Nonnegative signed Roman domination in graphs

Nasrin Dehgardi
Department of Mathematics and Computer Science
Sirjan University of Technology
Sirjan, I.R. Iran
n.dehgardi@sirjantech.ac.ir

L. Volkmann
Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Abstract

Let G be a finite and simple graph with vertex set $V(G)$. A nonnegative signed Roman dominating function (NNSRDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq 0$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, and (ii) every vertex u for which $f(u) = -1$ has a neighbor v for which $f(v) = 2$. The weight of an NNSRDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The nonnegative signed Roman domination number $\gamma_{sR}^{NN}(G)$ of G is the minimum weight of an NNSRDF on G. In this paper, we initiate the study of the nonnegative signed Roman domination number of graphs, and we present different bounds on $\gamma_{sR}^{NN}(G)$. We determine the nonnegative signed Roman domination number of some classes of graphs. If n is the order and m the size of the graph G, then we show that $\gamma_{sR}^{NN}(G) \geq \frac{3}{4}(\sqrt{8n+1} - 1) - n$ and $\gamma_{sR}^{NN}(G) \geq (8n - 12m)/7$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma_{sR}^{NN}(G) \geq \frac{3}{2}(\sqrt{4n+9} - 3) - n$, and we characterize the extremal graphs.

Keywords: nonnegative signed Roman dominating function, nonnegative signed Roman domination

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1 Introduction

In this paper, we continue the study of Roman dominating functions in graphs. Let G be a finite and simple graph with vertex set $V = V(G)$ and edge set $E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph G, respectively. We write $d_G(v) = d(v)$ for the degree of a vertex v. The minimum and maximum degree are $\delta(G) = \delta$ and $\Delta(G) = \Delta$. The sets $N_G(v) = N(v) = \{u \mid uv \in E(G)\}$ and $N_G[v] = N[v] = N(u) \cup \{v\}$ are called the open neighborhood and closed neighborhood of the vertex v, respectively. A graph G is regular or r-regular if $\Delta(G) = \delta(G) = r$. For disjoint subsets U and V of vertices, we denote by $[U, V]$ the set of edges between U and V. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. Also if $S \subseteq V(G)$, then $G[S]$ is the subgraph induced by S.

A cycle on n vertices is denoted by C_n, while a path on n vertices is denoted by P_n. We denote by K_n the complete graph on n vertices and by $K_{m,n}$ the complete bipartite graph with one partite set of cardinality m and the other of cardinality n. A star is a complete bipartite graph of the form $K_{1,n}$. A vertex of degree one is called a leaf. The complement of a graph G is denoted by $\overline{G}$.

For a real-valued function $f : V(G) \to R$, the weight of f is $\omega(f) = \sum_{v \in V(G)} f(v)$, and for $S \subseteq V(G)$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V(G))$. Consult [3] and [4] for notation and terminology which are not defined here.

For an integer $k \geq 1$, a signed Roman k-dominating function (SRkDF) on a graph G is defined in [6] as a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u \in N_G[v]} f(u) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an SRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed Roman k-domination number $\gamma_{sR}^k(G)$ of G is the minimum weight of an SRkDF on G. The special case $k = 1$ was introduced in [1]. Signed Roman domination in graphs and digraphs is well studied in the literature, see for example [2, 5, 7, 8, 9]. Following [6], we initiate the study of nonnegative signed Roman dominating functions on graphs G.

A nonnegative signed Roman dominating function (NNSRDF) on G is defined as a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u \in N[v]} f(u) \geq 0$ for every $v \in V(G)$ and every vertex u for which $f(u) = -1$ has a neighbor v for which $f(v) = 2$. For a vertex $v \in V$, we denote $f(N[v])$ by $f[v]$ for notational convenience. The weight of an NNSRDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The nonnegative signed Roman domination number $\gamma_{sR}^{NN}(G)$ of G is the minimum weight of an NNSRDF on G. A $\gamma_{sR}^{NN}(G)$-

function is a nonnegative signed Roman dominating function on G of weight $\gamma_{sR}^{NN}(G)$. For an NNSRDF f on G, let $V_i = V_i^f = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. An NNSRDF $f : V(G) \to \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$. Further, we let $n_{-1} = |V_{-1}|$, $n_1 = |V_1|$, $n_2 = |V_2|$, and so $n = n_2 + n_1 + n_{-1}$. Therefore $\gamma_{sR}^{NN}(G) = 2n_2 + n_1 - n_{-1}$.

We present different sharp lower and upper bounds on $\gamma_{sR}^{NN}(G)$. We determine the nonnegative signed Roman domination number of some classes of graphs. We show that $\gamma_{sR}^{NN}(G) \geq \frac{3}{4}(\sqrt{8n+1}-1) - n$ and $\gamma_{sR}^{NN}(G) \geq (8n - 12m)/7$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma_{sR}^{NN}(G) \geq \frac{3}{2}(\sqrt{4n+9}-3) - n$, and we characterize the extremal graphs.

2 Special classes of graphs

In this section, we determine the nonnegative signed Roman domination number of special classes of graphs. We start with an easy but useful observation

Observation 1. If G is a graph of order n with maximum degree $\Delta(G)$, then

$$\gamma_{sR}^{NN}(G) \geq \Delta(G) + 1 - n.$$

Proof. Let $v \in V(G)$ be a vertex of maximum degree, and let f be a $\gamma_{sR}^{NN}(G)$-function. Then the definitions imply

$$\begin{aligned}\gamma_{sR}^{NN}(G) &= \sum_{u \in V(G)} f(u) = \sum_{u \in N[v]} f(u) + \sum_{u \in V(G)-N[v]} f(u) \\ &\geq 0 + \sum_{u \in V(G)-N[v]} f(u) \geq -(n - (\Delta(G)+1)) = \Delta(G) - n + 1,\end{aligned}$$

and the proof is complete. □

Proposition 2. For $n \geq 1$, $\gamma_{sR}^{NN}(K_{1,n}) = 0$ with exception of the cases that $n = 1$ or $n = 3$, in which cases we have $\gamma_{sR}^{NN}(K_{1,1}) = \gamma_{sR}^{NN}(K_{1,3}) = 1$.

Proof. According to Observation 1, $\gamma_{sR}^{NN}(K_{1,n}) \geq 0$. Now let u be the central vertex, and let $\{u_1, u_2, \ldots, u_n\}$ be the leaves of the star $K_{1,n}$. First let n be even. Define the function $f : V(K_{1,n}) \to \{-1, 1, 2\}$ by $f(u) = 2$, $f(u_1) = f(u_2) = -1$ and $f(u_i) = (-1)^i$ for each vertex $u_i \in V - \{u, u_1, u_2\}$. Then the function f is an NNSRDF on $K_{1,n}$ of weight 0 and thus $\gamma_{sR}^{NN}(K_{1,n}) \leq 0$. This implies that $\gamma_{sR}^{NN}(K_{1,n}) = 0$ when n is even.

