Majestic t-Tone Colorings of Bipartite Graphs with Large Cycles

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Abstract

For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$, let $\mathcal{P}([k])$ denote the power set of the set $[k]$ and let $\mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$. For each integer t with $1 \le t < k$, let $\mathcal{P}_t([k])$ denote the set of t-element subsets of $\mathcal{P}([k])$. For an edge coloring $c : E(G) \to \mathcal{P}_t([k])$ of a graph G, where adjacent edges may be colored the same, $c' : V(G) \to \mathcal{P}^*([k])$ is the vertex coloring in which $c'(v)$ is the union of the color sets of the edges incident with v. If c' is a proper vertex coloring of G, then c is a majestic t-tone k-coloring of G. For a fixed positive integer t, the minimum positive integer k for which a graph G has a majestic t-tone k-coloring is the majestic t-tone index $\mathrm{maj}_t(G)$ of G. It is known that if G is a connected bipartite graph of order at least 3, then $\mathrm{maj}_t(G) = t+1$ or $\mathrm{maj}_t(G) = t+2$ for each positive integer t. It is shown that (i) if G is a 2-connected bipartite graph of arbitrarily large order n whose longest cycles have length ℓ where $n-5 \le \ell \le n$ and $t \ge 2$ is an integer, then $\mathrm{maj}_t(G) = t+1$ and (ii) there is a 2-connected bipartite graph F of arbitrarily large order n whose longest cycles have length $n-6$ and $\mathrm{maj}_2(F) = 4$. Furthermore, it is shown for integers $k, t \ge 2$ that there exists a k-connected bipartite graph G such that $\mathrm{maj}_t(G) = t+2$. Other results and open questions are also presented.

Key Words: majestic t-tone coloring, majestic t-tone index, bipartite graphs.

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1 Introduction

For a nontrivial connected graph G, an edge coloring c of G is a function $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ for some positive integer k. Such a coloring c is a k-edge coloring. An edge coloring c is *unrestricted* if no condition is placed on how the edges may be colored and so adjacent edges may be colored the same. If every two adjacent edges of G are colored differently, then c is a *proper edge coloring* and the minimum number of colors required of a proper edge coloring of G is its *chromatic index* $\chi'(G)$. A vertex coloring of G is a function $c' : V(G) \to [k]$ for some positive integer k. A vertex coloring c' of a graph G is *proper* if adjacent vertices are colored differently and the minimum number of colors required of a proper vertex coloring of G is its *chromatic number* $\chi(G)$. We refer to the book [5] for graph theory notation and terminology not described in this paper. All graphs under consideration are connected graph of order at least 3.

During the past three decades, a number of edge colorings of graphs have been described that give rise to various vertex colorings of the graphs (see [1, 4, 7, 11, 12, 13] for example). In [2, 3, 6, 10], the color of a vertex is defined as the set of colors of the edges incident with the vertex, where the goal is to minimize the number of colors assigned to the edges so that the resulting vertex coloring is proper.

For a nontrivial connected graph G on which has been defined an edge coloring $c : E(G) \to [k]$, the associated vertex coloring c' is defined by

$$c'(v) = \{c(e) : e \in E_v\}, \tag{1}$$

where E_v is the set of edges incident with v; that is, $c'(v)$ is the set of colors of those edges incident with v. The edge coloring c is called a *majestic k-edge coloring* of G (or, more simply a *majestic coloring* of G) if the induced vertex coloring c' is a proper vertex coloring of G. The minimum positive integer k for which G has a majestic k-edge coloring is the *majestic chromatic index* of G (or, more simply the *majestic index* of G) and denoted by $\text{maj}(G)$ (or $\chi'_m(G)$, which is used in [2]). In [6, 10], the majestic index of a graph G is called the *general neighbour-distinguishing index* of G and is denoted by *gndi*(G). Since there is no majestic edge coloring of K_2, the majestic index does not exist for K_2. Thus, we consider only connected graphs of order at least 3. If G is a connected graph of size $m \geq 2$, then $\text{maj}(G)$ exists and $2 \leq \text{maj}(G) \leq m$. In the case of bipartite graphs G of order at least 3, $\text{maj}(G)$ is either 2 or 3.

Theorem 1.1 [2, 6, 10] *If G is a connected bipartite graph of order 3 or more, then*

$$2 \leq \text{maj}(G) \leq 3.$$

A generalization of majestic colorings was introduced by Chartrand and studied in [8, 9]. For a positive integer k, let $\mathcal{P}([k])$ denote the power set of the set $[k]$ and let $\mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$ denote the set of nonempty subsets of $[k]$. For each integer t with $1 \le t < k$, let $\mathcal{P}_t([k])$ denote the set of t-element subsets of $\mathcal{P}([k])$; consequently, $|\mathcal{P}_t([k])| = \binom{k}{t}$. For each unrestricted edge coloring $c : E(G) \to \mathcal{P}_t([k])$ of a graph G, the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined as described in (1); that is, $c'(v)$ of a vertex v is the set of colors of those edges incident with v. If c' is a proper vertex coloring of G, then c is called a *majestic t-tone k-coloring* of G. An edge coloring of G is a *majestic t-tone coloring* if it is a majestic t-tone k-coloring for some integer $k > t$. For a fixed positive integer t, the minimum positive integer k for which a graph G has a majestic t-tone k-coloring is called the *majestic t-tone index* $\text{maj}_t(G)$ of G. In particular, a majestic 1-tone k-coloring is a majestic k-coloring and the majestic 1-tone index of a graph G is the majestic index of G.

