INVERSE-CONJUGATE COMPOSITIONS MODULO m

AUGUSTINE O. MUNAGI

ABSTRACT. We consider inverse-conjugate compositions of a positive integer n in which the parts belong to the residue class of 1 modulo an integer $m > 0$. It is proved that such compositions exist only for values of n that belong to the residue class of 1 modulo $2m$. An enumeration result is provided using the properties of inverse-conjugate compositions. This work extends recent results for inverse-conjugate compositions with odd parts.

1. INTRODUCTION

A composition of a positive integer n is a representation of n as a sequence of positive integers that sum to n. The terms are called parts while n is the *weight* of the composition. It is well known that there are $c(n) = 2^{n-1}$ compositions of n, and $c(n,k) = \binom{n-1}{k-1}$ compositions of n with exactly k parts which are also called k-compositions.

The conjugate of a composition may be obtained by drawing its zig-zag graph. Here each part is represented by a row of dots such that the first dot on a row is aligned with the last dot on the previous row. For example, the zig-zag graph of the composition $(5,3,1,2,2)$ is

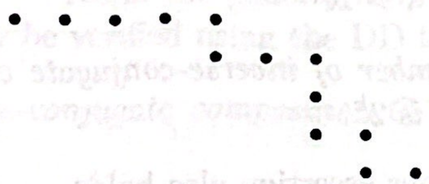

Figure 1

The conjugate is the composition corresponding to the columns of the graph, from left to right. Thus the conjugate of $(5,3,1,2,2)$, from Figure 1,

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is $(1,1,1,1,2,1,3,2,1)$. The conjugate of a composition C will be denoted by C'.

The *inverse* of a composition C, denoted by $\overline{C}$, is the composition obtained by reversing the order of the parts of C. A composition C is called *self-inverse* if it satisfies $C = \overline{C}$, and *inverse-conjugate* if $C' = \overline{C}$. For example one can verify that $(1,1,2,1,1,1,5,3)$ is an inverse-conjugate composition of 15.

A composition $C = (c_1, c_2, \dots)$ will be said to be congruent to an integer $t \geq 0$ modulo an integer $m > 0$, denoted by $C \equiv t \pmod m$, if each part c_i satisfies $c_i \equiv t \pmod m$.

We will study the enumeration properties of inverse-conjugate compositions that are $\equiv 1 \pmod m$. Note that an inverse-conjugate composition of $n > 1$ always contains 1 and a *big part*, i.e., a part > 1.

The number of compositions of n using only odd parts is known to be the nth Fibonacci number F_n ($F_n = F_{n-1} + F_{n-2}$, $F_0 = 0, F_1 = 1$) [2],[8, A000045]. The set of compositions of n that are $\equiv 1 \pmod m$ is in bijection with the set of compositions of $n + m - 1$ into parts greater that $m - 1$, where the common enumerator is the generalized Fibonacci number $\sum_{j \geq 0} \binom{n-1-(m-1)j}{j}$ (see [5]).

Specializing to inverse-conjugate compositions, Guo [1] recently proved that inverse-conjugate compositions using only odd parts exist for odd numbers of the type $4k + 1$ but not $4k + 3$. In the former case the inverse-conjugate compositions are enumerated by 2^k, $k > 0$.

We remark that Guo's result is a special case, $m = 2$, of the following results.

Theorem 1. *Let $m > 1$ be an integer. Then a positive integer n has an inverse-conjugate composition $C \equiv 1 \pmod m$ if and only if n has the form $n = 2mk + 1$ for some $k > 1$.*

Theorem 2. *The number of inverse-conjugate compositions of $2mk + 1$ that are $\equiv 1 \pmod m$ is 2^k.*

The following stronger assertion also holds.

Theorem 3. *Let $m > 1, k > 0$ be integers. Then a number of the form $2mk + 1$ has an inverse-conjugate composition $C \equiv 1 \pmod m$ with r big parts, where $1 \leq r \leq k$, and not $r > k$.*

Denoting the number of inverse-conjugate compositions of n that are $\equiv 1 \pmod m$ by $v(n, m)$, we have, for example, $v(9,3) = 0 = v(17,3)$ while $v(n,3) > 0$ for $n = 7, 13, 19, \dots$. We will prove Theorems 1 to 3 in Section 3 as well as the following adaptation of a classical identity of P. A. MacMahon between inverse-conjugate and self-inverse compositions (see [3, 6]).

