Spanning Trees and Hamiltonicity

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Abstract

The 3-path graph $\mathcal{P}_3(G)$ of a connected graph G of order 3 or more has the set of all 3-paths (paths of order 3) of G as its vertex set and two vertices of $\mathcal{P}_3(G)$ are adjacent if they have a 2-path in common. A Hamiltonian walk in a nontrivial connected graph G is a closed walk of minimum length that contains every vertex of G. With the aid of spanning trees and Hamiltonian walks in graphs, we provide sufficient conditions for the 3-path graph of a connected graph to be Hamiltonian.

Key Words: line graph, 3-path graph, Hamiltonian graph, spanning tree, Hamiltonian walk.

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1 Introduction

The *line graph* $L(G)$ of a nonempty graph G has the set of edges in G as its vertex set where two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Harary and Nash-Williams [4] characterized those graphs whose line graph is Hamiltonian. Their characterization primarily involved the existence of a circuit in a graph called a *dominating circuit* in which every edge of the graph is incident with a vertex of the circuit.

Theorem 1.1 [4] *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if G is the star $K_{1,t}$ for some integer $t \geq 3$ or G contains a dominating circuit.*

While a connected graph G with no vertices of degree 1 or 2 need not have a Hamiltonian line graph (see Figure 1), Chartrand and Wall [2] verified that if G is a connected graph with $\delta(G) \geq 3$, then $L(G)$ must have a spanning subgraph containing an Eulerian circuit, which is a dominating circuit of $L(G)$ and, consequently, gives the following result.

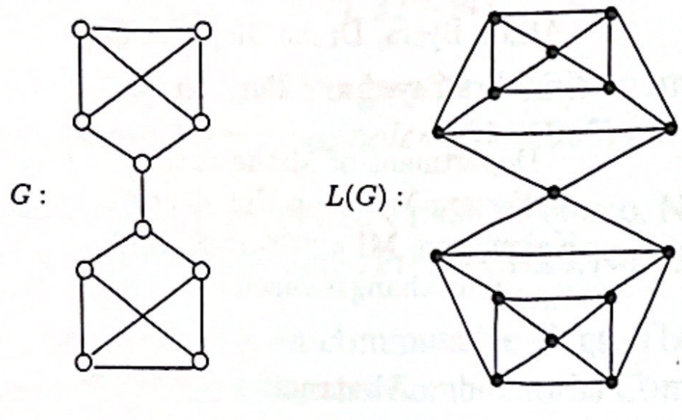

Figure 1: A connected 3-regular graph G and its line graph $L(G)$

Theorem 1.2 [2] *If G is a connected graph with $\delta(G) \geq 3$, then $L(L(G))$ is Hamiltonian.*

There are graphs possessing a variety of Hamiltonian properties where spanning trees or spanning walks play a major role. There is an alternative proof of Theorem 1.2 that can be given with the aid of spanning trees and Hamiltonian walks (see [1]). For the purpose of describing this technique, we refer to a concept introduced in [1] that provides us with another way to look at both $L(G)$ and $L(L(G))$ for a graph G. The vertex set of the line graph $L(G)$ is the set of 2-paths of a graph G (the paths P_2 of order 2) where two vertices of $L(G)$ are adjacent if the corresponding paths of G have a path P_1 in common. This observation leads us to a generalization of line graphs. Let $k \geq 2$ be an integer and let G be a graph containing k-paths. The *k-path graph* $\mathcal{P}_k(G)$ of G has the set of k-paths of G as its vertex set where two distinct vertices of $\mathcal{P}_k(G)$ are adjacent if the corresponding k-paths of G have a $(k-1)$-path in common. Thus, the 2-path graph of a nonempty graph is its line graph. Here, we are especially interested in the case when $k = 3$, that is, the 3-path graph $\mathcal{P}_3(G)$ of a connected graph G of order at least 3, which therefore has the set of 3-paths in G as its vertex set where two distinct vertices of $\mathcal{P}_3(G)$ are adjacent if the corresponding 3-paths of G have a 2-path (an edge) in common. Since every 3-path in a graph G is both a vertex of $\mathcal{P}_3(G)$ and an edge of $L(G)$ and every 3-path is obtained from a pair of adjacent edges of G, it follows that $\mathcal{P}_3(G) = L(L(G))$. In terms of 3-path graphs, Theorem 1.2 can be restated as follows.