The following results were obtained in [8, 9].

Theorem 1.2 *If G is a connected graph of order at least 3 and $t \ge 2$ is an integer, then $t+1 \le \text{maj}_t(G) \le \text{maj}(G) + (t-1)$.*

Proposition 1.3 *If G is a connected graph of order at least 3 and s and t are positive integers with $s \le t$, then*

$$\text{maj}_t(G) \le \text{maj}_s(G) + (t-s)$$

It is known for an integer $t \ge 2$ that only bipartite graphs have majestic t-tone index $t+1$.

Theorem 1.4 *If G is a connected graph such that $\text{maj}_t(G) = t+1$ for some integer $t \ge 2$, then G is bipartite.*

There are bipartite graphs having the majestic t-tone index $t+2$, however.

Theorem 1.5 *If G is a connected bipartite graph of order at least 3 and $t \ge 2$ is an integer, then $t+1 \le \text{maj}_t(G) \le t+2$.*

For a positive integer t, a connected bipartite graph G is of *type* 1 if $\text{maj}_t(G) = t+1$ and is of *type* 2 if $\text{maj}_t(G) = t+2$. Theorems 1.1 and 1.5 give rise to the natural question of determining which connected bipartite graphs are of which type. This question is answered for some well-known classes of bipartite graphs, namely complete bipartite graphs, trees and connected unicyclic bipartite graphs.

Proposition 1.6 *Let r, s, t be integers where $r, s, t \geq 2$. Then*

$$\mathrm{maj}_t(K_{r,s}) = t + 1.$$

Theorem 1.7 *Let T be a tree of order 3 or more and $t \geq 2$ an integer. Then $\mathrm{maj}_t(T) = t+1$ if and only if all end-vertices of T belong to the same partite set of T.*

A *unicyclic graph* is a connected graph containing exactly one cycle. In particular, the graph C_n is a unicyclic graph. A unicyclic graph G is therefore bipartite only when the unique cycle of G is an even cycle.

Theorem 1.8 *Let G be a unicyclic bipartite connected graph and $t \geq 2$ an integer. Then $\mathrm{maj}_t(G) = t+1$ if and only if all end-vertices of G belong to the same partite set of G.*

2 On 2-Connected Bipartite Graphs of Type 1

We now investigate majestic t-tone indices of connected bipartite graphs having large cycles. In order to do this, we first determine the majestic t-tone index of cycles for each positive integer t. For each integer $n \geq 3$, it is known (see [2]) that

$$\mathrm{maj}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \not\equiv 0 \pmod 4. \end{cases} \tag{2}$$

Thus, $\mathrm{maj}(C_n) = 2$ if and only if $n \equiv 0 \pmod 4$.

Proposition 2.1 *For integers $n \geq 3$ and $t \geq 2$,*

$$\mathrm{maj}_t(C_n) = \begin{cases} t+1 & \text{if } n \text{ is even} \\ t+2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a cycle of order $n \geq 3$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 4$ is even.* Thus, either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$, then $\mathrm{maj}(C_n) = 2$ by (2) and so $\mathrm{maj}_t(C_n) = t + 1$ by Theorem 1.2. In fact, the edge coloring $c : E(C_n) \to \mathcal{P}_t([t + 1])$ defined by

$$c(e) = \begin{cases} [t+1] - \{t+1\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \pmod 4 \\ & \text{and } 4 \leq i \leq n \\ [t+1] - \{t\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \pmod 4 \\ & \text{and } 2 \leq i \leq n-2 \end{cases}$$

is a majestic t-tone coloring of C_n. If $n \equiv 2 \pmod 4$, the edge coloring $c: E(C_n) \to \mathcal{P}_t([t+1])$ defined by

$$c(e) = \begin{cases} [t+1] - \{t+1\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \pmod 4 \\ & \text{and } 2 \le i \le n-2 \\ [t+1] - \{t\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \pmod 4 \\ & \text{and } 4 \le i \le n-2 \\ [t+1] - \{1\} & \text{if } e \text{ is incident with } v_n. \end{cases}$$

In each case, $c'(v_i) = [t+1]$ for all odd integers i with $1 \le i \le n-1$ and $|c'(v_i)| = t$ for all even integers i with $2 \le i \le n$. Since c' is proper, it follows that c is a majestic t-tone coloring of C_n and so $\text{maj}_t(C_n) \le t+1$. Therefore, $\text{maj}_t(C_n) = t+1$ when n is even.

Case 2. *$n \ge 3$ is odd.* Since C_n is an odd cycle, it follows by Theorem 1.4 that $\text{maj}_t(C_n) \ge t+2$. By Theorem 1.2 and (2), $\text{maj}_t(C_n) \le (t-1) + \text{maj}(C_n) = t+2$. Thus, $\text{maj}_t(C_n) = t+2$ if n is odd. ■

It is useful to observe that the majestic t-tone $(t+1)$-colorings of the even cycle C_n of order $n \ge 4$ in the proof of Proposition 2.1 use only three distinct t-element subsets of $[t+1]$. Furthermore, there is a majestic t-tone $(t+1)$-coloring c of $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, where $n \ge 4$ is even, that uses only three distinct t-element subsets of $[t+1]$ such that for $1 \le i \le n$ either

- ⋆ $c'(v_i) = [t+1]$ for all even integers i and $|c'(v_i)| = t$ for all odd integers i or
- ⋆ $|c'(v_i)| = t$ for all even integers i and $c'(v_i) = [t+1]$ for all odd integers i.

