# On Slopes of Cancellable Numbers

Jerome Manheim
& Hossein Shahmohamad
School of Mathematical Sciences
Rochester Institute of Technology, Rochester, NY 14623
Email: hxssma@rit.edu

**Abstract**

A cancellable number (CN) is a fraction in which a decimal digit can be removed ("canceled") in the numerator and denominator without changing the value of the number; examples include $\frac{64}{16}$ where the 6 can be canceled and $\frac{98}{49}$ where the 9 can be canceled. We show that the slope of the line of a cancellable number need not be negative.

## 1 Introduction

In 1933, Morley [2] posed the problem of illegitimate cancellation, identifying $\frac{16}{64} = \frac{64}{16}$, $\frac{65}{26} = \frac{65}{26}$, $\frac{95}{19} = \frac{95}{19}$ and $\frac{98}{49} = \frac{98}{49}$ as the only proper fractions with denominators less than one hundred which could be reduced by the indicated illegal cancellation. B·L· Schwartz [3] provided the solutions for three digit denominators in 1961. In [1], more results on cancellable numbers were presented.

## 2 A Result on Cancellation Slopes

In [1], it was stated that a $p_H$ CN for $\frac{L}{M}$ is a representative for $\frac{L}{M}$ such that $H$ can be cancelled 1,2,3, ..., $p$ times, each cancellation producing a representative of $\frac{L}{M}$. An examination of 1-CNs for an arbitrary $\frac{L}{M}$, $L \geq M$ suggests that the cancelling line never has a positive slope. Consider $\frac{64}{16}$, $\frac{864}{216}$, $\frac{20}{10}$, $\frac{238}{34}$, $\frac{16058}{4012}$.

Even if we look at the $p$-CNs, it seems that whenever there does exist a CN with a positive slope cancelling line, e.g., $\frac{7992}{6993}$, the positive slope is not necessary, e.g., $\frac{7992}{6993}$. Just how general is this observation? The general case for integers follows.

**Theorem 2.1**: Let $M = \frac{S}{T}$. If the coefficient of $10^k$ in the numerator cancels with the coefficient of $10^{k+r}$ in the denominator, then the expression is an $r$-CN for $M$.

**Proof:** Assume that the $(k+1)$st digit from the right in the numerator is cancellable with the $(k+r+1)$st digit from the right in the denominator, $r > 0$.

Let $M = \frac{S}{T}$ where
$S = 10^m a_m + \ldots + 10^{k+r+1} a_{k+r+1} + 10^{k+r} a_{k+r} + 10^{k+r-1} a_{k+r-1} + \ldots + 10^{k+1} a_{k+1} + 10^k a_k + 10^{k-1} a_{k-1} + \ldots + a_0$

$T = 10^n b_n + \ldots + 10^{k+r+1} b_{k+r+1} + 10^{k+r} H + 10^{k+r-1} a_{k+r-1} + \ldots + 10^{k+1} b_{k+1} + 10^k b_k + 10^{k-1} b_{k-1} + \ldots + b_0$

Now we can see that $M = \frac{P}{Q}$ where
$P = 10^{m-1} a_m + \ldots + 10^{k+r} a_{k+r+1} + 10^{k+r-1} a_{k+r} + 10^{k+r-2} a_{k+r-1} + \ldots + 10^k a_{k+1} + 10^{k-1} a_{k-1} + \ldots + a_0$

$Q = 10^{n-1} b_n + \ldots + 10^{k+r} b_{k+r+1} + 10^{k+r-1} b_{k+r-1} + 10^{k+r-2} a_{k+r-2} + \ldots + 10^k b_k + 10^{k-1} b_{k-1} + \ldots + b_0$

From the equation with $M = \frac{S}{T}$ we obtain the following set of equations:
$Mb_0 = a_0 + 10c_0$, $Mb_1 + c_0 = a_1 + 10c_1$, $\ldots$, $Mb_k + c_{k-1} = H + 10c_k$, $Mb_{k+1} + c_k = a_{k+1} + 10c_{k+1}, \ldots, Mb_{k+r-1} + c_{k+r-2} = a_{k+r-1} + 10c_{k+r-1}, \ldots, Mb_n + c_{n-1} = a_n + 10c_n = a_m$.

From the equation with $M = \frac{P}{Q}$ we obtain the following set of equations:
$Mb_0 = a_0 + 10c_0$, $Mb_1 + c_0 = a_1 + 10c_1$, $\ldots$, $Mb_k + c_{k-1} = a_{k+1} + 10c_k$, $Mb_{k+1} + c_k = a_{k+2} + 10c_{k+1}, \ldots, Mb_{k+r-1} + c_{k+r-2} =$

$a_{k+r} + c_{k+r-1}$.