Now let n be odd. It is easy to verify that $\gamma_{sR}^{NN}(K_{1,1}) = \gamma_{sR}^{NN}(K_{1,3}) = 1$. Let next $n \geq 5$. Now we distinguish three cases.

Case 1. Let $n = 6p - 1$ for an integer $p \geq 1$. Define the function $f : V(K_{1,n}) \to \{-1,1,2\}$ by $f(u) = 2$, $f(u_i) = 2$ for $1 \leq i \leq 2p-1$ and $f(u_i) = -1$ otherwise. Then the function f is an NNSRDF on $K_{1,n}$ of weight $\omega(f) = f(N[u]) = 4p - 4p = 0$ and so $\gamma_{sR}^{NN}(K_{1,n}) \leq 0$ in this case.

Case 2. Let $n = 6p + 1$ for an integer $p \geq 1$. Define the function $f : V(K_{1,n}) \to \{-1,1,2\}$ by $f(u) = 2$, $f(u_1) = 1$, $f(u_i) = 2$ for $2 \leq i \leq 2p$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on $K_{1,n}$ of weight $\omega(f) = f(N[u]) = 4p+1-(4p+1) = 0$ and so $\gamma_{sR}^{NN}(K_{1,n}) \leq 0$ in this case.

Case 3. Let $n = 6p + 3$ for an integer $p \geq 1$. Define the function $f : V(K_{1,n}) \to \{-1,1,2\}$ by $f(u) = 2$, $f(u_1) = f(u_2) = 1$, $f(u_i) = 2$ for $3 \leq i \leq 2p+1$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on $K_{1,n}$ of weight $\omega(f) = f(N[u]) = 4p+2-(4p+2) = 0$ and so $\gamma_{sR}^{NN}(K_{1,n}) \leq 0$.

Therefore $\gamma_{sR}^{NN}(K_{1,n}) = 0$ when $n \geq 5$ is odd, and the proof is complete. □

Proposition 3. For $n \geq 1$, $\gamma_{sR}^{NN}(K_n) = 1$ when $n = 1,2,4$ and $\gamma_{sR}^{NN}(K_n) = 0$ otherwise.

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. If $n = 1,2,4$, then it is easy to see that $\gamma_{sR}^{NN}(K_n) = 1$. Thus let $n \neq 1,2,4$. Using Observation 1, we have $\gamma_{sR}^{NN}(K_n) \geq 0$.

First let n be odd. Define the function $f : V(K_n) \to \{-1,1,2\}$ by $f(u_1) = 2$, $f(u_2) = f(u_3) = -1$ and $f(u_i) = (-1)^i$ for each vertex $u_i \in V - \{u_1, u_2, u_3\}$. Then the function f is an NNSRDF on K_n of weight 0 and thus $\gamma_{sR}^{NN}(K_n) \leq 0$. Hence $\gamma_{sR}^{NN}(K_n) = 0$ when n is odd and $n \neq 1$.

Now let n be even and $n \geq 6$. Define the function $f : V(K_n) \to \{-1,1,2\}$ by $f(u_1) = f(u_2) = 2$, $f(u_3) = f(u_4) = f(u_5) = f(u_6) = -1$ and $f(u_i) = (-1)^i$ for each $7 \leq i \leq n$. Then f is an NNSRDF on K_n of weight 0 and thus $\gamma_{sR}^{NN}(K_n) \leq 0$. So $\gamma_{sR}^{NN}(K_n) = 0$ when n is even and $n \neq 2,4$. □

Propositions 2 and 3 show that Observation 1 is sharp.

Proposition 4. For $n \geq 1$, $\gamma_{sR}^{NN}(P_n) = 0$ when $n \equiv 0 \pmod 3$ and $\gamma_{sR}^{NN}(P_n) = 1$ otherwise.

Proof. Let $P_n := u_1 u_2 \ldots u_n$. First let $n \equiv 0 \pmod 3$. Define the function $f : V(P_n) \to \{-1,1,2\}$ by $f(u_{3i+2}) = 2$ for $0 \leq i \leq \lfloor \frac{n-1}{3} \rfloor$ and $f(u_i) = -1$ otherwise. Then the function f is an NNSRDF on P_n of weight 0 and thus $\gamma_{sR}^{NN}(P_n) \leq 0$. To prove $\gamma_{sR}^{NN}(P_n) \geq 0$, we proceed by induction on n. If $n = 3$, then $P_3 = K_{1,2}$ and we have $\gamma_{sR}^{NN}(P_3) = 0$. Thus let $n \geq 6$, and let f be a $\gamma_{sR}^{NN}(P_n)$-function. Let $P_{n-3} = P_n - \{u_1, u_2, u_3\}$. If $f(u_3) = -1$, then the function $g : V(P_{n-3}) \to \{-1,1,2\}$ defined by $g(u_i) = f(u_i)$ for

$4 \le i \le n$ is an NNSRDF of P_{n-3} of weight at $\omega(f) - f[u_2]$. By the induction hypothesis, we have

$$\gamma_{sR}^{NN}(P_n) = \omega(f) = \omega(g) + f[u_2] \ge \gamma_{sR}^{NN}(P_{n-3}) + f[u_2] \ge 0.$$

If $f(u_3) \ne -1$, then $f[u_2] \ge 2$. If $f(u_4) = 2$, then the function $g : V(P_{n-3}) \to \{-1,1,2\}$ defined by $g(u_i) = f(u_i)$ for $4 \le i \le n$ is an NNSRDF of P_{n-3} of weight at $\omega(f) - f[u_2]$ and the result follows by the induction hypothesis. If $f(u_4) \ne 2$, then the function $g : V(P_{n-3}) \to \{-1,1,2\}$ defined by $g(u_4) = f(u_4) + 1$ when $f(u_4) = 1$ and $g(u_4) = f(u_4)+2$ when $f(u_4) = -1$ and $g(u_i) = f(u_i)$ for $5 \le i \le n$ is an NNSRDF of P_{n-3} of weight at most $\omega(f) - f[u_2] + 2$. By the induction hypothesis, we deduce that

$$\gamma_{sR}^{NN}(P_n) = \omega(f) \ge \omega(g) + f[u_2] - 2 \ge \gamma_{sR}^{NN}(P_{n-3}) + f[u_2] - 2 \ge 0.$$

Therefore we have, $\gamma_{sR}^{NN}(P_n) = 0$ when $n \equiv 0 \pmod 3$.