Figure 1 shows such colorings for $t = 2$ and $n \in \{10, 12\}$.

Remarks: With the previous comments and Proposition 1.3 in mind, we may therefore state the following theorems for an integer $t \ge 2$ in general but we need only verify them for $t = 2$. Let S_1, S_2, S_3 be the three distinct 2-element subsets of $[3] = \{1, 2, 3\}$.

Since the largest cycle that a graph can have is a Hamiltonian cycle, we begin with Hamiltonian bipartite graphs. The result below follows immediately from the proof of Proposition 2.1.

Corollary 2.2 *If G is a Hamiltonian bipartite graph of even order at least 4 and $t \ge 2$ is an integer, then* $\text{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a Hamiltonian cycle of G, where $n \ge 4$ is even, and let $c_C : E(C) \to \mathcal{P}_2([3])$ be a majestic 2-tone

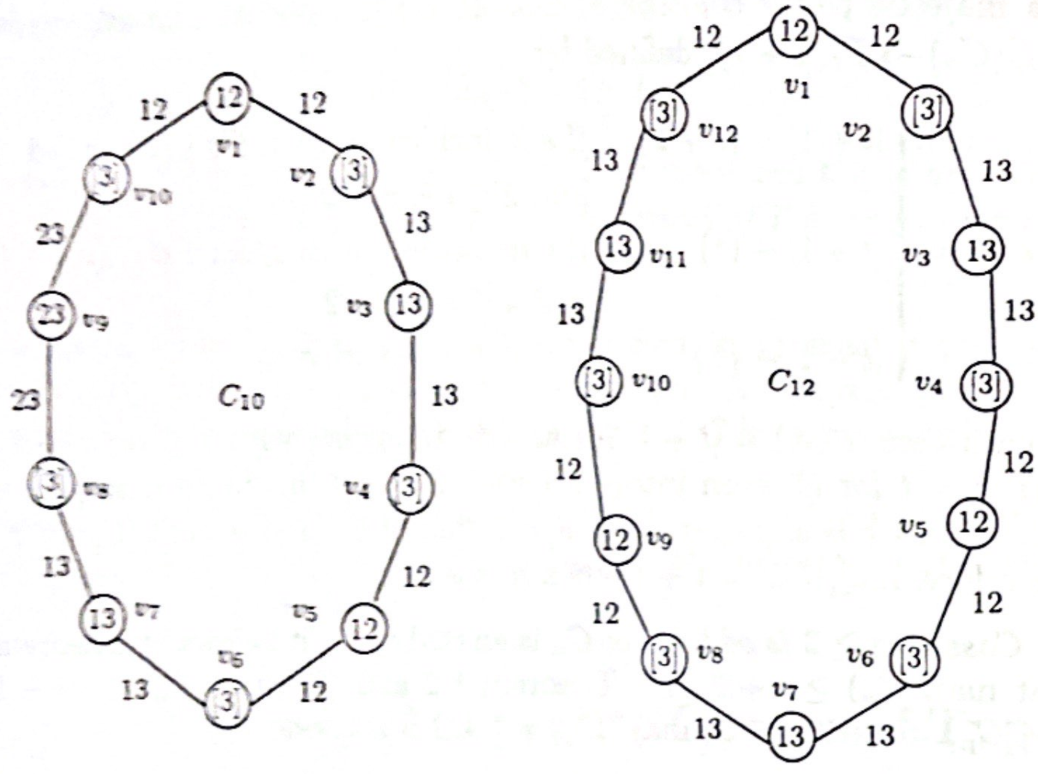

Figure 1: Majestic 2-tone 3-colorings of C_{10} and C_{12}

3-coloring of C. If $G \neq C$, then G contains chords. Each chord e of C joins vertices v_i and v_j of C such that either (1) $|c'_C(v_i)| = 2$ and $c'_C(v_j) = [3]$ or (2) $c'_C(v_i) = [3]$ and $|c'_C(v_j)| = 2$. We may assume, without of generality, that $|c'_C(v_i)| = 2$. In this case, define $c(v_iv_j) = c'_C(v_i)$. Coloring all chords of C in this way produces an edge coloring $c : E(G) \to \mathcal{P}_2([3])$ such that $c'_C(v_i) = c'(v_i)$ for all i with $1 \leq i \leq n$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t + 1$. ■

