The number of minimal prime ideals in a k-integral domain and an algorithm for constructing maximal k-zero divisors in Z_n

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Abstract

Let R be a commutative ring with identity. For any integer $k > 1$, an element is a k-zero divisor if there are k distinct elements including the given one, such that the product of all is zero but the product of fewer than all is nonzero. Let $Z(R, k)$ denote the set of the k-zero divisors of R. A ring with no k-zero divisors is called a k-domain. In this paper we define the hyper-graphic constant $HG(R)$ and study some basic properties of k-domains. Our main result is theorem 5.1 which is as follows:

Let R be a commutative ring such that the total ring of fraction $T(R)$ is zero dimensional. If R is a k-domain for $k \geq 2$, then R has finitely many minimal prime ideals.

Using this result and lemma 5.4, we improve a finiteness theorem proved in **[11]** to a more robust theorem 5.5 which says:

Suppose R is not a k-domain and has more than k minimal prime ideals. Further, suppose that $T(R)$ is a zero dimensional ring. Then $Z(R, k)$ is finite if and only if R is finite.

We end this paper with a proof of an algorithm describing the maximal k-zero divisor hyper-graphs on $\mathbb{Z}_n$.

Keywords: Hyper-graph, commutative rings, ideals, k-zero divisors, total ring of fractions.

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1 Introduction

A simple graph is an ordered pair (V, E), where V is a vertex set and E is an edge set with edges of the form $\{v_1, v_2\}$ where v_1, v_2 are two distinct vertices. A zero divisor graph on a commutative ring R is a simple graph $\Gamma(R)$ whose vertex set is the set of zero divisors $Z(R) = \{a \in R|$ there exists $b \neq 0$ with $ab = 0\}$. Two distinct zero divisors form an edge if $a \cdot b = 0$. The study of a zero divisor graph on a commutative ring was first introduced in 1988 by Beck in **[5]**. Anderson and Livingston **[2]** modified this definition by removing zero from the vertex set. We give an example of a zero divisor graph that follows Anderson and Livingston.

Example 1.1. Suppose $R = \mathbb{Z}_{12}$. The vertex set is

$$Z(R)^* = \{2, 3, 4, 6, 8, 9, 10\}$$

and $\Gamma(R)$ is given below, where the edges are represented by lines joining pairs of vertices.

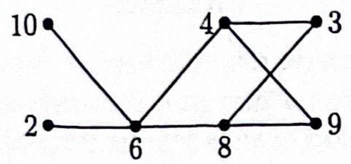

Figure 1: Zero divisor graph on $\mathbb{Z}_{12}$.

More examples and classifications can be found in **[2, 3, 7]** and the references cited there. It is clear that the graph is finite if $\Gamma(R)$ has finitely many vertices. Ganesan proved in **[9]** that for a ring R which is not an integral domain, $\Gamma(R)$ is a finite graph if and only if R is a finite ring.

A hyper-graph is a natural generalization of simple graph. As in any graph, a hyper-graph is also a pair (V, E) with V a vertex set and E an edge set. But edges of a hyper-graph can contain more than two vertices (see Berge **[6]** for detailed definition). If all edges have an equal number of elements, say k, then the hyper-graph is called a **k-uniform** hyper-graph. Formally, a k-uniform hyper-graph H is an ordered pair (V, E) where V is a vertex set and E is a set of k-element subsets of V called edges. In 2007, Eslahchi and Rahimi in **[8]** used the graphs to hyper-graphs relationship to generalize the concept of zero divisors and introduced k-zero divisors. This enables us to associate a k uniform hyper graph $H_k(R)$ to a commutative ring R for every k. Example 2.1 shows the k-uniform hypergraph $H_2(\mathbb{Z}_{12})$. In the same paper, Eslahchi and Rahimi posed a finiteness question which was partially answered by the author of this paper in **[11]**. In this paper, we give fuller answer to this question by counting the number of minimal prime ideals of a ring which has no k-zero divisors. In section 2 we record definitions and introduce the notations used in this paper. Section 3 contains some elementary results on reduced rings. Section 4 studies the effect of localization and quotient modulo the nilradical on k-zero divisors. In section 5, we prove the main theorem and derive a more robust form of the finiteness theorem from **[11]**. In section 6 we design an algorithm to describe maximal k-zero divisor hyper-graphs of $\mathbb{Z}_n$.