Theorem 1.3 *If G is a connected graph with $\delta(G) \geq 3$, then $\mathcal{P}_3(G)$ is Hamiltonian.*

A *Hamiltonian walk* in a connected graph G is a closed walk of minimum length that contains every vertex of G. This concept was introduced by Goodman and Hedetniemi [3] who showed that if G is a connected graph of order n and size m, then the length of Hamiltonian walk W in G is at least n and at most $2m$. Furthermore, every edge of G occurs at most twice in W. The length of W is n if and only if G is Hamiltonian (in which case W is a Hamiltonian cycle) and the length of W is $2m$ if and only if G is a tree (in which case each edge of G appears exactly twice in W).

Every embedding of a tree T in the plane gives rise to a Hamiltonian walk in T. For example, let T be the star $K_{1,4}$ of order 5 whose four edges are labeled a, b, c, d. Figures 2(a) and 2(b) show two different embeddings of T in the plane. By tracing the walk as shown in Figure 2(c) using the embedding of T in Figure 2(a), we construct the Hamiltonian walk $W_1 = (w, v, x, v, y, v, z, v, w)$ or, in terms of edges of T, the walk $W_1 = (a, b, b, c, c, d, d, a)$. While every edge of T occurs exactly twice on W_1, the 3-path $(w, v, x) = (a, b) = ab$ occurs once in W_1 but the 3-path $(w, v, y) = ac$ does not occur at all in W_1. On the other hand, the embedding of T shown in Figure 2(b) gives rise to the Hamiltonian walk $W_2 = (w, v, y, v, x, v, z, v, w) = (a, c, c, b, b, d, d, a)$, which contains the 3-path (w, v, y) but not the 3-path (w, v, x).

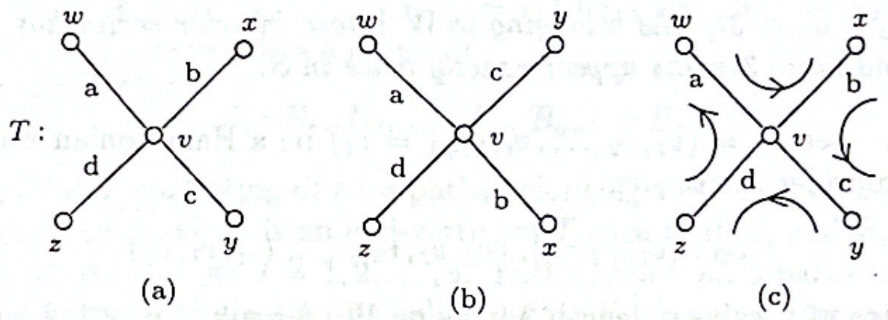

Figure 2: Two embeddings of $K_{1,4}$ in the plane

2 A More General Result

With the aid of spanning trees and Hamiltonian walks in graphs, we can not only extend Theorem 1.3 but apply this technique to establish sufficient conditions for the 3-path graph of a connected graph to possess stronger Hamiltonian properties. First, we present a few lemmas.

Lemma 2.1 *Let $\{f_1, f_2, \ldots, f_k\}$ be the edge set of a star F of size $k \geq 2$ in a connected graph. For $\ell = \binom{k}{2}$, there is a sequence $s : H_1, H_2, \ldots, H_\ell$ of*

3-paths in F consisting of ℓ distinct ordered pairs $f_i f_j$ where $1 \le i \ne j \le k$ such that

(i) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(ii) $H_1 = f_1 f_i$ and $H_\ell = f_j f_1$ for some integers $i, j \ge 2$ where $i \ne j$

Lemma 2.2 *Let $\{f_1, f_2, \ldots, f_k\}$ be the edge set of a star F of size $k \ge 3$ in a connected graph. For $\ell = \binom{k}{2} - 1$, there is a sequence $H_1, H_2, \ldots, H_\ell$ of 3-paths in F consisting of ℓ ordered pairs $f_i f_j$ where $1 \le i \ne j \le k$ and $\{i, j\} \ne \{1, k\}$ such that*

(i) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, k - 1$ and

(ii) $H_1 = f_1 f_i$ and $H_\ell = f_j f_k$ for integers i and j with $i \ne 2$ and $j \ne 1$.