Theorem 2.3 *If G is a connected bipartite graph of odd order $n \geq 5$ whose longest cycles have order $n-1$ and $t \geq 2$ is an integer, then* $\text{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1)$ be a longest cycle of G and let v be the vertex of G that is not on C. Then the subgraph $H = G - v$ is Hamiltonian. By Corollary 2.2, $\text{maj}_t(H) = t + 1$. Let $c_H : E(H) \to \mathcal{P}_2([3])$ be a majestic 2-tone 3-coloring of H. We now construct a majestic 2-tone 3-coloring $c : E(G) \to \mathcal{P}_2([3])$ of G as follows. Necessarily, the neighbors of v in G all have an odd subscript or all have an even subscript. Hence, either each neighbor of v has vertex color [3] or each neighbor of v has a 2-element subset of [3] as its vertex color. Let S_1, S_2, S_3 be the three distinct 2-element subsets of $[3] = \{1, 2, 3\}$. If each neighbor of v has vertex color [3], then assign each edge incident with v the color S_1. Then $c'(v) = S_1$ and the resulting edge coloring is a majestic 2-tone 3-coloring of G. If each neighbor of v has a 2-element subset of [3] as its vertex color, then relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-1$ and reapply the majestic coloring c_H to the new labeling of H. Thus, each neighbor of v has vertex color [3] and we proceed as above.

In each case, the graph G has a majestic 2-tone 3-coloring and so $\text{maj}_t(G) = t+1$. ■

We now show that if G is a 2-connected bipartite graph of sufficiently large order n which fails to contain either an n-cycle (when n is even) or an $(n-1)$-cycle(when n is odd) but does contain an $(n-2)$-cycle, then we still have the same conclusion as in Theorem 2.3.

In the proofs of the next two results, let S_1, S_2, S_3 be the three distinct 2-element subsets of $[3] = \{1, 2, 3\}$.

Theorem 2.4 *If G is a 2-connected bipartite graph of even order $n \geq 12$ whose longest cycles have order $n-2$ and $t \geq 2$ is an integer, then* $\text{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-2}, v_{n-1} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $u, v \in V(G)$ such that $H = G - u - v$. Thus, H is a Hamiltonian bipartite graph of order $n-2$. By Corollary 2.2, $\text{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_2([3])$ be a majestic 2-tone 3-coloring of H. We show that there is a majestic 2-tone 3-coloring c of G.

Since G is bipartite, the vertex u is adjacent to vertices v_i $(1 \leq i \leq n-2)$ of C such that all integers i are odd or all integers i are even. The same is true of the vertex v. Consequently, either (1) each neighbor of u has a 2-element subset of [3] as its vertex color or (2) each neighbor of u has [3] as its vertex color. The same is true of the vertex v. We consider the following two cases:

Case 1. *u and v are adjacent in G.* Without loss of generality, we may assume that the neighbors of u in H are colored [3] and the neighbors of v are colored with 2-element subsets of [3]. Furthermore, we may assume some neighbor of v is colored S_1. We define a coloring of G as follows: For each edge e incident with u define $c(e) = S_2$, for each edge vv_i $(1 \leq i \leq n-2)$, define $c(vv_i) = c_H(v_i)$, and for each edge $e \in E(H)$ define $c(e) = c_H(e)$. Then $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-2$, $c'(u) = S_2$ and $c'(v) = [3]$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Case 2. *u and v are not adjacent in G.* We consider three possible situations here.

Subcase 2.1. *The neighbors of u and v are all colored* [3]. We define a coloring of G as follows: For each edge e incident with u or v, define $c(e) = S_1$, and for each edge $e \in E(H)$ define $c(e) = c_H(e)$. Then $c'(v_i) = c'_H(v_i)$

for $1 \leq i \leq n-2$ and $c'(u) = c'(v) = S_1$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 2.2. Without loss of generality, the neighbors of u are colored $[3]$ *and the neighbors of v are colored with 2-element subsets of* $[3]$.

Subcase 2.2.1. Two neighbors of v are colored differently. For each edge e incident with u define $c(e) = S_1$, for each edge vv_i $(1 \leq i \leq n-2)$, define $c(vv_i) = c_H(v_i)$, and for each edge $e \in E(H)$ define $c(e) = c_H(e)$. Then $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-2$, $c'(u) = S_1$ and $c'(v) = [3]$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 2.2.2. All neighbors of v are colored with the same 2-element subset of $[3]$. We may assume that every neighbor of v is colored S_1. Since G is 2-connected, it follows that v is adjacent to two or more vertices of the cycle C in G. We may assume that v_5 is one of the neighbors of v and $c'_H(v_5) = S_1$. Thus, $c_H(v_4v_5) = c_H(v_5v_6) = S_1$. Furthermore, $c'_H(v_3) = c_H(v_2v_3) = c_H(v_3v_4) \neq S_1$ and $c'_H(v_7) = c_H(v_6v_7) = c_H(v_7v_8) \neq S_1$.

- ⋆ If $c'_H(v_3) = c'_H(v_7)$, say $c'_H(v_3) = c'_H(v_7) = S_2$, then we recolor the edges v_4v_5 and v_5v_6 with the color S_3 as well as recolor any edge of H not on C that is incident with v_5 with the color S_3. We then proceed as in Subcase 2.2.1.

- ⋆ If $c'_H(v_3) \neq c'_H(v_7)$, then we may assume that $c'_H(v_3) = c_H(v_2v_3) = c_H(v_3v_4) = S_2$ and $c'_H(v_7) = c_H(v_6v_7) = c_H(v_7v_8) = S_3$. If $c'_H(v_9) = c_H(v_8v_9) = c_H(v_9v_{10}) = S_1$, then we recolor the edges v_4v_5 and v_5v_6 (and any other edges with incident with v_5) with the color S_3 and recolor the edges v_6v_7 and v_7v_8 (and any other edges with incident with v_7) with the color S_2. We then proceed as in Subcase 2.2.1. If $c'_H(v_9) = c_H(v_8v_9) = c_H(v_9v_{10}) = S_2$, then we recolor the edges v_4v_5 and v_5v_6 (and any other edges with incident with v_5) with the color S_3 and recolor the edges v_6v_7 and v_7v_8 (and any other edges with incident with v_7) with the color S_1. We then proceed as in Subcase 2.2.1.

