In PG(3,q), $q=p^h$ a prime power and h having no odd factor, any (q^2+1)–set of class $[0,m,n]_1$ is an ovoid and any (q^2+q+1)–set of class $[1,m,n]_2$ containing at least two lines is either a plane or a cone projecting an oval from a point.

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Abstract. In PG$(3,p^{2^h})$, ovoids and cones projecting an oval from a point are characterized as three character sets with respect to lines and planes, respectively.

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1. Introduction and motivation.

An algebraic variety K can be thought as a set of points in a projective space that has a certain behavior with respect to subspaces. In a finite projective space PG(r,q) the variety K contains a finite number k of points and the intersection varieties have also a finite number of points. To characterize a variety we mean to reconstruct or describe the structure of a k–set K of PG(r,q) starting from few arithmetic or geometric properties. In PG$(3,q)$, the projective space of dimension three and order q, with $q=p^h$, a prime power, let K denote a k–set, i.e. a set of k points. For each integer i such that $0 \le i \le q+1$ (respectively $0 \le i \le q^2+q+1$), let us denote by $t_i=t_i(K)$ the number of lines (respectively planes) of PG$(3,q)$ meeting K in exactly i points. The numbers t_i are called the *characters* of K with respect to the lines (resp. planes), see [11]. Let $m_1,m_2,\dots,m_s$ be s integers such that $0 \le m_1 < m_2 <,\dots,< m_s \le q+1$ (resp. $0 \le m_1 < m_2 <,\dots,< m_s \le q^2+q+1$). A set K is said to be *of class* $[m_1,m_2,\dots,m_s]_{1 \text{ (resp. 2)}}$ if $t_i \ne 0$ only if $i \in \{m_1, m_2, \dots, m_s\}$. Moreover, K is said to be *of type* $(m_1, m_2, \dots, m_s)_{1 \text{ (resp. 2)}}$ if $t_i \ne 0$ if and only if $i \in \{m_1, m_2, \dots, m_s\}$. The

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integers $m_1, m_2, \ldots, m_s$ are called *intersection numbers* with respect to the lines (resp. planes). Of particular interest are sets with few intersection numbers, see [1] and [6]. The most studied sets are those with only two intersection numbers, see [3], [7], [8], [9], [12], [13], [14] and [17], and not much seems to be known in the general case of sets with more than two, [6], [15] and [16]. In particular, conics were characterized as *ovals*, i.e. $(q+1)$–sets of type $(0,1,2)_1$ in $PG(2,q)$, q odd, see [4] Theorem 8.14 page 180, elliptic quadrics were characterized as *ovoids*, i.e. (q^2+1)–sets of type $(0,1,2)_1$ in $PG(3,q)$, q odd, see [5] Theorem 16.1.7 page 35. Moreover, quadratic cones were characterized as (q^2+q+1)–sets of type $(1,q+1,2q+1)_2$ containing at least two lines in $PG(3,q)$, q odd, see [2] Theorem 1.1 page 69. These results motivate us to investigate, in $PG(3,q)$, (q^2+1)–set of class $[0,m,n]_1$ and (q^2+q+1)–set of class $[1,m,n]_2$. It would be nice to establish under what orders q, these pure incidence properties, without other extra conditions, are sufficient to identify ovoids and cones projecting an oval from a point. We prove the following

Theorem. *In* $PG(3,p^{2^h})$,

- *any* $(p^{2^{h+1}}+1)$*–set of class* $[0,m,n]_1$ *is an ovoid;*
- *any* $(p^{2^{h+1}}+p^{2^h}+1)$*–set of class* $[1,m,n]_2$ *containing at least two lines is either a plane or a cone projecting an oval from a point.*

2.- The proof of the Theorem.

In order to prove the first part of the Theorem, let K denote a (q^2+1)–set of class $[0,m,n]_1$ in $PG(3,q)$, with $m\geq 1$ and $2\leq n\leq q+1$.

Lemma 1. *The intersection numbers m and n are equal to* 1 *and* p^e+1, *respectively, where e is a nonnegative integer with* $0\leq e<\frac{h}{2}$.