Theorem 4. *The number of inverse-conjugate compositions of* $2mk+1$ *that are* $\equiv 1 \pmod m$ *is equal to the number of self-inverse compositions* C *of* $2mk+1$ *such that both* C *and* C' *are* $\equiv 1 \pmod m$.

We begin, in Section 2, by summarizing additional properties of compositions that will be used in the proofs.

2. Properties of compositions and preliminary results

We recall an alternative method of obtaining the conjugate of a composition that is known as the Direct Detection (DD) technique, [5, 6]. It will be convenient to write compositions symbolically by representing a maximal string of 1's of length x by 1^x, where two adjacent big parts are assumed to be separated by 1^0. Then the general composition has the following two forms up to inversion.

(1) $C = (1^{a_1}, b_1, 1^{a_2}, b_2, \dots)$, $a_1 > 0, a_i \geq 0, i > 1$, $b_i \geq 2\ \forall i$;

(2) $E = (b_1, 1^{a_1}, b_2, 1^{a_2}, \dots)$, $a_i \geq 0, b_i \geq 2$.

The conjugate, in each case, is given by the rule:

(1′) $C' = (a_1+1, 1^{b_1-2}, a_2+2, 1^{b_2-2}, \dots)$.

(2′) $E' = (1^{b_1-1}, a_1+2, 1^{b_2-2}, a_2+2, \dots)$.

For example, the conjugate of $(5,3,1,2,2) = (5,1^0,3,1,2,1^0,2)$ is given by $(1^4,2,1,3,1^0,2,1)$, that is, $(5,3,1,2,2)' = (1^4,2,1,3,2,1)$.

If C is a composition of n with f parts, then it is known that C' has $n-f+1$ parts; and if $n > 1$, then $C \neq C'$ (there is no self-conjugate composition besides (1)). But when C is inverse-conjugate, then $f = n - f + 1$ or $n = 2f - 1$. Thus inverse-conjugate compositions are defined only for odd weights. We recall the general form of an inverse-conjugate composition which may be verified using the DD technique (see [4, 6]):

Lemma 5. *An inverse-conjugate composition* C *(or its inverse) has the form:*

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \dots, b_{r-1}, 1^{b_1-2}, b_r),\ b_i \geq 2. \quad (1)$$

The following properties follow at once from Lemma 5.

(i) The length of C is given by $f = b_1 + \cdots + b_r - r + 1$.
(ii) The sum of big parts of C is $b_1 + \cdots + b_r = f + r - 1$, $1 \leq r < f$.
(iii) An inverse-conjugate composition C of $n = 2f - 1 > 1$ is completely determined by the sequence of its big parts. Lemma 5 implies an obvious algorithm for obtaining the inverse-conjugate composition corresponding to a finite sequence of big parts. Accordingly for each integer $s > 0$ we define a function InvConj from the set of all s-tuples $(b_1, \dots, b_s)$, $b_i > 1\ \forall\ i$, to

the set of all sequences of the form (1), that is,

$$InvConj : (b_1, \dots, b_s) \longmapsto (1^{b_s-1}, b_1, 1^{b_{s-1}-2}, b_2, \dots, b_{s-1}, 1^{b_1-2}, b_s). \quad (2)$$

In particular the set of inverse-conjugate compositions of $n = 2f - 1$ with r big parts is given by the image of the restriction of InvConj to the set of r-tuples $(b_1, \dots, b_r)$ satisfying property (ii).

We will also use the operations of *concatenation* '|' and *join* '⊎' which are defined for two compositions $A = (a_1, \dots, a_i)$ and $B = (b_1, \dots, b_j)$ by

$$A|B = (a_1, \dots, a_i, b_1, \dots, b_j) \quad \text{and} \quad A \uplus B = (a_1, \dots, a_{i-1}, a_i + b_1, b_2, \dots, b_j).$$

The following theorem from [6] gives a useful method of dissecting inverse-conjugate compositions.