Now let $n \equiv 1 \pmod 3$. Define the function $f : V(P_n) \to \{-1,1,2\}$ by $f(u_{3i+2}) = 2$ for $0 \le i \le \lfloor \frac{n-2}{3} \rfloor$, $f(u_n) = 1$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on P_n of weight 1 and thus $\gamma_{sR}^{NN}(P_n) \le 1$. Using an argument similar to that described in the case above, we can see that $\gamma_{sR}^{NN}(P_n) \ge 1$ and thus $\gamma_{sR}^{NN}(P_n) = 1$ when $n \equiv 1 \pmod 3$.

Finally let $n \equiv 2 \pmod 3$. Define the function $f : V(P_n) \to \{-1,1,2\}$ by $f(u_{3i+2}) = 2$ for $0 \le i \le \lfloor \frac{n}{3} \rfloor$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on P_n of weight 1 and thus $\gamma_{sR}^{NN}(P_n) \le 1$. Using an argument similar to that described above, we can see that $\gamma_{sR}^{NN}(P_n) \ge 1$ and thus $\gamma_{sR}^{NN}(P_n) = 1$ when $n \equiv 2 \pmod 3$ and this completes the proof. □

Proposition 5. For $n \ge 3$,

$$\gamma_{sR}^{NN}(C_n) = \begin{cases} 0 & n \equiv 0 \pmod 3 \\ 2 & n \equiv 1 \pmod 3 \\ 1 & n \equiv 2 \pmod 3). \end{cases}$$

Proof. Let $C_n := (u_1 u_2 \dots u_n)$. By Proposition 3, the result is valid for $n = 3$. Let now $n \ge 4$. First let $n \equiv 0 \pmod 3$. Define the function $f : V(C_n) \to \{-1,1,2\}$ by $f(u_{3i+2}) = 2$ for $0 \le i \le \lfloor \frac{n-1}{3} \rfloor$ and $f(u_i) = -1$ otherwise. Then the function f is an NNSRDF on C_n of weight 0 and thus $\gamma_{sR}^{NN}(C_n) \le 0$. On other hand if g is a $\gamma_{sR}^{NN}(C_n)$-function, then

$$\gamma_{sR}^{NN}(C_n) = \omega(g) = \sum_{0 \le i \le \lfloor \frac{n-1}{3} \rfloor} g[u_{3i+2}] \ge 0,$$

Thus $\gamma_{sR}^{NN}(C_n) = 0$ when $n \equiv 0 \pmod 3$.

Next let $n \equiv 1 \pmod 3$. Define the function $f : V(C_n) \to \{-1, 1, 2\}$ by $f(u_{3i+2}) = 2$ for $0 \le i \le \lfloor \frac{n-2}{3} \rfloor$, $f(u_n) = 2$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on C_n of weight 2 and thus $\gamma_{sR}^{NN}(C_n) \le 2$. Now let g be a $\gamma_{sR}^{NN}(C_n)$-function. If $g(u_i) \neq 2$ for every i, then

$$\gamma_{sR}^{NN}(C_n) = \omega(g) = \sum_{1 \le i \le n} g(u_i) \ge 4,$$

a contradiction. Thus assume, without loss of generality, that $g(u_n) = 2$. Then we observe that

$$\gamma_{sR}^{NN}(C_n) = \omega(g) = \sum_{0 \le i \le \lfloor \frac{n-2}{3} \rfloor} g[u_{3i+2}] + g(u_n) \ge 2,$$

and so $\gamma_{sR}^{NN}(C_n) = 2$ when $n \equiv 1 \pmod 3$.

Finally let $n \equiv 2 \pmod 3$. Define the function $f : V(C_n) \to \{-1, 1, 2\}$ by $f(u_{3i+2}) = 2$ for $0 \le i \le \lfloor \frac{n}{3} \rfloor$ and $f(u_i) = -1$ otherwise. Then f is an NNSRDF on C_n of weight 1 and thus $\gamma_{sR}^{NN}(C_n) \le 1$. Now let g be a $\gamma_{sR}^{NN}(C_n)$-function. If $g(u_i) \neq 2$ for every i, then

$$\gamma_{sR}^{NN}(C_n) = \omega(g) = \sum_{1 \le i \le n} g(u_i) \ge 5,$$

a contradiction. Thus assume, without loss of generality, that $g(u_n) = 2$. This implies that

$$\gamma_{sR}^{NN}(C_n) = \omega(g) = \sum_{0 \le i \le \lfloor \frac{n-3}{3} \rfloor} g[u_{3i+2}] + g(u_{n-1}) + g(u_n) \ge 1.$$

Thus $\gamma_{sR}^{NN}(C_n) = 1$ when $n \equiv 2 \pmod 3$ and this completes the proof. □

In Proposition 2, we determined exact values of the nonnegative signed Roman domination number of $K_{1,n}$. In the following, we determine exact values of the nonnegative signed Roman domination number of $K_{m,n}$ for $n, m \ge 2$.

Proposition 6. For $n \ge m \ge 2$,

$$\gamma_{sR}^{NN}(K_{m,n}) = \begin{cases} 3 & m = 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. First assume that $m = 2$. If n is even, then define the function $f : V(K_{2,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = -1$ and $f(y_i) = (-1)^{i+1}$ for $2 \le i \le n$. Then the function f is an NNSRDF on $K_{2,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{2,n}) \leq 2$. If n is odd, then define the function $f : V(K_{2,n}) \to \{-1,1,2\}$ by $f(x_1) = 2$, $f(x_2) = 1$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$. Then f is an NNSRDF on $K_{2,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{2,n}) \leq 2$. Now let g be a $\gamma_{sR}^{NN}(K_{2,n})$-function. If $g(x_1), g(x_2) \neq 2$, then for each i, $g(y_i) \neq -1$. It follows that

$$\gamma_{sR}^{NN}(K_{2,n}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 2.$$

Let now $g(x_1) = 2$. Then

$$\gamma_{sR}^{NN}(K_{2,n}) = \omega(g) = g(x_1) + g[x_2] \geq 2.$$

Now assume that $m = 3$. If n is even, then define the function $f : V(K_{3,n}) \to \{-1,1,2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = -1, f(x_3) = 1$ and $f(y_i) = (-1)^{i+1}$ for $2 \leq i \leq n$. Then f is an NNSRDF on $K_{3,n}$ of weight 3 and thus $\gamma_{sR}^{NN}(K_{3,n}) \leq 3$. If n is odd, then define the function $f : V(K_{3,n}) \to \{-1,1,2\}$ by $f(x_1) = 2$, $f(x_2) = f(x_3) = 1$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$. Then f is an NNSRDF on $K_{3,n}$ of weight 3 and thus $\gamma_{sR}^{NN}(K_{3,n}) \leq 3$. Now let g be a $\gamma_{sR}^{NN}(K_{3,n})$-function. If $g(x_1), g(x_2), g(x_3) \neq 2$, then for each i, $g(y_i) \neq -1$. Thus

$$\gamma_{sR}^{NN}(K_{3,n}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 3.$$