Lemma 2.3 *Let G be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let T be a spanning tree of G, where W is a Hamiltonian walk of T. There exists a sequence $\mathcal{S}$: $A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ of 3-paths of G consisting of all 3-paths belonging to W and all 3-paths of G whose interior vertex is an end-vertex of T such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p$. Furthermore, all 3-paths of $\mathcal{S}$ are distinct except for those 3-paths belonging to W whose interior vertex has degree 2 in T and those 3-paths appear exactly twice in $\mathcal{S}$.*

Proof. Let $W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1)$ be a Hamiltonian walk of T. First, consider the sequence

$$\mathcal{S}_0 : (v_1, v_2, v_3), (v_2, v_3, v_4), \ldots, (v_t, v_1, v_2)$$

consisting of t walks of length 2 lying on W. A walk of length 2 on W is a 3-path on W if its interior vertex is not an end-vertex of T. Furthermore, a 3-path on W appears exactly once in $\mathcal{S}_0$ if its interior vertex has degree at least 3 and a 3-path on W appears exactly twice in $\mathcal{S}_0$ if its interior vertex has degree 2. For each end-vertex v_j of T, the walk (v_{j-1}, v_j, v_{j+1}) belongs to $\mathcal{S}_0$ and so $v_{j-1} = v_{j+1}$. Thus, (v_{j-2}, v_{j-1}, v_j), (v_{j-1}, v_j, v_{j+1}), (v_j, v_{j+1}, v_{j+2}) are three consecutive terms in $\mathcal{S}_0$, where $v_{j-1} = v_{j+1}$. Let $x = v_{j-2}v_{j-1}$, $a = v_{j-1}v_j = v_j v_{j+1}$ and $y = v_{j+1}v_{j+2}$. Then xa, aa, ay are three consecutive terms in $\mathcal{S}_0$ and aa appears exactly once in $\mathcal{S}_0$.

We claim that $\deg_G v_j \ne 2$; for otherwise, the two edges incident with v_j in G are bridges of G and hence they must belong to T. However then, $\deg_T v_j = 2$, which is a contradiction. Thus, either $\deg_G v_j = 1$ or $\deg_G v_j = k \ge 3$. If $\deg_G v_j = 1$, then we delete aa from $\mathcal{S}_0$. If $\deg_G v_j = k \ge 3$, then let a=e_1, e_2, e_3, $\ldots$, e_{k-1}, e_k be the k distinct edges incident with v_j in G where then only $a = e_1$ belongs to T. There are $\binom{k}{2} = \ell$

3-paths of G with interior vertex v_j that do not belong to $\mathcal{S}_0$. Each of these 3-paths has the form $e_r e_s$ where $r, s \in \{1, 2, \ldots, k\}$ and $r \neq s$. By Lemma 2.1, there is an ordering $H_1, H_2, \ldots, H_\ell$ of these ℓ 3-paths such that $H_1 = e_1 e_i$ and $H_\ell = e_1 e_j$, where $i, j \geq 2$ and $i \neq j$. We replace $aa = e_1 e_1$ in $\mathcal{S}_0$ by $H_1, H_2, \ldots, H_\ell$ or, equivalently, replace xe_1, $e_1 e_1$, $e_1 y$ in $\mathcal{S}_0$ by xe_1, $H_1, H_2, \ldots, H_\ell$, $e_1 y$. Applying this procedure to every end-vertex of T, we obtain a sequence $\mathcal{S}$ with the desired property. ■

Lemma 2.4 *Let G be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let T be a spanning tree of G containing vertices of degree 2, where W is a Hamiltonian walk of T. There exists a sequence S: A_1, A_2, ..., A_p, $A_{p+1} = A_1$ of distinct 3-paths of G consisting of*