Theorem 6. *If $C = (c_1, \dots, c_k)$ is an inverse-conjugate composition of $n = 2k - 1 > 1$, or its inverse, then there is an index j such that $c_1 + \cdots + c_j = k - 1$ and $c_{j+1} + \cdots + c_k = k$ with $c_{j+1} > 1$.*

Moreover,

$$\overline{(c_1, \dots, c_j)} = (c_{j+1} - 1, c_{j+2}, \dots, c_k)' \quad (3)$$

Thus C can be written in the form

$$C = A|(1) \uplus B \quad \text{such that} \quad B' = \overline{A}, \quad (4)$$

where A and B are generally different compositions of $k - 1$.

The following lemma will play a useful role in the next section.

Lemma 7. *The number of r-compositions of N into parts greater than m that are $\equiv 1 \pmod{m}$ is equal to the number of r-compositions of $\frac{N-r}{m}$.*

Proof. Let $(b_1, \dots, b_r)$ be a composition of N with $m < b_i \equiv 1 \pmod{m}$ for all i. Then applying the operation $b_i \to \frac{b_i - 1}{m}$ to each part we obtain the map

$$(b_1, \dots, b_r) \longmapsto \left(\frac{b_1 - 1}{m}, \dots, \frac{b_r - 1}{m} \right), \quad (5)$$

where the image is a composition of $\frac{b_1-1}{m} + \cdots + \frac{b_r-1}{m} = \frac{N-r}{m}$. This map is clearly one-to-one. □

The numbers $v(2mk + 1, m)$ fulfill the following recurrence relation.

Proposition 8. *We have*

$$v(2mk + 1, m) = 2v(2m(k - 1) + 1, m), \ n > 1, \ v(1, m) = 1. \quad (6)$$

Proof. Let $V(n)$ denote the set of compositions enumerated by $v(n, m)$ and let $V_1(n)$ be the subset containing only compositions with first part 1, with $V_2(n) = V(n) \setminus V_1(n)$. Then we define two maps $\theta_i : V(2m(k - 1) + 1) \to V_i(2mk + 1)$, $i = 1, 2$, as follows.

$$\theta_1 : C \mapsto (1^m)|C \uplus (m). \quad (7)$$

$$\theta_2 : C \mapsto (m) \uplus |C|(1^m). \tag{8}$$

Note that $\theta_i(C) \in V_i(2mk+1)$. To illustrate θ_1 we consider

$C = (1^{mc_t}, mc_1+1, 1^{mc_{t-1}-1}, \ldots, 1^{mc_1-1}, mc_t+1)$. Then

$\theta_1(C) = (1^{m(c_t+1)}, mc_1+1, 1^{mc_{t-1}-1}, \ldots, 1^{mc_1-1}, m(c_t+1)+1)$,

and for $\overline{C} = (mc_t+1, 1^{mc_1-1}, \ldots, \ldots, mc_1+1, 1^{mc_t}) \neq C$, we have

$\theta_1(\overline{C}) = (1^m, mc_t+1, 1^{mc_1-1}, \ldots, \ldots, mc_1+1, 1^{mc_t-1}, m+1)$.

This map is clearly invertible. Thus for instance, if $T = (1^{ms_r}, ms_1+1, \ldots, 1^{ms_1-1}, ms_r+1) \in V_1(2mk+1)$, then

$\theta_1^{-1} : T \mapsto (1^{m(s_r-1)}, ms_1+1, \ldots, 1^{ms_1-1}, m(s_r-1)+1)$, $s_r > 1$,

and when $s_r = 1$, we have

$\theta_1^{-1} : T \mapsto (ms_1+1, 1^{ms_{r-1}-1}, \ldots, ms_{r-1}+1, 1^{ms_1})$.

Thus θ_1 is a bijection. By a similar reasoning one can show that θ_2 is a bijection.

Therefore (7) and (8) imply

$2|V(2m(k-1)+1)| = |Im(\theta_1)| + |Im(\theta_2)| = |V_1(2mk+1)| + |V_2(2mk+1)| = |V(2mk+1)|$, which gives the recurrence (6). □

3. Proofs of the main Theorems

In this section we present two proofs of Theorems 1 and 2, and a proof of Theorems 3 and 4.