Now let, without loss of generaltiy, $g(x_1) = 2$. If $g(x_2) \neq -1$ ($g(x_3) \neq -1$ is similar), then

$$\gamma_{sR}^{NN}(K_{3,n}) = \omega(g) = g[x_3] + g(x_1) + g(x_2) \geq 0 + 2 + 1 = 3.$$

Thus let $g(x_2) = g(x_3) = -1$. Since $g[y_i] \geq 0$ for each i, we deduce that $g(y_i) \neq -1$ for each i. Hence

$$\gamma_{sR}^{NN}(K_{3,n}) = \omega(g) = \sum_{1 \leq i \leq 3} g(x_i) + \sum_{1 \leq i \leq n} g(y_i) \geq 0 + 3 = 3.$$

Let $m \geq 4$. We first show that $\gamma_{sR}^{NN}(K_{m,n}) \leq 2$. Assume that m is even. If n is even, then define the function $f : V(K_{m,n}) \to \{-1,1,2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_i) = (-1)^{i+1}$ for $2 \leq i \leq m$ and $f(y_j) = (-1)^{j+1}$ for $2 \leq j \leq n$. Then f is an NNSRDF on $K_{m,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{m,n}) \leq 2$. If n is odd, then define the function $f : V(K_{m,n}) \to \{-1,1,2\}$ by $f(x_1) = f(y_1) = f(y_2) = 2$, $f(x_2) = f(y_3) = f(y_4) = -1$, $f(x_i) = (-1)^i$ for $3 \leq i \leq m$ and $f(y_j) = (-1)^j$ for $5 \leq j \leq n$. Then f is an NNSRDF on $K_{m,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{m,n}) \leq 2$. Finally let $m \geq 5$ be odd. If n is even, then define the function $f : V(K_{m,n}) \to \{-1,1,2\}$ by

$f(x_1) = f(x_2) = f(y_1) = 2$, $f(x_3) = f(x_4) = f(y_2) = -1$, $f(x_i) = (-1)^i$ for $5 \le i \le m$ and $f(y_j) = (-1)^j$ for $3 \le j \le n$. Then f is an NNSRDF on $K_{m,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{m,n}) \le 2$. If n is odd, then define the function $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(x_2) = f(y_1) = f(y_2) = 2$, $f(x_3) = f(x_4) = f(y_3) = f(y_4) = -1$, $f(x_i) = (-1)^i$ for $5 \le i \le m$ and $f(y_j) = (-1)^j$ for $5 \le j \le n$. Then f is an NNSRDF on $K_{m,n}$ of weight 2 and thus $\gamma_{sR}^{NN}(K_{m,n}) \le 2$. Therefore $\gamma_{sR}^{NN}(K_{m,n}) \le 2$ when $m \ge 4$.

To prove $\gamma_{sR}^{NN}(K_{m,n}) \ge 2$, assume that g is a $\gamma_{sR}^{NN}(K_{m,n})$-function. If $g(u) \ne 2$ for every $u \in X$ ($u \in Y$ is similar), then $g(y_i) \ge 1$ for $1 \le i \le n$. This yields to

$$\gamma_{sR}^{NN}(K_{m,n}) = \omega(g) = \sum_{2 \le i \le n} g(y_i) + g[y_1] \ge n - 1 \ge 3,$$

a contradiction. Thus we assume, without loss of generality, that $g(x_1) = g(y_1) = 2$. If $g(x_i) \ne -1$, $2 \le i \le m$ ($g(y_j) \ne -1$, $2 \le j \le n$ is similar), then

$$\gamma_{sR}^{NN}(K_{m,n}) = \omega(g) = g[x_1] + \sum_{2 \le i \le m} g(x_i) \ge 3,$$

a contradiction. Thus we may assume that $g(x_2) = g(y_2) = -1$. Since $g[x_2] \ge 0$, we observe that $\sum_{1 \le i \le n} g(y_i) \ge 1$, and since $g[y_2] \ge 0$, we have $\sum_{1 \le i \le m} g(x_i) \ge 1$. Hence

$$\gamma_{sR}^{NN}(K_{m,n}) = \omega(g) = \sum_{1 \le i \le m} g(x_i) + \sum_{1 \le i \le n} g(y_i) \ge 1 + 1 = 2,$$

and this completes the proof. □

3 Bounds on $\gamma_{sR}^{NN}(G)$

In this section we start with some simple upper bounds on the nonnegative signed Roman domination number of a graph. Furthermore, we show that $\gamma_{sR}^{NN}(G) \ge \frac{3}{4}(\sqrt{8n+1} - 1) - n$ and $\gamma_{sR}^{NN}(G) \ge (8n - 12m)/7$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma_{sR}^{NN}(G) \ge \frac{3}{2}(\sqrt{4n+9} - 3) - n$, and we characterize the extremal graphs.

Proposition 7. If G is a graph of order n, then

$$\gamma_{sR}^{NN}(G) \le n,$$

with equality if and only if $G = \overline{K_n}$.

Proof. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = 1$ for each vertex $v \in V(G)$. Then the function f is an NNSRDF on G of weight n

and thus $\gamma_{sR}^{NN}(G) \le n$. If $G = \overline{K_n}$, then obviously $\gamma_{sR}^{NN}(G) = n$. Now let $\gamma_{sR}^{NN}(G) = n$. If $G \ne \overline{K_n}$, then $\delta(G) \ge 1$. Let u be a vertex of minimum degree in G and let v be a neighbor of u. Then, $f = (\{u\}, V - \{u,v\}, \{v\})$ is an NNSRDF in G, and so $\gamma_{sR}^{NN}(G) \le n-1$. Hence $G = \overline{K_n}$. □

Theorem 8. Let G be a connected graph of order $n \ge 2$. Then $\gamma_{sR}^{NN}(G) = n-1$ if and only if $n = 2$.

Proof. Clearly, if $n = 2$, then $\gamma_{sR}^{NN}(G) = 1 = n-1$. Conversely, assume that $\gamma_{sR}^{NN}(G) = n-1$.

If $\mathrm{diam}(G) = 1$, then G is the complete graph, and Proposition 3 implies the desired result.

Let now $\mathrm{diam}(G) \ge 3$, and let $u_1u_2\dots u_p$ be a diametral path. Define the function $f : V(G) \to \{-1,1,2\}$ by $f(u_1) = f(u_p) = -1$, $f(u_2) = f(u_{p-1}) = 2$ and $f(x) = 1$ otherwise. Since $p \ge 4$, it is easy to verify that f is an NNSRDF on G of weight $n-2$, a contradiction.