(i) *all 3-paths belonging to W,*

(ii) *all 3-paths of G whose interior vertex has degree 1 or 2 in T and*

(iii) *all 3-paths of G whose interior vertex is a neighbor of a vertex of degree 2 in G such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p$.*

Proof. Let $W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1)$ be a Hamiltonian walk of T. By Lemma 2.3, there exists a sequence

$$\mathcal{S}_1 : B_1, B_2, \ldots, B_q, B_{q+1} = B_1$$

of 3-paths of G consisting of all 3-paths belonging to W and all 3-paths of G whose interior vertex is an end-vertex of T such that B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, q$. Furthermore, all 3-paths of $\mathcal{S}_1$ are distinct except for those 3-paths belonging to W whose interior vertex has degree 2 in T and those 3-paths appear exactly twice in $\mathcal{S}$.

Let v be a vertex of degree 2 in T and let a and b denote the two edges incident with v in T. Since v appears exactly twice on W, the 3-path ab appears exactly twice in $\mathcal{S}_1$. Suppose that v appears first as v_j in W and so the 3-path $ab = (v_{j-1}, v_j, v_{j+1})$ appears first in $\mathcal{S}_1$ where $a = v_{j-1}v_j$ and $b = v_j v_{j+1}$. Let $x = v_{j-2}v_{j-1}$ and $y = v_{j+1}v_{j+2}$. Thus, xa, ab, by are three consecutive terms in $\mathcal{S}_1$. We consider two cases according to whether $\deg_G v_j = 2$ or $\deg_G v_j = k \geq 3$.

Case 1. $\deg_G v_j = 2$. Then v_j is adjacent to an end-vertex and a vertex of degree at least 3 in G. We may assume, without loss of generality, that v_{j+1} is an end vertex of G and v_{j-1} is a vertex of degree at least 3 in G. There are two subcases, according to whether $\deg_T v_{j-1} = 2$ or $\deg_T v_{j-1} \geq 3$.

Subcase 1.1. $\deg_T v_{j-1} = 2$. Since $\deg_G v_{j-1} \geq 3$, there are edges $f_1, f_2, \ldots, f_t$ of G that are incident with v_{j-1} but not in T, where $t = \deg_G v_{j-1} - 2 \geq 1$. Then S_1 contains

$$zx, \underline{xa, ab}, ba, ax \tag{1}$$

as consecutive terms for some edge z of T. [Note that the walk bb of length 2 in W was deleted from the sequence in the proof of Lemma 2.3.] Since $\deg_T v_{j-1} = 2$ and $\deg_G v_{j-1} = t+2$, there are $\binom{t+2}{2} - 1 = \ell$ 3-paths of G with interior vertex v_{j-1} that do not belong to S_1. Applying Lemma 2.2 to the set $\{x, f_1, \ldots, f_t, a\}$, we obtain an ordering

$$s_1 : H_1, H_2, \ldots, H_\ell$$

of distinct 3-paths such that $H_1 = xf_1$ and $H_\ell = af_t$. Now, we replace the consecutive terms xa, ab in (1) by the sequence s_1, resulting in a new sequence of 3-paths that contains the consecutive terms

$$zx, H_1 = xf_1, H_2, \ldots, H_\ell = af_t, ba, ax.$$

Subcase 1.2. $\deg_T v_{j-1} \geq 3$. Then S_1 contains

$$zx, xa, \underline{ab}, ba, ay \tag{2}$$

as consecutive terms for some edges y and z of T, where $x \neq y$. If every edge incident with v_{j-1} belongs to T, that is, $\deg_G v_{j-1} = \deg_T v_{j-1}$, then every 3-path whose interior vertex is v_{j-1} belongs to S_1. Thus, we may assume that there are edges of G incident with v_{j-1} that do not belong to T. Let $f_1, f_2, \ldots, f_t$ be the edges of G incident with v_{j-1} that do not belong to T where $t = \deg_G v_{j-1} - \deg_T v_{j-1} \geq 1$ and let $x, a, e_1, e_2, \ldots, e_k$ be the edges of G incident with v_{j-1} that belong to T, where $k = \deg_T v_{j-1} - 2 \geq 1$. We may assume, without loss of generality, that T is embedded in the plane so that the edges $x, a, e_1, e_2, \ldots, e_k$ of T incident with v_{j-1} appear in this order in W (not necessarily as consecutive terms). Hence, for every integer i with $1 \leq i \leq d$, each edge e_i appears in the consecutive terms $x_i e_i, e_i e_{i+1}$ for some edge x_i of T in S_1. Let $\ell = \binom{t+2}{2} - 1$. There are $\binom{t+2}{2} - 1 + kt = \ell + kt$ 3-paths of G with interior vertex v_{j-1} that do not belong to S_1. Next, we add these 3-paths to S_1 and delete the 3-path ab in (2) as follows:

- ⋆ Applying Lemma 2.2 to the set $\{x, f_1, \ldots, f_t, a\}$ gives rise to an ordering s_0: H_1, H_2, …, H_ℓ of distinct 3-paths of G such that $H_1 = xf_1$ and $H_\ell = af_t$. We replace ab in (2) by the sequence s_0.
- ⋆ For $1 \leq i \leq k$, let s_i : $e_i f_1, e_i f_2, \ldots, e_i f_t$. We insert s_i between the two consecutive terms $x_i e_i$ and $e_i e_{i+1}$ in S_1.

This produces a new sequence consisting of all 3-paths in $\mathcal{S}_1$ and all distinct 3-paths of G whose interior vertex is v_{j-1}.

Case 2. $\deg_G v_j = k \geq 3$. Let a=e_1, e_2, e_3, ..., e_{k-1}, $e_k = b$ be the k distinct edges incident with v_j in G where then only $a = e_1$ and $b = e_k$ belong to T. There are $\binom{k}{2} - 1 = \ell$ 3-paths of G with interior vertex v_j that do not belong to $\mathcal{S}_1$. Each of these 3-paths has the form $e_r e_s$ where $r, s \in \{1, 2, \ldots, k\}$ and $\{r, s\} \neq \{1, k\}$. By Lemma 2.2, there is an ordering $H_1, H_2, \ldots, H_\ell$ of these ℓ 3-paths such that $H_1 = e_1 e_r$ and $H_\ell = e_s e_k$, where $r \neq k$ and $s \neq 1$. We replace $ab = e_1 e_k$ in $\mathcal{S}_1$ by $H_1, H_2, \ldots, H_\ell$, (or, equivalently, replace xe_1, $e_1 e_k$, $e_k y$ in $\mathcal{S}_1$ by xe_1, $H_1, H_2, \ldots, H_\ell$, $e_k y$).

Applying this procedure to every vertex of degree 2 in T, we obtain a sequence $\mathcal{S}$ with the desired property. ■

Lemma 2.5 *Let G be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let T be a spanning tree of G containing vertices of degree 3 and let W be a Hamiltonian walk of T. There exists a sequence*

$$S\text{: } A_1,\ A_2,\ \ldots,\ A_p,\ A_{p+1} = A_1$$

of distinct 3-paths of G consisting of

(i) all 3-paths belonging to W,

(ii) all 3-paths of G whose interior vertex has degree 1, 2 or 3 in T and

(iii) all 3-paths of G whose interior vertex is a neighbor of a vertex of degree 2 in G such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p$.

Proof. Let $W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1)$ be a Hamiltonian walk of T. By Lemma 2.4, there exists a sequence $\mathcal{S}_1$: B_1, B_2, ..., B_q, $B_{q+1} = B_1$ of distinct 3-paths of G consisting of (i) all 3-paths belonging to W, (ii) all 3-paths of G whose interior vertex has degree 1 or 2 in T and (iii) all 3-paths of G whose interior vertex is a neighbor of a vertex of degree 2 in G such that such that B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, q$.

Let v be a vertex of degree 3 in T. If v is a neighbor of a vertex of degree 2 in G, then we needn't do anything. Thus, we may assume that v is not a neighbor of a vertex of degree 2 in G. Let e_1, e_2 and e_3 denote the three edges incident with v in T. We may assume that the sequence $\mathcal{S}_1$ contains

(1) three consecutive terms ae_1, e_1e_2, e_2b,

(2) three consecutive terms ce_2, e_2e_3, e_3d and

(3) three consecutive terms fe_3, e_3e_1, e_1g.