First Proof of Theorem 1. Let the inverse-conjugate composition C in (1) satisfy $C \equiv 1 \pmod m$. Then $b_i = mt_i + 1$, $t_i > 0$ for each i. Thus by property (i) after Lemma 5, and because $n = 2f - 1$, we have

$$n = 2(mt_1 + 1 + \cdots + mt_r + 1 - r + 1) - 1 = 2m(t_1 + \cdots + t_r) + 1.$$

Setting $k = t_1 + \cdots + t_r$ gives $n = 2mk + 1$.

Conversely we show that if $n = 2mk+1, k > 0$, then n has an inverse-conjugate composition $C \equiv 1 \pmod m$. It will suffice to choose $C = (1^{mk}, mk+1)$. □

Second Proof of Theorem 1. The proof will be deduced from Lemma 7. Assume that $n = 2f-1$ has an inverse-conjugate composition $C \equiv 1 \pmod m$ with C as in (1). Then $b_1 + \cdots + b_r = f + r - 1$, and by Lemma 7, the sum $\frac{(f+r-1)-r}{m} = \frac{f-1}{m}$ is integral, equal to some k. That is, $\frac{f-1}{m} = k$ or $f = mk+1$ which implies that $n = 2mk+1$.

Conversely, we show that when $n = 2mk+1$, $k > 0$, then n has an inverse-conjugate composition $\equiv 1 \pmod m$. We construct such composition C. Consider any composition $(c_1, \ldots, c_r)$ of k. Then reversing the map (5), we have $(c_1, \ldots, c_r) \mapsto (mc_1+1, \ldots, mc_r+1)$, where the latter is a composition of $mk+r$. Lastly we invoke InvConj and find that

$InvConj((mc_1+1, \ldots, mc_r+1)) = (1^{mc_r}, mc_1+1, 1^{mc_{r-1}-1}, mc_2+1 \ldots, 1^{mc_1-1}, mc_r+1)$, which is a composition whose weight is $mk+r+\sum_{i=1}^{r}(mc_i-1)+1 = 2mk+1$. Hence the proof. $\square$

First Proof of Theorem 2. The proof may be deduced from the second proof of Theorem 1. The construction there implies that there are as many standard compositions of k as there are inverse-conjugate compositions C of $2mk+1$ that are $\equiv 1 \pmod{m}$ and have first part 1. The inverses (i.e., conjugates), $\overline{C} = C' \neq C$ are similarly identified by compositions of k via the construction

$$(mc_1+1, \dots, mc_r+1) \longmapsto (mc_1+1, 1^{mc_r-1}, \dots, mc_r+1, 1^{mc_1}). \qquad (9)$$

Hence the total number of inverse-conjugate compositions is $2c(k) = 2 \cdot 2^{k-1} = 2^k$. $\square$

Example 9. *Let $m = 3$, $k = 7$ so that $2mk+1 = 43$, and consider* $(7,4,10,4) \equiv 1 \pmod 3$. *Then we have*

$$InvConj((7,4,10,4)) = (1^3, 7, 1^8, 4, 1^2, 10, 1^5, 4)$$

while (9) *gives*

$$(7,4,10,4) \longmapsto (7, 1^2, 4, 1^8, 10, 1^2, 4, 1^6).$$

Similarly $(4,10,4,7)$ *produces* $(1^6, 4, 1^2, 10, 1^8, 4, 1^2, 7)$ *and* $(4, 1^5, 10, 1^2, 4, 1^8, 7, 1^3)$.

Remark 10. *It is easy to deduce from the foregoing proofs that the number of inverse-conjugate compositions of $2mk+1$ with exactly r big parts $\equiv 1 \pmod{m}$ is $2c(k,r) = 2\binom{k-1}{r-1}$.*

Second Proof of Theorem 2. The proof is obtained by iterating the recurrence (6) to obtain

$$v(2mk+1, m) = 2^j v(2m(k-j)+1, m), \ 1 \le j \le k,$$

which leads to $v(2mk+1,m) = 2^k v(1,m)$ or $v(2mk+1,m) = 2^k$ as desired. $\square$

Proof of Theorem 3. We will give an explicit construction of an inverse-conjugate composition C of $n = 2mk+1$ with $C \equiv 1 \pmod{m}$ such that C has r big parts, $1 \le r \le k$. Let the sequence of r big parts of C be $B = (m+1, m+1, \dots, m+1, y)$, where

$$y = (mk+r) - (r-1)(m+1) = m(k-r+1)+1 \equiv 1 \pmod{m}.$$

That is, $B = (m+1, m+1, \dots, m+1, m(k-r+1)+1)$.