Finally, let $\mathrm{diam}(G) = 2$, and let uvw be a diametral path. Let $v_1, v_2, \dots, v_t$ be the vertices of degree two with the property that $N(v_i) = \{u,w\}$ with $v_i \ne v$ for $1 \le i \le t$.

If there is no such vertex of degree two, then define the function $f : V(G) \to \{-1,1,2\}$ by $f(u) = f(w) = -1$, $f(v) = 2$ and $f(x) = 1$ otherwise.

If $t \ge 1$, then define the function $f : V(G) \to \{-1,1,2\}$ by $f(u) = f(w) = 2$, $f(v) = f(v_1) = -1$ and $f(x) = 1$ otherwise.

In both cases it is easy to check that f is an NNSRDF on G of weight at most $n-2$, a contradiction. □

Corollary 9. Let G be a graph of order $n \ge 2$. Then $\gamma_{sR}^{NN}(G) = n-1$ if and only if G consists of a K_2 and $n-2$ isolated vertices.

Corollary 10. Let G be a graph of order $n \ge 2$ such that $G \ne \overline{K_n}$ and $G \ne K_2 \cup \overline{K_{n-2}}$. Then $\gamma_{sR}^{NN}(G) \le n-2$.

Theorem 11. If G is a graph of order n with minimum degree $\delta(G) \ge 1$, then

$$\gamma_{sR}^{NN}(G) \le n + 1 - 2\left\lfloor \frac{\delta(G)+1}{2} \right\rfloor.$$

Proof. Define $t = \left\lfloor \frac{\delta(G)+1}{2} \right\rfloor$. Let $v \in V(G)$ be a vertex of maximum degree, and let $A = \{u_1, u_2, \dots, u_t\}$ be a set of t neighbors of v. Define the function $f : V(G) \to \{-1,1,2\}$ by $f(v) = 2$, $f(u_i) = -1$ for $1 \le i \le t$ and $f(w) = 1$ for $w \in V(G) - (A \cup \{v\})$. If $x \in V(G) - (A \cup \{v\})$, then

$$f[x] \ge -t + 1 + (\delta(G) - t) = \delta(G) + 1 - 2t = \delta(G) + 1 - 2\left\lfloor \frac{\delta(G)+1}{2} \right\rfloor \ge 0.$$

If $x \in A$, then

$$f[x] \geq -t + (\delta(G) - (t-1)) = \delta(G) + 1 - 2t = \delta(G) + 1 - 2\left\lfloor \frac{\delta(G)+1}{2} \right\rfloor \geq 0.$$

Now if $x = v$, then

$$f[x] = -t + 2 + (\Delta(G) - t) = \Delta(G) + 2 - 2t = \Delta(G) + 2 - 2\left\lfloor \frac{\delta(G)+1}{2} \right\rfloor \geq 0.$$

Therefore f is an NNSRDF on G of weight $2 - t + (n - t - 1) = n + 1 - 2t$ and thus $\gamma_{sR}^{NN}(G) \leq n + 1 - 2t = n + 1 - 2\left\lfloor \frac{\delta(G)+1}{2} \right\rfloor$. □

In [6], we have proved the following proposition for the signed Roman k-domination function when $k \geq 1$.

Proposition 12. *[6]* Let $k \geq 1$ be an integer. Assume that $f = (V_{-1}, V_1, V_2)$ is an SRkDF on a graph G of order n. If $\delta \geq k - 1$, then

1. $(\Delta + \delta + 2)\omega(f) \geq (\delta + 2k - \Delta)n + (\delta - \Delta)|V_2|$.
2. $\omega(f) \geq \frac{(\delta+2k-2\Delta-1)n}{2\Delta+\delta+3} + |V_2|$.

It is a simple matter to verify that Proposition 12 remains valid for $k = 0$. Hence we have the following useful result.

Proposition 13. If $f = (V_{-1}, V_1, V_2)$ is an NNSRDF on a graph G of order n, then

1. $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta)n + (\delta - \Delta)|V_2|$.
2. $\omega(f) \geq \frac{(\delta-2\Delta-1)n}{2\Delta+\delta+3} + |V_2|$.

As an application of the 1. inequality in Proposition 13, we obtain a lower bound on the nonnegative signed Roman domination number for regular graphs.

Corollary 14. If G is an regular graph, then $\gamma_{sR}^{NN}(G) \geq 0$.

Propositions 3 and 5 demonstrate that Corollary 14 is sharp.

Corollary 15. If G is a graph with $\delta < \Delta$, then

$$\gamma_{sR}^{NN}(G) \geq \frac{2n(\delta - \Delta)}{2\Delta + \delta + 3}$$

Proof. Multiplying both sides of the inequality 2. in Proposition 13 by $\Delta-\delta$ and adding the resulting inequality to the inequality 1. in Proposition 13, we obtain

$$\gamma_{sR}^{NN}(G) \geq \frac{(-2\Delta^2+2\Delta\delta-2\Delta+2\delta)n}{(\Delta+1)(2\Delta+\delta+3)} = \frac{2n(\delta-\Delta)}{2\Delta+\delta+3}.$$

□

Example 16. Let $x_1, x_2, \ldots, x_{2p}$ be the leaves of the star $K_{1,2p}$ with $p \geq 2$. If we add the edges $x_1x_2, x_3x_4, \ldots, x_{2p-1}x_{2p}$ to the star $K_{1,2p}$, then denote the resulting graph by H. Now let $H_1, H_2, \ldots, H_p$ be p copies of H with the central vertices $v_1, v_2, \ldots, v_p$. Define the graph G as the disjoint union of $H_1, H_2, \ldots, H_p$ such that all central vertices are pairwise adjacent. Then $\delta(G) = 2$, $\Delta(G) = 3p-1$ and $n(G) = p(2p+1)$. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v_i) = 2$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. It is easy to verify that $\sum_{x\in N[u]} f(x) = 0$ for every vertex $u \in V(G)$. Therefore f is an NNSRDF on G of weight

$$\omega(f) = -2p(p-1) = \frac{2n(G)(\delta(G)-\Delta(G))}{2\Delta(G)+\delta(G)+3}.$$

Example 16 shows that Corollary 15 is sharp.