Proof. Let P be a point of K, since the points of K different from P are partitioned by the lines on P we get: $q^2+1\geq(q^2+q+1)(m-1)$ and so $m=1$. Moreover, by a Result in [13], $n\leq\sqrt{q}$. Counting again the size of K by the lines through the point P, we get that $(n-1)\mid q^2$. Since $q=p^h$ is a prime power, $n-1$ is a power of p. Let us denote by p^e the power of p equal to $n-1$, with $0\leq e<\frac{h}{2}$. So, $n=p^e+1$. □

By counting in double way the number of lines, the number of pairs (P,r), where $P\in K$ and r is a line through P, and the number of pairs $((P, Q),r)$, where $\{P, Q\}\subset K$ and r is the line through P and Q, we get the following equations:

$$t_0 + t_1 + t_n = (q^2+q+1)(q^2+1)$$
$$t_1 + nt_n = (q^2+q+1)(q^2+1)$$
$$n(n-1)t_n = q^2(q^2+1)$$

Solving these equations we obtain the characters of K with respect to the lines:

(I) $t_0 = q^2(q^2+1)/n$

(II) $t_1 = (q^2+1)(q^2+q+1)-[q^2(q^2+1)/(n-1)]$

(III) $t_n = q^2(q^2+1)/[n(n-1)]$

We have that $t_0>0$, $t_1>0$ and $t_n > 0$. Therefore, K is of type $(0,1,n)_1$.

If $e=0$ then $n=2$ and K is an ovoid, see [5] Theorem 16.1.7 page 35. Assume $e\neq 0$. Since $(n-1) \mid q^2$ and two consecutive integers are coprime, from (I) we get $n \mid q^2+1$. So, $p^e+1 \mid p^{2h}+1$. Thus, $e \mid 2h$ and $2h=ef$, with the integer e even and the integer f odd. Since h has no odd factor, we have a contradiction. This completes the proof of the first part of the Theorem.

Now, in order to prove the second part of the Theorem, let K denote a (q^2+q+1)–set of class $[1,m,n]_2$ containing at least two lines in PG$(3,q)$, with $m\geq 2$ and $3\leq n\leq q^2+q+1$.

Lemma 2. *The intersection number m is equal to* $q+1$.

Proof. By counting in double way the number of planes, the number of pairs (P,α), where $P\in K$ and α is a plane through P, and the number of pairs $((P, Q),\alpha)$, where $\{P, Q\}\subset K$ and α is a plane through P and Q, we get the following equations:

$t_1 + t_m + t_n = (q+1)(q^2+1)$

$t_1 + mt_m + nt_n = (q^2+q+1)^2$

$m(m-1)t_m + n(n-1)t_n = q(q^2+q+1)(q+1)^2$

Solving these equations we obtain the characters of K with respect to the planes:

$t_1 = (q+1)(q^2+1)-[q(q^3+q^2+2q+1)/(n-1)]+q[(q^2+q+1)(q+1)^2-n(q^3+q^2+2q+1)]/[(m-1)(n-1)]$

$t_m = q[n(q^3+q^2+2q+1)-(q^2+q+1)(q+1)^2]/[(m-1)(n-m)]$

$t_n = q[(q^2+q+1)(q+1)^2-m(q^3+q^2+2q+1)]/[(n-1)(n-m)]$

If $n \leq q+1$, then $t_m < 0$, a contradiction. So $n > q+1$.

Since $t_n\geq 0$, then $m\leq q+1$. Assume $m<q+1$, all the $q+1$ planes through a line l contained in K are n–planes. Since the points of K not on l are partitioned by the planes on l we get: $(q+1)[n-(q+1)]=q^2$. Since $(q+1)$ and q^2 are coprime, we have that $n-(q+1)=1$ and $(q+1)=q^2$, a contradiction, because $q\geq 2$. Therefore, $m=q+1$. □

Now, we prove

Lemma 3. *The set K is either a plane or of type* $(1,q+1,n)_2$.