Comparing the above set of equations, it follows that $a_{k+1} = a_{k+2} = \ldots = a_{k+r} = H$. To show that $b_k = b_{k+1} = \ldots = b_{k+r-1} = H$, we examine the division process instead of multiplication. Since division proceeds from left to right, while multiplication begins with the rightmost digit , the first remainder is called $t_n$, the second $t_{n-1}$, etc. Noting that $a_m = 10c_n + a_n$, and writing $\frac{10c_n+a_n}{M} = b_n + t_n$, we have:

$a_n + 10C_n = M(b_n + t_n)$, $10Mt_n + a_{n-1} = M(b_{n-1} + t_{n-1})$, $\ldots$, $10Mt_{k+r+1} + a_{k+r} = M(H + t_{k+r})$, $10Mt_{k+r} + a_{k+r-1} = M(b_{k+r-1} + t_{k+r-1})$, $\ldots$, $10Mt_{k+1} + H = M(b_k + t_k), \ldots$

Similarly, we obtain: $\ldots$, $10Mr_{k+r+1} + a_{k+r} = M(b_{k=r-1} + t_{k+r})$, $10Mr_{k+r} + a_{k+r-1} = M(b_{k+r-2} + t_{k+r-1})$, $\ldots$, $10Mt_{k+1} + H = M(b_{k-1} + t_k), \ldots$

Comparing the above sets of equations shows that $b_{k+r-1} = b_{k+r-2} = \ldots = b_k = H$. It follows that it is always possible to avoid positive slope canceling lines and this completes the proof.

## 3 The case for Irrationals

If $I$ is an irrational number expressed as a quotient, then either the numerator or denominator or both have an infinite decimal part. Assuming that $I$ can be recaptured after cancellation of a finite number $n$ of digits, the numerator and denominator can then be multiplied by a sufficiently large power of 10 so that the cancellable digits all lie to the left of the decimal point. The assumption of cancellability states

$$I = \frac{(10^{k_1+1}A_1 + 10^{k_1}H_1 + 10^{k_2+1}A_2 + 10^{k_2}H_2 + \ldots + A_n +).A_{n+1}}{(10^{l_1+1}B_1 + 10^{l_1}H_1 + 10^{l_2+1}B_2 + 10^{l_2}H_2 + \ldots + B_n +).B_{n+1}}$$

$$= \frac{(10^{k_1+1-n}A_1 + 10^{k_2+2-n}A_2 + \ldots + A_n+).A_{n+1}}{(10^{l_1+1-n}B_1 + 10^{l_2+2-n}B_2 + \ldots + B_n+).B_{n+1}}$$

where either $.A_{n+1}$ or $.B_{n+1}$ or both are infinite decimals. But then, since $\frac{r}{s} = \frac{u}{v} \rightarrow \frac{r}{s} = \frac{r-u}{s-v}$, the irrational $I = \frac{P(A_i,H_i)}{Q(B_i,H_i)}$, where $P$ and $Q$ are finite-polynomials in their arguments, which is a contradiction. This leads to the following:

**Theorem 3.1**: No irrational number has a CN representation in which a finite number of digits can be cancelled.

This argument essentially states that $I = \frac{A}{B} = \frac{A-x}{B-y} \rightarrow I = \frac{x}{y}$. But, if the number of cancellations is finite both $x$ and $y$ are necessarily rational. In the infinite case one or both of $\{x, y\}$ may be irrational, as may be their quotient.

## 4 Different Bases

In base 10, zero and nine are preferential digits for cancellation. In base $k$, zero and $(k-1)$ are preferential. The following theorem expresses this fact and suggests that results concerning CNs have analogs in any base.

**Theorem 4.1**: If $L$ has $l$ digits and $M$ has $m$ digits in base $K$ and $l \geq m$, then

$$\frac{(L-1)K^{l+p} + (K-1)K^{l+p-1} + (K^{l+p-1} - L)}{(M-1)K^{l+p} + (K-1)K^{l+p-1} + (K^{l+p-1} - M)}$$

in base $K$ is a $P_{K-1}$-CN for $\frac{L}{M}$ in base $K$.

The proof consists of showing first that the expression does indeed equal $\frac{L}{M}$ and second that $j$ cancellations produce an identity, i.e., that

$$\frac{(L-1)K^{l+p} + (K-1)K^{l+p-1} + (K^{l+p-1} - L)}{(M-1)K^{l+p} + (K-1)K^{l+p-1} + (K^{l+p-1} - M)} =$$

$$\frac{(L-1)K^{l+p-j} + (K-1)K^{l+p-1-j} + (K^{l+p-1-j} - L)}{(M-1)K^{l+p-j} + (K-1)K^{l+p-1-j} + (K^{l+p-1-j} - M)}$$

in base $K$. Each of these is a simple reduction. Most CN results expressed in base $J$ have obvious analogs in base $K$. Accordingly, generality is not lost if $J = 10$ and this convention is continued in what follows.

## 5 Acknowledgment

The second author declares Dr. Jerome Henry Manheim to be the (main) author of this research endeavor and humbly pays homage and tribute to his memory and his family. We are grateful for the tremendous feedback from the referee as well.

## References

[1] Aliabadi, Manheim, Nategh, Shahmohamad, "Cancellable Numbers", JCMCC **102** (2017), pp. 45-54

[2] Morely, "Problems and Solutions", American Mathematical Monthly (August-September 1933), Vol. 40, No. 7, 425-26.

[3] Schwartz, "Illegal Cancellation," proposal 434; Mathematical magazine (September-October, 1961), Vol. 34, No. 6, 367-68