Theorem 17. Let G be a graph of order $n \geq 2$ with maximum degree $\Delta(G) \leq n-2$. If $\delta(G)$ is the minimum degree, then

$$\gamma_{sR}^{NN}(G) \geq \delta(G) + 4 - n.$$

Proof. Let f be a $\gamma_{sR}^{NN}(G)$-function. If $f(x) = 1$ for each vertex $x \in V(G)$, then $\gamma_{sR}^{NN}(G) = n \geq \delta(G) + 4 - n$. Now assume that there exists a vertex w with $f(w) = -1$. Then w has a neighbor v with $f(v) = 2$. Since $d(v) \leq \Delta(G) \leq n-2$, there exists a vertex u not adjacent to v. Therefore we obtain the desired bound as follows.

$$\begin{aligned}\gamma_{sR}^{NN}(G) &= \sum_{x\in V(G)} f(x) = f(v) + \sum_{x\in N[u]} f(x) + \sum_{x\in V(G)-(N[u]\cup\{v\})} f(x) \\ &\geq 2 + 0 - (n - d(u) - 2) = 4 + d(u) - n \geq \delta(G) + 4 - n.\end{aligned}$$

□

Corollary 18. Let G be an r-regular graph of order n. If $r = n-2$, then $\gamma_{sR}^{NN}(G) \geq 2$, and if $r = n-3$, then $\gamma_{sR}^{NN}(G) \geq 1$.

Corollary 18 is an improvement of Corollary 14 for the special case that G is $(n-2)$-regular or $(n-3)$-regular. The cycles C_4 and C_5 show that equality in Corollary 18 is possible. Combining Corollary 18 with Theorem 11, we arrive at the next result.

Corollary 19. Let G be an r-regular graph of order n. If $r = n - 2$, then $2 \leq \gamma_{sR}^{NN}(G) \leq 3$. If $r = n - 3$ and n is even, then $1 \leq \gamma_{sR}^{NN}(G) \leq 3$, and if $r = n - 3$ and n is odd, then $1 \leq \gamma_{sR}^{NN}(G) \leq 4$.

We call a set $S \subseteq V(G)$ a 2-packing of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices of $u, v \in S$. The maximum cardinality of a 2-packing is the 2-packing number of G, denoted by $\rho(G)$.

Theorem 20. If G is a graph of order n such that $\delta(G) \geq 1$, then

$$\gamma_{sR}^{NN}(G) \geq (\delta(G) + 1) \cdot \rho(G) - n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of G, and let f be a $\gamma_{sR}^{NN}(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(G)} N[v_i]$ then, since $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a 2-packing of G, we have

$$|A| = \sum_{i=1}^{\rho(G)} (d(v_i) + 1) \geq (\delta(G) + 1) \cdot \rho(G).$$

It follows that

$$\begin{aligned} \gamma_{sR}^{NN}(G) &= \sum_{u \in V(G)} f(u) = \sum_{i=1}^{\rho(G)} f[v_i] + \sum_{u \in V(G)-A} f(u) \\ &\geq \sum_{u \in V(G)-A} f(u) \geq -n + |A| \\ &\geq (\delta(G) + 1) \cdot \rho(G) - n. \end{aligned}$$

□

Corollary 21. If G is a graph of order n such that $\delta(G) \geq 1$, then

$$\gamma_{sR}^{NN}(G) \geq (\delta(G) + 1)(1 + \left\lfloor \frac{\text{diam(G)}}{3} \right\rfloor) - n.$$

Proof. Let $d = \text{diam(G)} = 3t + r$ with integers $t \geq 0$ and $0 \leq r \leq 2$, and let $\{v_1, v_2, \ldots, v_d\}$ be a diametral path. Then $A = \{v_0, v_3, \ldots, v_{3t}\}$ is a 2-packing of G such that $|A| = 1 + \left\lfloor \frac{\text{diam(G)}}{3} \right\rfloor$. Since $\rho(G) \geq |A|$, Theorem 20 implies that

$$\gamma_{sR}^{NN}(G) \geq (\delta(G) + 1) \cdot \rho(G) - n \geq (\delta(G) + 1)(1 + \left\lfloor \frac{\text{diam(G)}}{3} \right\rfloor) - n.$$

□

If $n \geq 5$, then Proposition 3 shows that Theorem 20 and Corollary 21 are sharp.

Now we determine a lower bound on the nonnegative signed Roman domination number of a graph. For this purpose, we define a family of graphs as follows. For $k \geq 1$, let $\mathcal{F}_k = \{F_k \mid k \geq 1\}$ be a family of graph as follows. Let X vertex set of the complete graph K_k and let F_k be the graph obtained from K_k by adding $2k$ new vertices to each vertex of the complete graph such that for each new vertex x, $1 \leq d(x) \leq 2$ and for every $u \in X$, $d(u) = 3k - 1$. We note that F_k has order $n = k(2k+1) = 2k^2 + k$. Let $\mathcal{F} = \bigcup_{k \geq 1} \mathcal{F}_k$.

Theorem 22. If G is a graph of order n, then

$$\gamma_{sR}^{NN}(G) \geq \frac{3}{4}(\sqrt{8n+1} - 1) - n,$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{sR}^{NN}(G)$-function. If $V_{-1} = \emptyset$, then $\gamma_{sR}^{NN}(G) = n \geq \frac{3}{4}(\sqrt{8n+1} - 1) - n$. Hence, we may assume that $V_{-1} \neq \emptyset$. Since each vertex in V_{-1} has at least one neighbor in V_2, it follows from the Pigeonhole Principle that at least one vertex v of V_2 has at least $\frac{|V_{-1}|}{|V_2|} = \frac{n_{-1}}{n_2}$ neighbors in V_{-1}. Therefore, $0 \leq f[v] \leq 2n_2 + n_1 - \frac{n_{-1}}{n_2}$, and so $2n_2^2 + n_1 n_2 - n_{-1} \geq 0$. Since $n = n_2 + n_1 + n_{-1}$, we have equivalently that $2n_2^2 + n_1 n_2 + n_2 + n_1 - n \geq 0$. Since $n_2 \geq 1$ and n_1 is a non-negative integer, $\frac{5}{3}n_1 n_2 - \frac{1}{3}n_1 \geq 0$. Therefore

$$\begin{aligned} 2(n_2 + \frac{2}{3}n_1 + \frac{1}{4})^2 - \frac{1}{8} - n &= 2n_2^2 + \frac{8}{9}n_1^2 + \frac{8}{3}n_1 n_2 + n_2 + \frac{2}{3}n_1 - n \\ &\geq (2n_2^2 + n_1 n_2 + n_2 + n_1 - n) + (\frac{5}{3}n_1 n_2 - \frac{1}{3}n_1) \geq 0. \end{aligned}$$

or equivalently, $3n_2 + 2n_1 \geq \frac{3}{4}(\sqrt{8n+1} - 1)$. Thus

$$\gamma_{sR}^{NN}(G) = 3n_2 + 2n_1 - n \geq \frac{3}{4}(\sqrt{8n+1} - 1) - n.$$

which establishes the desired lower bound.