Subcase 2.3. The neighbors of u and v are colored with 2-element subsets of $[3]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-2$ and reapply the coloring c_H to the new labeling of H. Thus, each neighbor of u and v has vertex color $[3]$ and we proceed as in Subcase 2.1.

In each case, the graph G has a majestic 2-tone 3-coloring and so $\text{maj}_t(G) = t+1$. ■

The next three results show that Theorem 2.4 can be extended even further. We will only provide a proof of the first result since the proofs for the other two results are similar but much more complicated and lengthy.

Theorem 2.5 *If G is a 2-connected bipartite graph of odd order $n \geq 13$ whose longest cycles have order $n-3$ and $t \geq 2$ is an integer, then* $\text{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-3}, v_{n-2} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $u, v, w \in V(G)$ such that $H = G - \{u, v, w\}$. Thus, H is a Hamiltonian bipartite graph of order $n-3$. By Corollary 2.2, $\text{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_2([3])$ be a majestic 2-tone 3-coloring of H. We show that there is a majestic 2-tone 3-coloring c of G. We now consider three cases. Recall that S_1, S_2, S_3 are the three distinct 2-element subsets of $[3]$.

Case 1. *No two of u, v, w are adjacent.* There are four subcases.

Subcase 1.1. *Each neighbor of u, v and w is colored* $[3]$. We define a coloring c of G as follows: For each edge e incident with u, v and w, define $c(e) = S_1$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$ and $c'(u) = c'(v) = c'(w) = S_1$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 1.2. *Each neighbor of u, v and w is colored with a 2-element subset of* $[3]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-3$ and reapply the coloring c_H to the new labeling of H. We then proceed as in Subcase 1.1.

Subcase 1.3. *Each neighbor of two of u, v and w is colored* $[3]$ *and each neighbor of the other, say w, is colored with a 2-element subset of* $[3]$.

Subcase 1.3.1. *Two neighbors of w are colored differently.* We define a coloring c of G as follows: For each edge e incident with u or v, define $c(e) = S_1$, for each edge $e = v_i w$ where $1 \leq i \leq n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = c'(v) = S_1$ and $c'(w) = [3]$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 1.3.2. *All neighbors of w are colored with the same 2-element subset of* $[3]$. We may assume that v_4 is one of the neighbors of w and $c'_H(v_4) = S_1$. Thus, $c_H(v_3v_4) = c_H(v_4v_5) = S_1$. Hence, $c_H(v_1v_2) = c_H(v_2v_3) \neq S_1$ and $c_H(v_5v_6) = c_H(v_6v_7) \neq S_1$. We may assume, without loss of generality, that $c_H(v_1v_2) = c_H(v_2v_3) = S_2$.

- ⋆ First, suppose that $c_H(v_5v_6) = c_H(v_6v_7) = S_2$. We recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3. We then proceed as in Subcase 1.3.1.

- ⋆ Next, suppose that $c_H(v_5v_6) = c_H(v_6v_7) \neq S_2$, say $c_H(v_5v_6) = c_H(v_6v_7) = S_3$. Thus, $c_H(v_7v_8) = c_H(v_8v_9) \neq S_3$. If $c_H(v_7v_8) =$

$c_H(v_8v_9) = S_1$, then we (1) recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3 and (2) recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) with the color S_2. We then proceed as in Subcase 1.3.1. If $c_H(v_7v_8) = c_H(v_8v_9) \neq S_1$, then $c_H(v_7v_8) = c_H(v_8v_9) = S_2$ and we (1) recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3 and (2) recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) with the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.4. *Each neighbor of two of u, v and w is colored with a 2-element subset of* [3] *and each neighbor of the other, say w, is colored* [3]. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-3$ and reapply the coloring c_H to the new labeling of H. We then proceed as in Subcase 1.3.

Case 2. $G[\{u,v,w\}] = K_2 + K_1$. We may assume that $uv \in E(G)$.

Subcase 2.1. *Each neighbor of w is colored* [3]. Since $uv \in E(G)$, we may assume that each neighbor of u is colored [3] and each neighbor of v is colored with a 2-element subset of [3], at least one of which is colored S_1. We define a coloring c of G as follows: For each edge e incident with u or w, define $c(e) = S_2$, for each edge $e = v_iv$ where $1 \leq i \leq n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = c'(w) = S_2$ and $c'(v) = [3]$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 2.2. *Each neighbor of w is colored with a 2-element subset of* [3]. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-3$ and reapply the coloring c_H to the new labeling of H. We then proceed as in Subcase 2.1.

Case 3. $G[\{u,v,w\}] = P_3$. We may assume that (u,v,w) is a path in G.

