union for all $e \in F_i$, $e \cup e_i$ for $i = 1, \cdots, 6$. The remaining blocks are 5 copies of G_2 as blocks and the blocks of BIBD(5, 4, 3) on G_3.

The following is the second construction of a GDD(2, 4, 5, 4; 12, 6). Two copies of $\{\{x, y\} \bigcup e \mid e$ is an edge of K_4 on G_2; $\{\{x\} \bigcup b \mid b$ is a block of BIBD(5, 3, 3) on $G_3\}$; $\{\{y\} \bigcup b \mid b$ is a block of BIBD(5, 3, 3) on $G_3\}$; These blocks together with the blocks of BIBD(4, 4, 5) on G_2 and the blocks of BIBD(5, 4, 3) on G_3 and the following blocks as written in columns.

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	3	4	5	1	2	3	4	5	1
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	c	c	c	c	c	d	d	d	d	d

and

1	3	5	2	4	1	3	5	2	4	1	3	5	2	4
3	5	2	4	1	3	5	2	4	1	3	5	2	4	1
b	b	b	b	b	b	b	b	b	b	c	c	c	c	c
c	c	c	c	c	d	d	d	d	d	d	d	d	d	d

The third construction of a GDD(2, 4, 5, 4; 12, 6) is as follows. A BIBD (6, 4, 6) on $G_1 \cup G_2$, a BIBD(7, 4, 6) on $G_1 \cup G_3$ and a BIBD(9, 4, 6) on $G_2 \cup G_3$. This example motivates the following general theorem.

THEOREM 9. *A GDD*$(2, n, n+1, 4; 12, 6)$ *for all* $n \geq 4$ *exists.*

PROOF. Use the blocks of a BIBD$(n+2, 4, 6)$ on $G_1 \cup G_2$, BIBD$(n+3, 4, 6)$ on $G_1 \cup G_3$ and BIBD$(2n+1, 4, 6)$ on $G_2 \cup G_3$. ■

A construction of a GDD(2, 4, 5, 4; 18, 6) is as follows: $\{a, b\} \cup e$ for all $e \in 2K_4$ on G_2, $\{a, b\} \cup e$ for all $e \in K_4$ on $G_3 \setminus \{5\}$, $\{a, 5\} \cup e$ for all $e \in K_4$ on $\{1, 2, 3, 4\}$ and $\{b, 5\} \cup e$ for all $e \in K_4$ on $\{1, 2, 3, 4\}$, 3 copies of $\{1, 2, 3, 4\}$ (Note the element 5 has come 6 times with $\{1, 2, 3, 4\}$), a BIBD(5, 3, 3) with each $x \in G_2$, and 16 copies of $\{x, y, z, w\}$.

4. Summary

In this paper we obtained necessary conditions for the existence of $GDD(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$ for $n_1 = 1$ and $n_1 = 2$. We proved nonexistence of these designs when we require equal number of blocks with Configuration $(1,1,2)$ and $(2,2)$ for $n_1 = 1$, and for $n_1 = 2$ except possibly for $n = 7$. Also, we proved that the necessary conditions are sufficient for the existence of $GDD(1, n, n+1, 4; \lambda_1, \lambda_2)$ whenever $\lambda_1 \geq \lambda_2$ except for two cases. Note that for a $GDD(1, n, n+1, 4; \lambda_1, \lambda_2)$ where $\lambda_2 > \lambda_1$, from Table 1, if $n = 6t+1$ and $\lambda_1 = 6s+1$, then $\lambda_2 \leq \frac{2(6t+1)}{6t+2} \times (6s+1) < 2(6s+1)$. Hence, if $t \geq 2s$, then a $GDD(1, 6t+1, 6t+2, 4; 6s+1, 12s+1)$ may exist. We also obtain several examples for $n_1 = 2$.

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