Suppose that $\gamma_{sR}^{NN}(G) = \frac{3}{4}(\sqrt{8n+1}-1)-n$. Then all the above inequalities must be equalities. In particular, $n_1 = 0$ and $2n_2^2 = n_{-1}$. Furthermore, each vertex of V_{-1} is adjacent to exactly one vertex of V_2 and therefore has degree one or two in G, while each vertex of V_2 is adjacent to all other $n_2 - 1$ vertices of V_2 and to $2n_2$ vertices of V_{-1}. Therefore, $G \in \mathcal{F}$.

On the other hand, suppose that $G \in \mathcal{F}$. Then $G \in \mathcal{F}_k$ and $G = F_k$ such that $k \geq 1$. Assigning to the every vertex of K_k the value 2, and to all other vertices the value -1, we produce an NNSRDF f of weight

$$f(V)=\sum_{v\in V} f(v)=2k-k(2k)=-2k^2+2k=\frac{3}{4}(\sqrt{8n+1}-1)-n.$$

Therefore,

$$\gamma_{sR}^{NN}(G)\le f(V)=\frac{3}{4}(\sqrt{8n+1}-1)-n.$$

Consequently,

$$\gamma_{sR}^{NN}(G)=\frac{3}{4}(\sqrt{8n+1}-1)-n.$$

□

Theorem 23. If G is a connected graph of order $n\ge 2$ and size m, then

$$\gamma_{sR}^{NN}(G)\ge\frac{8n-12m}{7}.$$

Proof. Let $f=(V_{-1},V_1,V_2)$ be a $\gamma_{sR}^{NN}(G)$-function, $|V_i|=n_i$, $m(G[V_i])=m_i$ for $i\in\{-1,1,2\}$ and $|V_1\cup V_2|=n_{12}$ and $m(G[V_1\cup V_2])=m_{12}$. If $V_{-1}=\emptyset$, then $\gamma_{sR}^{NN}(G)=n\ge\frac{8n-12m}{7}$. Now we assume that $V_{-1}\neq\emptyset$. Since each vertex of V_{-1} is adjacent to at least one vertex of V_2, we have

$$\sum_{v\in V_2}||[v,V_{-1}]||=||[V_{-1},V_2]||\ge n_{-1}.$$

Furthermore, for each $v\in V_2$, we observe that $0\le f[v]=f(v)+2||[v,V_2]||+||[v,V_1]||-||[v,V_{-1}]||$ and thus $||[v,V_{-1}]||\le 2||[v,V_2]||+||[v,V_1]||+2$. We deduce that

$$\begin{aligned}n_{-1} &\le \sum_{v\in V_2}||[v,V_{-1}]||\le\sum_{v\in V_2}(2||[v,V_2]||+||[v,V_1]||+2)\\ &= 4m_2+||[V_1,V_2]||+2n_2=4m_{12}+2n_2-4m_1-3||[V_1,V_2]||,\end{aligned}$$

and thus $m_{12}\ge(n_{-1}-2n_2+4m_1+3||[V_1,V_2]||)/4$. This inequality and $n_{-1}\le||[V_{-1},V_2]||$ lead to

$$\begin{aligned}m &\ge m_{12}+||[V_{-1},V_2]||+||[V_1,V_{-1}]||\\ &\ge \frac{1}{4}(n_{-1}-2n_2+4m_1+3||[V_1,V_2]||)+n_{-1}+||[V_1,V_{-1}]||\\ &= \frac{1}{4}(5n_{-1}-2n_{12}+2n_1+4m_1+3||[V_1,V_2]||+4||[V_1,V_{-1}]||)\\ &= \frac{1}{4}(5n-7n_{12}+2n_1+4m_1+3||[V_1,V_2]||+4||[V_1,V_{-1}]||).\end{aligned}$$

It follows that

$$n_{12} \geq \frac{1}{7}(5n - 4m + 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|),$$

and so

$$\begin{aligned}
\gamma_{sR}^{NN}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\
&\geq \frac{3}{7}(5n - 4m + 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) - n - n_1 \\
&= \frac{1}{7}(8n - 12m) + \frac{1}{7}(12m_1 + 9|[V_1, V_2]| + 12|[V_1, V_{-1}]| - n_1).
\end{aligned}$$

Let

$$\mu(n_1) = 12m_1 + 9|[V_1, V_2]| + 12|[V_1, V_{-1}]| - n_1.$$

It suffices to show that $\mu(n_1) \geq 0$, because then $\gamma_{sR}^{NN}(G) \geq \frac{8n-12m}{7}$, which establish the desired lower bound. If $n_1 = 0$, then $\mu(n_1) = 0$. Now we assume that that $n_1 \geq 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$. Since G is connected, each component H_i contains a vertex adjacent to a vertex of V_2 or to a vertex of V_{-1} for $1 \leq i \leq t$. This implies

$$\begin{aligned}
m_1 + |[V_1, V_2]| + |[V_1, V_{-1}]| &\geq (h_1 - 1) + (h_2 - 1) + \ldots + (h_t - 1) + t \\
&= h_1 + h_2 + \ldots + h_t = n_1.
\end{aligned}$$

This leads to

$$\begin{aligned}
\mu(n_1) &= 12m_1 + 9|[V_1, V_2]| + 12|[V_1, V_{-1}]| - n_1 \\
&> m_1 + |[V_1, V_2]| + |[V_1, V_{-1}]| - n_1 \geq 0,
\end{aligned}$$

and the proof is complete. □

Corollary 24. If T is a tree of order $n \geq 2$, then

$$\gamma_{sR}^{NN}(T) \geq \frac{12 - 4n}{7}.$$

Next example demonstrates that the lower bounds in Theorem 23 and Corollary 24 are sharp.

Example 25. For $k \geq 1$, let F_k be the graph obtained from a connected graph F of order k by adding $2d_F(v) + 2$ pendant edges to each vertex v of F. Then

$$n(F_k) = n(F) + \sum_{v \in V(F)} (2d_F(v) + 2) = 3n(F) + 4m(F)$$

and

$$m(F_k) = m(F) + \sum_{v\in V(F)} (2d_F(v)+2) = 2n(F) + 5m(F).$$

Assigning to every vertex in $V(F)$ the weight 2 and to every vertex in $V(F_k) - V(F)$ the weight -1 produces an NNSRDF f of weight

$$\omega(f) = 2n(F) - \sum_{v\in V(F)} (2d_F(v)+2) = -4m(F) = \frac{8n(F_k) - 12m(F_k)}{7}.$$

Using Theorem 23, we obtain $\gamma_{sR}^{NN}(F_k) = \frac{8n(F_k)-12m(F_k)}{7}$.