If every edge incident with v belongs to T, then we needn't do anything. On the other hand, suppose that there are $k \geq 1$ edges incident with v that do not belong to T, say $f_1, f_2, \ldots, f_k$. Applying Lemma 2.1 to the set $\{e_1, f_1, \ldots, f_k\}$, there is a sequence

$$s_1 : H_1, H_2, \ldots, H_\ell$$

consisting of $\ell = \binom{k+1}{2}$ distinct ordered pairs xy where $x, y \in \{e_1, f_1, \ldots, f_k\}$ such that

(*i*) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(*ii*) $H_1 = e_1f_1$ and $H_\ell = f_ke_1$

Next, we

⋆ insert s_1 between ae_1 and e_1e_2 in $\mathcal{S}_1$,

⋆ insert $e_2f_1, e_2f_2, \ldots, e_2f_k$ between e_1e_2 and e_2b and

⋆ insert $e_3f_1, e_3f_2, \ldots, e_3f_k$ between e_2e_3 and e_3d.

Applying this procedure to every vertex of degree 3 in T, we obtain a sequence $\mathcal{S}$ with the desired property. ∎

We are now prepared to prove the following theorem.

Theorem 2.6 *If G is a connected of order at least 4 such that each vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3, then $\mathcal{P}_3(G)$ is Hamiltonian.*

Proof. It suffices to show that there exists an ordering

$$\mathcal{S}\colon A_1, A_2, \ldots, A_p, A_{p+1} = A_1$$

of all 3-paths A_i $(1 \leq i \leq p)$ of G such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p$. We verify this by showing the following statement.

For every spanning tree T of G and a Hamiltonian walk W of T, there exists a sequence $\mathcal{S}$: $A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ of distinct 3-paths of G consisting of (i) all 3-paths belonging to W (ii) all 3-paths of G whose interior vertex has degree k or less for every integer k with $k = 3, 4, \ldots, \Delta(T)$ in T and (iii) all 3-paths of G whose interior vertex is a neighbor of a vertex of degree 2 in G such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p$.

We verify this statement by induction on $k \geq 3$. Let T be a spanning tree of G. By Lemma 2.5, the result is true for $k = 3$. Therefore, if $\Delta(T) = 2$ or $\Delta(T) = 3$, then the result follows. Hence, we may assume that $\Delta(T) \geq 4$. Assume, for an integer k with $3 \leq k < \Delta(T)$, that there exists a sequence

$$\mathcal{S}_1 : B_1,\ B_2,\ \ldots,\ B_q,\ B_{q+1} = B_1$$

of distinct 3-paths of G consisting of (i) all 3-paths belonging to W, (ii) all 3-paths of G whose interior vertex whose interior vertex has degree 1, $2, \ldots, k$ in T and (iii) all 3-paths of G whose interior vertex is a neighbor of a vertex of degree 2 in G such that B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, q$. If T has no vertex of degree $k+1$ in T, then the result follows. Hence, we may assume that T contains one or more vertices of degree $k+1 \geq 4$ in T.

Let v be a vertex of degree $k+1 \geq 4$ in T. If v is a neighbor of a vertex of degree 2 in G, then we needn't do anything. Thus, we may assume that v is not a neighbor of a vertex of degree 2 in G. Let $e_1, e_2, \ldots, e_{k+1}$ denote the $k+1$ edges incident with v in T. We may assume, by relabeling $e_1, e_2, \ldots, e_{k+1}$ if necessary, that

(i) $\mathcal{S}_1$ contains the 3-paths $e_i e_{i+1}$ for $1 \leq i \leq k$ and $e_{k+1}e_1$ and

(ii) these 3-paths appear in $\mathcal{S}_1$ in the order $e_1e_2, e_2e_3, \ldots, e_ke_{k+1}, e_{k+1}e_1$ not necessarily consecutive terms in $\mathcal{S}_1$.