We now apply InvConj to obtain $InvConj(B) = C$, where

$$C = (1^{m(k-r+1)}, m+1, 1^{m-1}, m+1, 1^{m-1}, \dots, m+1, 1^{m-1}, m(k-r+1)+1)$$

which is an inverse-conjugate composition of $2mk+1$ with the first $r-1$ big parts equal to $m+1$.
Lastly, note that we cannot have an inverse-conjugate composition $C \equiv 1 \pmod{m}$ of $2mk+1$ with $r > k$ big parts, otherwise its weight would be

$$2(b_1+\cdots+b_r-r+1)-1 \geq 2(r(m+1)-r+1)-1 = 2rm+1 > 2mk+1. \qquad \square$$

We now turn to Theorem 4 which we restate in the extended form

Theorem 11. *The following sets of compositions contain the same number of objects.*
(i) Compositions C of $mk+1$ where both C and C' are $\equiv 1 \pmod{m}$.
(ii) Self-inverse compositions T of $n = 2mk+1$ where both T and T' are $\equiv 1 \pmod{m}$.
(iii) Inverse-conjugate compositions of $n = 2mk+1$ which are $\equiv 1 \pmod{m}$.

(Note that Theorem 4 corresponds to $(ii) \iff (iii)$ in Theorem 11).

Proof. The proof will be given in the order: $(i) \implies (ii) \implies (iii) \implies (i)$. Let the corresponding sets be denoted by

$$(i): C(mk+1) \quad (ii): SI(2mk+1) \quad (iii): IC(2mk+1).$$

$(i) \implies (ii)$: Define a map $C(mk+1) \mapsto SI(2mk+1)$ by

$$(c_1,\ldots,c_j) \mapsto (c_1,\ldots,c_{j-1},2c_j-1,c_{j-1},\ldots,c_1).$$

This map is clearly injective. The weight of the image composition is $2((mk+1)-c_j)+2c_j-1 = 2mk+1$. Since $c_j = ms+1$, $s \geq 0$, we have $2c_j-1 = 2ms+1$, and the inherited parts insure that the image and its conjugate are $\equiv 1 \pmod{m}$.
$(ii) \implies (iii)$: If $T = (a_1,\ldots,a_j,d,a_j,\ldots,a_1) \in SI(2mk+1)$, then d is odd and satisfies $d \equiv 1 \pmod{m}$, that is, $d = 2ms+1$, $s \geq 0$. So T has the form $T = A|(2ms+1)|\overline{A}$, where A is a composition of $M \leq mk$. We define a map $SI(2mk+1) \mapsto IC(2mk+1)$ and use Theorem 6 to clarify the resulting images:
If $s=0$, then $T = A|(1)|\overline{A} \mapsto A|\left((1)|\overline{A}\right)'$ which is a member of $IC(2mk+1)$ of type $A|(1) \uplus \overline{A'}$ (from Theorem 6).
If $s>0$, then $T = A|(2ms+1)|\overline{A} \mapsto A|(ms) \uplus \left((ms+1)|\overline{A}\right)'$ which is a member of $IC(2mk+1)$ of type $A \uplus (1)|\overline{A'}$.
Each image consists of inherited parts from T and T' except the two middle strings of parts, 1^{ms-1}, and $ms+1$. But these also fulfill the required properties. The second case is illustrated below.
Let $s>0$ and $T = (1^{a_1},b_1,\ldots,b_{t-1},1^{a_t},2ms+1,1^{a_t},b_{t-1},\ldots,b_1,1^{a_1})$, where $a_1 \equiv 0 \pmod{m}$, $a_i \equiv -1 \pmod{m}, i>1$, $b_i \equiv 1 \pmod{m}$. Then $T \in SI(2mk+1)$, and