Next we determine a lower bound on the nonnegative signed Roman domination number of a bipartite graph. For this purpose, we define a family of bipartite graphs as follows. For $k \geq 1$, let $\mathcal{B}_k = \{B_k \mid k \geq 1\}$ be a family of bipartite graph as follows. Let X and Y be the partite sets of the complete bipartite graph $K_{k,k}$ and let B_k be the bipartite graph obtained from $K_{k,k}$ by adding $2k+2$ new vertices to each vertex of the complete bipartite graph such that for each new vertex x, $1 \leq d(x) \leq 2$ and for every $u \in X \cup Y$, $d(u) = 3k+2$. We note that B_k has order $n = 2k(2k+3) = 4k^2 + 6k$. Let $\mathcal{B} = \bigcup_{k\geq 1} \mathcal{B}_k$.

Theorem 26. If G is a bipartite graph of order n, then

$$\gamma_{sR}^{NN}(G) \geq \frac{3}{2}(\sqrt{4n+9} - 3) - n,$$

with equality if and only if $G \in \mathcal{B}$.

Proof. Let X and Y be the partite sets of the bipartite graph G. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{sR}^{NN}(G)$-function and let X_{-1}, X_1, and X_2 be the set of vertices in X that are assigned the value -1, 1 and 2, respectively under f. Let Y_{-1}, Y_1, and Y_2 be defined analogously. Let $|X_{-1}| = s$, $|X_1| = s_1$, $|X_2| = s_2$, $|Y_{-1}| = t$, $|Y_1| = t_1$, $|Y_2| = t_2$. Thus, $n_{-1} = s + t$, $n_1 = s_1 + t_1$ and $n_2 = s_2 + t_2$. We First show that

$$s \leq t_2(2s_2 + s_1 + 2), \qquad t \leq s_2(2t_2 + t_1 + 2). \tag{1}$$

For each vertex $y \in Y_2$, we have that $f(y)+2d_{X_2}(y)+d_{X_1}(y)-d_{X_{-1}}(y) = f[y] \geq 0$, and so $d_{X_{-1}}(y) \leq 2d_{X_2}(y)+d_{X_1}(y)+f(y) = 2d_{X_2}(y)+d_{X_1}(y)+2 \leq 2s_2+s_1+2$. By the definition of an NNSRDF, each vertex in X_{-1} is adjacent to at least one vertex in Y_2, and so

$$\begin{aligned} s = |X_{-1}| \leq ||[X_{-1}, X_2]|| &= \sum_{y\in Y_2} d_{X_{-1}}(y) \\ &\leq \sum_{y\in Y_2} (2s_2 + s_1 + 2) \\ &\leq t_2(2s_2 + s_1 + 2). \end{aligned}$$

Analogously, we have that $t \leq s_2(2t_2 + t_1 + 2)$. Now we show that

$$s_1 + s_2 + t_1 + t_2 \geq \sqrt{n + \frac{9}{4}} + \frac{1}{3}(s_1 + t_1) - \frac{3}{2}. \qquad (2)$$

We note that for integers s and t, we have $s^2 + t^2 \geq 2st$, with equality if and only if $s = t$. Hence by simple algebra and by inequality (1), we have that

$$\begin{aligned}
&(\frac{2}{3}s_1 + s_2 + \frac{2}{3}t_1 + t_2 + \frac{3}{2})^2 \\
&\geq s_2^2 + t_2^2 + 2s_2t_2 + \frac{4}{3}s_2t_1 + \frac{4}{3}s_1t_2 + 3s_2 + 3t_2 + 2s_1 + 2t_1 + \frac{9}{4} \\
&\geq 4s_2t_2 + s_2t_1 + s_1t_2 + 3s_2 + 3t_2 + s_1 + t_1 + \frac{9}{4} \\
&\geq s + t + s_2 + t_2 + s_1 + t_1 + \frac{9}{4} \\
&= n + \frac{9}{4}.
\end{aligned}$$

The desired inequality now follows by taking squaring roots on both sides and rearranging terms. We now return to the proof of Theorem 26. By inequality (2), we have

$$\begin{aligned}
\gamma_{sR}^{NN}(G) &= 2n_2 + n_1 - n_{-1} \\
&= 3n_2 + 2n_1 - n \\
&= 3(n_2 + n_1) - n_1 - n \\
&= 3(s_2 + t_2 + s_1 + t_1) - (s_1 + t_1) - n \\
&\geq 3(\sqrt{n + \frac{9}{4}} + \frac{1}{3}(s_1 + t_1) - \frac{3}{2}) - (s_1 + t_1) - n \\
&= 3\sqrt{n + \frac{9}{4}} - \frac{9}{2} - n \\
&= \frac{3}{2}(\sqrt{4n + 9} - 3) - n.
\end{aligned}$$

which establishes the desired lower bound.

Suppose that $\gamma_{sR}^{NN}(G) = \frac{3}{2}(\sqrt{4n+9} - 3) - n$. Then all the above inequalities, including the inequalities in (1) and (2), must be equalities. In particular, $s_1 = t_1 = 0$ and $s^2 + t^2 = 2st$, implying that $s_2 = t_2$. Equality (1), implies that $s = t_2(2s_2 + 2)$, $t = s_2(2t_2 + 2)$ and that every vertex in X_2 is adjacent to every vertex in Y_2 and vice versa. Further, every vertex x in X_{-1} has $1 \leq d(x) \leq 2$ and is adjacent to exactly one vertex of Y_2 while every vertex in Y_2 is adjacent to exactly $2s_2 + 2$ vertices in X_{-1}.

Analogously, every vertex y in Y_{-1} has $1 \leq d(y) \leq 2$ and is adjacent to exactly one vertex of X_2 while every vertex in X_2 is adjacent to exactly $2t_2 + 2$ vertices in Y_{-1}. Thus, $G = B_k$, and so $G \in \mathcal{B}$.

On the other hand, suppose $G \in \mathcal{B}$. Then $G \in \mathcal{B}_k$ and $G = B_k$ such that $k \geq 1$. Assigning to the every vertex of $K_{k,k}$ the value 2, and to all other vertices the value -1, we produce an NNSRDF f of weight

$$f(V) = \sum_{v \in V} f(v) = 2(2k) - 2k(2k+2) = -4k^2 = \frac{3}{2}(\sqrt{4n+9} - 3) - n.$$

Therefore,

$$\gamma_{sR}^{NN}(G) \leq f(V) = \frac{3}{2}(\sqrt{4n+9} - 3) - n.$$

Consequently,

$$\gamma_{sR}^{NN}(G) = \frac{3}{2}(\sqrt{4n+9} - 3) - n.$$

□

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