Hence, there are 3-paths having interior vertex v do not belong to $\mathcal{S}_1$. Let X be the set of all such 3-paths and so

$$|X| = \binom{k+1}{2} - (k+1) \geq 2.$$

Divide X into $k-1$ subsets $X_1, X_2, \ldots, X_{k-1}$ where

$$X_1 = \{e_1e_s :\ 3 \leq s \leq k\} \text{ and } X_r = \{e_re_s \in X :\ r+2 \leq s \leq k+1\}$$

for $2 \leq r \leq k-1$. In particular, $X_1 = \{e_1e_3, e_1e_4, \ldots, e_1e_k\}$, $X_2 = \{e_2e_4, e_2e_5, \ldots, e_2e_k, e_2e_{k+1}\}$ and $X_{k-1} = \{e_{k-1}e_{k+1}\}$. Next, let s_r be an ordering of vertices of X_r for $1 \leq r \leq k-1$. The sequence $\mathcal{S}_1$ contains the following $k+1$ three consecutive terms

$$a_re_r, e_re_{r+1}, e_{r+1}b_r \text{ for } 1 \leq r \leq k \text{ and } a_{k+1}e_{k+1}, e_{k+1}e_1, e_1b_{k+1}.$$

for some edges a_r, b_r, a_{k+1} and b_{k+1} of T. If every edge incident with v belongs to T, then we insert s_r between a_re_r and e_re_{r+1} for $1 \leq r \leq k$. On the other hand, suppose that there are $d \geq 1$ edges incident with v that do not belong to T, say $f_1, f_2, \ldots, f_d$. Applying Lemma 2.1 to the set $\{e_1, f_1, \ldots, f_d\}$, there is a sequence

$$s_0 : H_1, H_2, \ldots, H_\ell$$

consisting of $\ell = \binom{d+1}{2}$ distinct ordered pairs xy where $x, y \in \{e_1, f_1, \ldots, f_d\}$ such that

(*i*) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(*ii*) $H_1 = e_1 f_i$ and $H_\ell = f_j e_1$ where $1 \le i \ne j \le d$ and $H_1 = H_\ell$ if $d = 1$.

Next, we insert s_1, s_0 between $a_1 e_1$ and $e_1 e_2$ in $\mathcal{S}_1$ and insert s_r between $a_r e_r$ and $e_r e_{r+1}$ for $2 \le r \le k-1$. Applying this procedure to every vertex of degree $k+1$ in T, we obtain a sequence $\mathcal{S}$ with the desired property. ■

Theorem 2.6 is best possible in the sense that if a connected graph G contains even one vertex of degree 2 that does not satisfy the conditions in Theorem 2.6, then $\mathcal{P}_3(G)$ need not be Hamiltonian. For example, the graph G of Figure 3 has exactly one vertex of degree 2, each of whose neighbors have degree 3. Since the 3-path ab of G is a cut-vertex of $\mathcal{P}_3(G)$, it follows that $\mathcal{P}_3(G)$ is not Hamiltonian. If H is the graph obtained from the graph G of Figure 3 by subdividing the edge a exactly once, then H contains a vertex of degree 2 that is adjacent to another vertex of degree 2 and a vertex of degree 3 in H and so H contains a path (u, v, w, x) where v and w have degree 2 in H and u and x have degree 3. In this case, $\mathcal{P}_3(H)$ contains a bridge joining the two 3-paths (u, v, w) and (v, w, x) and so $\mathcal{P}_3(H)$ is not Hamiltonian. Furthermore, if F is a connected graph containing a vertex v of degree 2 such that v is adjacent to an end-vertex u and another vertex w of degree 2, the the 3-path (u, v, w) is an end-vertex in $\mathcal{P}_3(F)$ and so $\mathcal{P}_3(F)$ is not Hamiltonian.

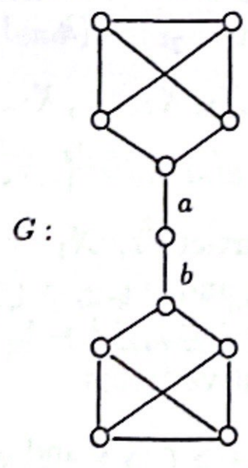

Figure 3: Showing Theorem 2.6 is best possible

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