Subcase 3.1. *Each neighbor of v is colored with a 2-element subset of* [3]. Thus, every neighbor of u and w is colored [3]. We define a coloring c of G as follows: For each edge e incident with u, define $c(e) = S_1$, for each edge e incident with w, define $c(e) = S_2$ for each edge $e = v_iv$ where $1 \leq i \leq n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = S_1$, $c'(w) = S_1$ and $c'(v) = [3]$. Thus, c is a majestic 2-tone 3-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 3.2. *Each neighbor of v is colored* [3]. Thus, every neighbor of u and w is colored with a 2-element subset of [3]. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-3$ and reapply the coloring c_H to the new labeling of H. We then proceed as in Subcase 3.1. ■

Theorem 2.6 *If G is a 2-connected bipartite graph of even order $n \geq 14$ whose longest cycles have order $n-4$ and $t \geq 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

Theorem 2.7 *If G is a 2-connected bipartite graph of odd order $n \geq 15$ whose longest cycles have order $n-5$ and $t \geq 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

In summary, it follows by Corollary 2.2 and Theorems 2.3–2.7 that if G is a 2-connected bipartite graph of arbitrarily large order n whose longest cycles have length ℓ where $n-5 \leq \ell \leq n$ and $t \geq 2$ is an integer, then $\mathrm{maj}_t(G) = t+1$. There is a limit, however, on how small the length of a longest cycle can be in terms of the order of a 2-connected bipartite graph G to guarantee that G has a majestic 2-tone 3-coloring. As we will see in the following section, there are infinitely many 2-connected bipartite graphs of arbitrarily large order n having maximum cycle length $n-6$ and 2-tone majestic index 4. We are not aware of any 2-connected bipartite graph of arbitrarily large order n having maximum cycle length $n-6$ and t-tone majestic index $t+2$ for all integers $t \geq 2$, however. This gives rise to the following more general question.

Problem 2.8 *For given integers t and r with $t \geq 2$ and $r \geq 6$, does there exist a 2-connected bipartite graph G of arbitrarily large order n having maximum cycle length $n-r$ such that* $\mathrm{maj}_t(G) = t+2$*?*

3 On 2-Connected Bipartite Graphs of Type 2

We now present a class of 2-connected bipartite graphs having majestic t-tone index $t+2$.

Theorem 3.1 *For each integer $t \geq 2$, there exists a 2-connected bipartite graph G such that* $\mathrm{maj}_t(G) = t+2$.

Proof. Let $s = \binom{t+2}{2}$. First, we construct a bipartite graph G of order $2s + 2(t+2) = 2(s+t+2)$ as follows. Let $X = \{x_1, x_2, \ldots, x_{t+2}\}$ and $Y = \{y_1, y_2, \ldots, y_{t+2}\}$. The partite sets of G are

$$\begin{aligned} U &= \{u_1, u_2, \ldots, u_s\} \cup Y \\ W &= \{w_1, w_2, \ldots, w_s\} \cup X. \end{aligned}$$

The subgraph $G[X \cup Y]$ of G induced by $X \cup Y$ is $K_{t+2,t+2}$. Let $X_1, X_2, \ldots, X_s$ be the s distinct 2-element subsets of X and $Y_1, Y_2, \ldots, Y_s$ the s distinct 2-element subsets of Y. For each integer i with $1 \leq i \leq s$, the vertex u_i

is joined to the two vertices in X_i and the vertex w_i is joined to the two vertices in Y_i. This completes the construction of G, which is illustrated in Figure 2 for $t = 2$. Thus, $\deg u_i = \deg w_i = 2$ for $1 \leq i \leq s$. The graph G is 2-connected but not Hamiltonian (as a vertex in $X \cup Y$ is adjacent to three vertices of degree 2 in G).