Proof. Since $m=q+1$, we get

$t_1 = [q^2(q+1)/(n-1)]-q$

$t_{q+1} = [n(q^3+q^2+2q+1)-(q^2+q+1)(q+1)^2]/(n-q-1)$

$t_n = q^3(q+1)/[(n-1)(n-q-1)]$

If $n=q^2+q+1$, we have that $t_1=0$, and K is a (q^2+q+1)–set of type $(q+1,q^2+q+1)_2$, i.e. a plane of PG(3,q). If $n<q^2+q+1$, we have that $t_1>0$, $t_{q+1}>0$ and $t_n>0$, then K is of type $(1,q+1,n)_2$. □

Now, we prove

Lemma 4. *There is a nonnegative integer e, with $0\le e\le 2h$ such that $n=p^e+q+1$.*

Proof. Since K contains at least two lines, let r denote a line contained in K. Counting the size of $K-r$ by the planes through the line r, we get that $(n-q-1) \mid q^2$. Since $q=p^h$ is a prime power, $n-q-1$ is a power of p. Let us denote by p^e the power of p equal to $n-q-1$, with $0\le e\le 2h$. So, $n=p^e+q+1$. □

Lemma 5. *The case $e=0$ is not possible.*

Proof. In this case K is a (q^2+q+1)–set of type $(1,q+1,q+2)_2$ in PG(3,q). Let us consider the line r through two points P and Q belonging to K. Let h denote the size of $r\cap K$, with $2\le h\le q+1$. A plane through r meets K either in $q+1$ points or in $q+2$ points. Let x denote the number of planes through r which meet K in $q+1$ points. Counting the size of $K-r$ by the planes through r, we get that $q^2+q+1-h=(q+1-h)x+(q+2-h)(q+1-x)$. Thus, $x=(2-h)+1$, which implies $h\le 2$ since there is at least one plane through r meeting K in $q+1$ points. Therefore, K is a cap, i.e. a set of points no three of which are collinear, a contradiction because the size of a cap is at most q^2+1, see Lemma 16.1.1 pag. 33 of [5]. □

Remark 6. *If $e=h$ K is a cone projecting an oval from a point.*

Proof. In this case K is a (q^2+q+1)–set of type $(1,q+1,2q+1)_2$ in PG(3,q), containing at least two lines, i.e. a cone projecting an oval from a point, see [2] Theorem 1.1 page 69. □

Remark 7. *If $e=2h$, K is a plane.*

Proof. In this case K is a (q^2+q+1)–set of type $(q+1,q^2+q+1)_2$ in PG(3,q), i.e. a plane. □

Since $n-1=p^e+p^h$, we write $n-1=p^f(p^g+1)$ where either $f=e$ and $g=h-e$ if $e<h$, or $f=h$ and $g=e-h$ if $e>h$. Since $q+1$ and q^2 are coprime, by $t_1 = [q^2(q+1)/(n-1)]-q$, we get that $p^g+1 \mid q+1$. Therefore, it follows that $p^g+1 \mid p^h+1$. If $g\neq 0$, by doing the division, $g \mid h$ and $h=gd$, with the integer g even and the integer d odd. Indeed, in this case, we have the factorization

$$p^h+1=(p^g+1)(p^{g(d-1)}-p^{g(d-2)}+p^{g(d-3)}-\ldots+1).$$

Thus, if the exponent h has no odd factor, i.e. h is a power of 2, then $e=0$, $e=h$ and $e=2h$ are the unique solutions and this completes the proof of the Theorem.

3.- Conclusion.

The work described in this paper is finalized to establish for which orders q pure incidence conditions are sufficient to identify ovoids and cones projecting an oval from a point. We have shown that, if the exponent h of the order $q=p^h$ has no odd factor, then, in PG(3,q), any (q^2+1)–set of class $[0,m,n]_1$ is an ovoid and any (q^2+q+1)–set of class $[1,m,n]_2$, containing at least two lines, is either a plane or a cone projecting an oval from a point. In the case in which the exponent h of the order $q=p^h$ has odd factors, there are other arithmetically feasible parameter sets for that order, and extra conditions need, so that this case is not interesting for our purpose. Indeed, we have proved that if $q=p^h$ and $2h=ef$, with the integer e even and the integer $f>1$ odd, a possible parameter of a (q^2+1)–set of type $(0,1,n)_1$ is $n=p^e+1$, but it is not known if such a set can exist. Moreover, we have proved that if $q=p^h$ and $h=gd$, with the integer g either equal to 1 or even and the integer $d>1$ odd, a possible parameter of a (q^2+q+1)–set of class $[1,q+1,n]_2$ is $n=p^e+p^h+1$, where either $e=h-g$ or $e=h+g$, but it is not known if such a set can exist. We simply want to point out the usefulness of the paper to show that certain assumptions in some characterizations are essential.

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