$$T = (1^{a_1}, b_1, \dots, b_{t-1}, 1^{a_t})|(2ms+1)|(1^{a_t}, b_{t-1}, \dots, b_1, 1^{a_1}) \mapsto E,$$

where $E = (1^{a_1}, b_1, \dots, b_{t-1}, 1^{a_t})|(ms) \uplus (ms+1, 1^{a_t}, b_{t-1}, \dots, 1^{a_2}, b_1, 1^{a_1})'$. Using the DD technique we obtain

$$E = (1^{a_1}, b_1, \dots, b_{t-1}, 1^{a_t})|(ms) \uplus (1^{ms}, a_t+2, 1^{b_{t-1}-2}, \dots, a_2+2, 1^{b_1-2}, a_1+1).$$

$$= (1^{a_1}, b_1, \dots, b_{t-1}, 1^{a_t}, ms+1, 1^{ms-1}, a_t+2, 1^{b_{t-1}-2}, \dots, a_2+2, 1^{b_1-2}, a_1+1).$$

which may be classified as

$$E = (1^{a_1}, b_1, \dots, b_{t-1}, 1^{a_t}, ms) \uplus (1)|(1^{ms-1}, a_t+2, 1^{b_{t-1}-2}, \dots, a_2+2, 1^{b_1-2}, a_1+1).$$

$$= A \uplus (1)|\overline{A'}.$$

(iii) $\implies$ *(i)*: Let $E = (e_1, \dots, e_{mk+1}) \in IC(2mk+1)$. Then Theorem 6 implies that E satisfies either of the following properties:

(a) $e_1 + \cdots + e_j = mk$, $e_{j+1} + \cdots + e_n = mk+1$ with $e_{j+1} > 1$, and $(e_1, \dots, e_j, 1)' = \overline{(e_{j+1}, \dots, e_n)}$.

(b) $e_1 + \cdots + e_j = mk+1$, $e_{j+1} + \cdots + e_n = mk$ with $e_j > 1$, and $(e_1, \dots, e_j - 1)' = \overline{(e_{j+1}, \dots, e_n)}$.

So if E satisfies (a), then $E \mapsto (e_1, \dots, e_j, 1)$, and if it satisfies (b), then $E \mapsto (e_1, \dots, e_j)$. In either case we obtain a unique member of $C(mk+1)$. □

As an illustration of the three maps in the proof of Theorem 11 consider $m = 4$ and $k = 3$, i.e., $mk+1 = 13$ and $2mk+1 = 25$. Then we have $C(13) = 8 = SI(25) = IC(25)$. The details of correspondences of the compositions are given in Table 1.

$C(13)$	$\to$	$SI(25)$	$\to$	$IC(25)$
(13)	$\mapsto$	(25)	$\mapsto$	$(13, 1^{12})$
$(1^4, 9)$	$\mapsto$	$(1^4, 17, 1^4)$	$\mapsto$	$(1^4, 9, 1^7, 5)$
$(5, 1^3, 5)$	$\mapsto$	$(5, 1^3, 9, 1^3, 5)$	$\mapsto$	$(5, 1^3, 5, 1^3, 5, 1^4)$
$(9, 1^4)$	$\mapsto$	$(9, 1^7, 9)$	$\mapsto$	$(9, 1^3, 5, 1^8)$
$(1^8, 5)$	$\mapsto$	$(1^8, 9, 1^8)$	$\mapsto$	$(1^8, 5, 1^3, 9)$
$(1^4, 5, 1^4)$	$\mapsto$	$(1^4, 5, 1^7, 5, 1^4)$	$\mapsto$	$(1^4, 5, 1^3, 5, 1^3, 5)$
$(5, 1^8)$	$\mapsto$	$(5, 1^{15}, 5)$	$\mapsto$	$(5, 1^7, 9, 1^4)$
(1^{13})	$\mapsto$	(1^{25})	$\mapsto$	$(1^{12}, 13)$

TABLE 1. The maps in the proof of Theorem 11 when $m = 4$ and $k = 3$.

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Augustine Munagi, The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa

E-mail address: `Augustine.Munagi@wits.ac.za`