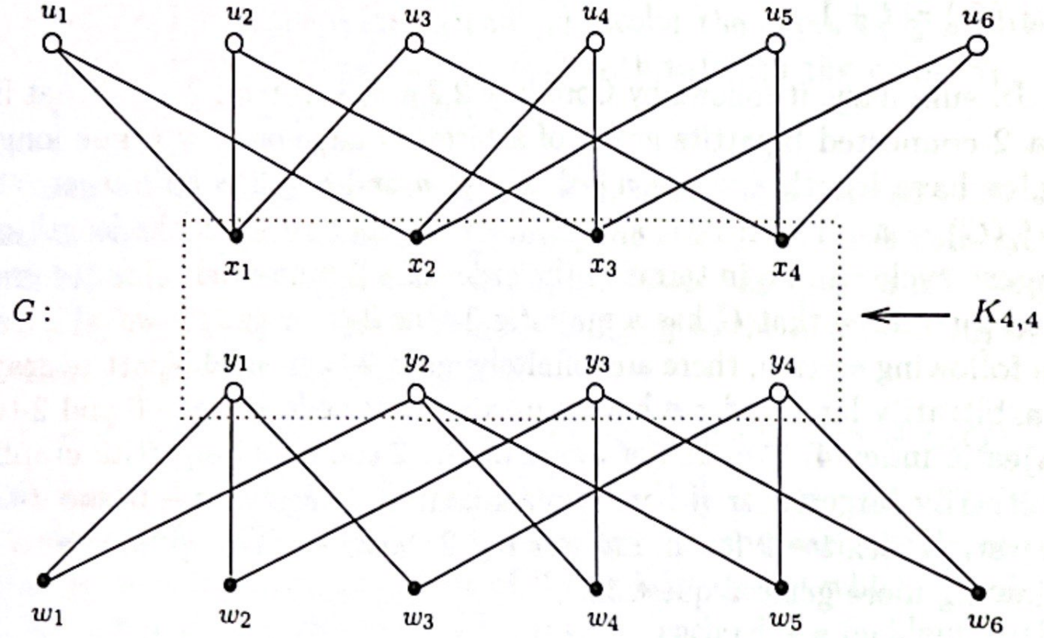

Figure 2: A 2-connected bipartite graph G with $\mathrm{maj}_2(G) = 4$

Next, we show that $\mathrm{maj}_t(G) = t + 2$. Assume, to the contrary, that $\mathrm{maj}_t(G) = t + 1$. Then there exists a majestic t-tone $(t + 1)$-coloring $c : E(G) \to [t + 1]$ of G. Thus, either $|c'(u)| = t$ for each $u \in U$ or $|c'(w)| = t$ for each $w \in W$. Because of the symmetry of G, we may assume that $|c'(u)| = t$ for each $u \in U$. Therefore, $c'(w) = [t + 1]$ for each $w \in W$. In particular, $c'(w_i) = [t + 1]$ for $i = 1, 2, \ldots, s$. Since $\deg_G w_i = 2$ and $c'(w_i) = [t+1]$ for each integer i with $1 \leq i \leq s$, the two neighbors of w_i have distinct vertex colors. Since $|c'(u)| = t$ for each $u \in U$ and there are exactly $t+1$ distinct t-element sets of $[t + 1]$, there are two vertices in Y that have the same vertex color, say $c'(y_p) = c'(y_q)$, where $p, q \in \{1, 2, \ldots, t+2\}$ and $p \neq q$. Let $w \in W$ such that w is adjacent only to y_p and y_q in G. However then, $|c'(w)| = t$, which is a contradiction. Therefore, $\mathrm{maj}_t(G) = t + 2$. ■

As an example, we provide a majestic 2-tone 4-coloring c of the graph G in Figure 2 as follows: Color each edge incident with u_1 and u_6 with the 2-element set $\{1, 4\}$, color each edge incident with w_1 and w_6 with the 2-element set $\{3, 4\}$ and color all remaining edges with the 2-element set $\{2, 4\}$. Observe that

* ⋆ $c'(u_1) = c'(u_6) = \{1, 4\}$, $c'(u_i) = \{2, 4\}$ for $2 \leq i \leq 5$ and $c'(y_j) = \{2, 3, 4\}$ for $1 \leq j \leq 4$ and

* $\star$ $c'(w_1) = c'(w_6) = \{3,4\}$, $c'(w_i) = \{2,4\}$ for $2 \le i \le 5$ and $c'(x_j) = \{1,2,4\}$ for $1 \le j \le 4$.

Thus, c' is a proper vertex coloring of G and so c is a majestic 2-tone 4-coloring of G. Redrawing the graph G as in Figure 3, we see that G has maximum cycle length $n-6$.

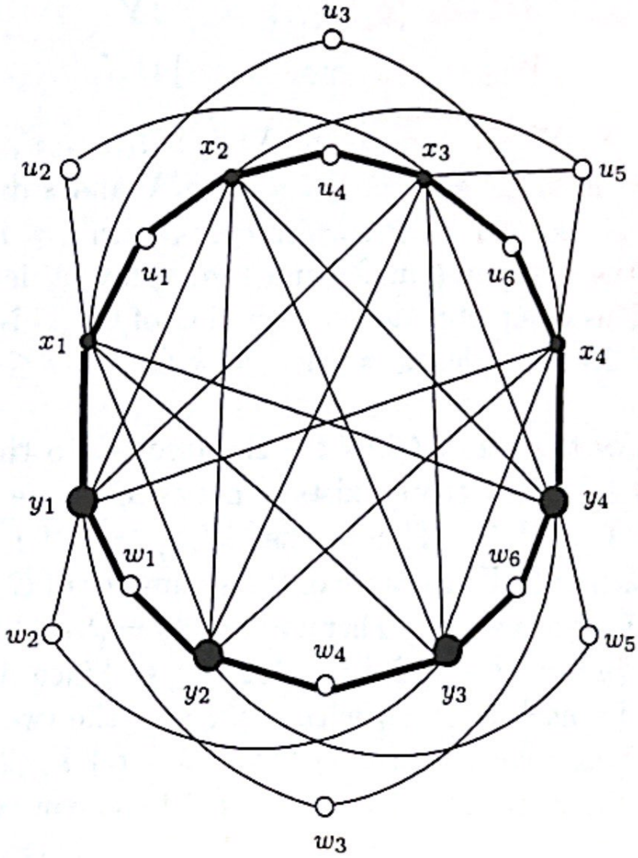

Figure 3: A 2-connected bipartite graph G of order 20 whose longest cycle has length 14 with $\text{maj}_2(G) = 4$

If we wanted to continue in the same vein as Theorems 2.3–2.7 to classify the 2-connected bipartite graphs of type 1 having maximum cycle length $n-6$, it would be necessary to require n to be at least 22. However, the graph G provides us with a means of constructing infinitely many 2-connected bipartite graphs having maximum cycle length $n-6$ and 2-tone majestic index 4. Let $P = (v_1, v_2, \ldots, v_{2k})$ be a path of length $2k \ge 2$ and define the graph H by $V(H) = V(G) \cup V(P)$ and $E(H) = E(G) \cup E(P) \cup \{x_1v_1, y_1v_{2k}\}$. Then H is a 2-connected bipartite graph of order $n = 20 + 2k$ with maximum cycle length $n-6$. An argument similar to the one given in the proof of Theorem 3.1 shows that $\text{maj}_2(H) = 4$. Theorem 3.1 can, in fact, be extended from 2-connected bipartite graphs to k-connected bipartite graphs for all integers $k \ge 2$.

Theorem 3.2 *Let k and t be integers such that $k, t \geq 2$. Then there exists a k-connected bipartite graph G such that* $\text{maj}_t(G) = t + 2$.

Proof. Let $s = \binom{tk-t+k}{k}$. First, we construct a bipartite graph G of order $2s+2(tk-t+k) = 2(s+tk-t+k)$ as follows. Let $X = \{x_1, x_2, \ldots, x_{tk-t+k}\}$ and $Y = \{y_1, y_2, \ldots, y_{tk-t+k}\}$. The partite sets of G are

$$U = \{u_1, u_2, \ldots, u_s\} \cup Y$$
$$W = \{w_1, w_2, \ldots, w_s\} \cup X.$$

The subgraph $G[X \cup Y]$ of G induced by $X \cup Y$ is $K_{tk-t+k,tk-t+k}$. There are s distinct k-element subsets X_i $(1 \leq i \leq s)$ of X and s distinct k-element subsets Y_i $(1 \leq i \leq s)$ of Y. For each integer i with $1 \leq i \leq s$, the vertex u_i is joined to the k vertices in X_i and the vertex w_i is joined to the k vertices in Y_i. This completes the construction of G. This is illustrated in Figure 2 for $t = 2$. Thus, $\deg u_i = \deg w_i = k$ for $1 \leq i \leq s$. The graph G is k-connected.

Next, we show that $\text{maj}_t(G) = t + 2$. Assume, to the contrary, that $\text{maj}_t(G) = t + 1$. Then there exists a majestic t-tone $(t + 1)$-coloring $c : E(G) \to [t + 1]$ of G. Thus, either $|c'(u)| = t$ for each $u \in U$ or $|c'(w)| = t$ for each $w \in W$. Because of the symmetry of G, we may assume that $|c'(u)| = t$ for each $u \in U$. Therefore, $c'(w) = [t + 1]$ for each $w \in W$. In particular, $c'(w_i) = [t + 1]$ for $i = 1, 2, \ldots, s$. Since $\deg_G w_i = k$ and $c'(w_i) = [t + 1]$ for each integer i with $1 \leq i \leq s$, the two neighbors of w_i have distinct vertex colors. Since (1) $|Y| = tk - t + k$, (2) $|c'(u)| = k$ for each $u \in U$, (3) there are exactly $t+1$ distinct t-element sets of $[t + 1]$ and (4) $tk - t + k > tk - t + k - 1 = (k - 1)(t + 1)$, it follows that there are k vertices in Y that have the same vertex color, say Y_j consists of these k vertices of Y for some integer j with $1 \leq j \leq s$. However, w_j is only adjacent to the k vertices of Y_j in G, which implies that $|c'(w_i)| = t$, which is a contradiction. Therefore, $\text{maj}_t(G) = t + 2$. ■

The proofs of Theorems 2.7 and 3.1 suggest the following conjecture.

Conjecture 3.3 *Let G be a connected bipartite graph of sufficiently large order n whose longest cycles have order $n-6$. If C is a cycle of length $n-6$ in G and the subgraph induced by the set $V(G) - V(C)$ of six vertices is not an empty graph, then* $\text{maj}_t(G) = t + 1$ *for each integer $t \geq 2$.*

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References

[1] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B.* **75** (1999) 288-301.

[2] Z. Bi, S. English, I. Hart and P. Zhang, Majestic colorings of graphs. *J. Combin. Math. Combin. Comput.* **102** (2017) 123-140.

[3] Z. Bi, I. Hart and P. Zhang, On set-defined colorings of Cartesian products of graphs. *J. Combin. Math. Combin. Comput.* To appear.

[4] A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge colorings. *J. Graph Theory.* **26** (1997) 73-82.

[5] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton (2009).

[6] E. Györi, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph. *Discrete Math.* **308** (2008) 827-831.

[7] F. Harary and M. Plantholt, The point-distinguishing chromatic index. *Graphs and Applications.* Wiley, New York (1985) 147-162.

[8] I. Hart and P. Zhang, Majestic 2-tone colorings of graphs. *Ars Combin.* To appear.

[9] I. Hart and P. Zhang, Majestic t-tone colorings of graphs. *Congr. Numer.* **228** (2017) 141-154.

[10] M. Horňák and R. Soták, General neighbour-distinguishing index via chromatic number. *Discrete Math.* **310** (2010) 1733-1736.

[11] P. Zhang, *Color-Induced Graph Colorings.* Springer, New York (2015).

[12] P. Zhang, *A Kaleidoscopic View of Graph Colorings.* Springer, New York (2016).

[13] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs. *Appl. Math. Lett.* **15** (2002) 623-626.