2 Definitions and notations

All rings considered in this paper are commutative with identity. An element a is called **regular** if $a \cdot b = 0$ is true only when $b = 0$. The set of regular elements, denoted by R^o, is closed under multiplication. Hence R^o forms a multiplicative set and localization with respect to this set results in a total ring of fractions (see exercise 9 on page 44 in **[4]**). We will denote the total ring of fractions of R by $T(R)$. Elements of $T(R)$ are of the form a/r where $r \in R^o$. By identifying element a of R with $a/1$ in $T(R)$, we can view R as a subring of $T(R)$.

Non regular elements are called zero-divisors. Thus, a is a zero divisor if there is a nonzero element b such that $ab = 0$. We will denote the set of zero divisors by $Z(R)$. An element a of R is called a **nilpotent** element if $a^m = 0$ for some positive integer m. If a is nilpotent, then $1 + a$ is a unit (see exercise 1, p.10 in **[4]**). We define the **nilpotent degree** $d_{nil}(a)$ of a nilpotent element a to be the smallest positive integer n for which $a^n = 0$. The **Nilradical** of R, denoted by N is the set of all the nilpotent elements of R. If $N = \{0\}$ then R is called a **reduced** ring. We recall the following definition from [8].

Definition 2.1. Let R be a commutative ring and $k \geq 2$ be a fixed integer. Element $v_1 \in R$ is called a **k-zero divisor** if there exist $v_2, v_3, \ldots, v_k$ in R such that (1) $\{v_1, v_2, \ldots, v_k\}$ are all distinct elements, (2) $\Pi_1^k v_i = 0$ and (3) $\Pi_{i \neq j} v_i \neq 0$ for any j, $1 \leq j \leq k$.

The set of all k-zero divisors is denoted by $Z(R, k)$. Clearly $Z(R, k) \subset Z(R)$. A ring in which $Z(R, k)$ is a null set is called a **k-integral domain** or just a **k-domain**.

Condition (3) implies that for each i, v_i is a nonzero element which is not regular. Further, if $\{v'_1, v'_2, \ldots, v'_r\}$ is a proper subset of $\{v_1, v_2, \ldots, v_k\}$, then $\Pi_{i=1}^r v'_i \neq 0$. In fact, (3) is equivalent to the statement that the product of fewer than all v_i is nonzero.

We define a k uniform hyper-graph $H_k(R)$ on R as follows. The vertex set is $Z(R, k)$. Elements $v_1, v_2, \ldots, v_k$ which appear in definition 2.1(2) form a **k-edge** (edge with k number of vertices) of the hyper-graph. We also note that a k-domain is a ring without k-zero divisors. Since k-zero divisors come in groups of k, we may say that a k-domain is a ring for which $H_k(R)$ is a null graph.

We give some examples.

Example 2.1. We consider the ring $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ with operations addition and multiplication modulo 12. The graph $H_3(\mathbb{Z}_{12})$ has the vertex set $Z(\mathbb{Z}_{12}, 3) = \{2, 3, 9, 10\}$. Notice that 6 is not a vertex (because if $6ab = 0$, then $2|a$ or $2|b$ resulting in either $6a = 0$ or $6b = 0$). There are two 3-edges, $\{2, 3, 10\}$ and $\{2, 9, 10\}$ which are represented by the two ovals in the following hyper-graph $H_3(\mathbb{Z}_{12})$: Note that $\mathbb{Z}_{12}$ is a 4- domain.

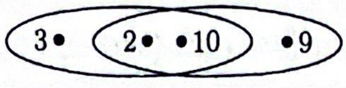

Figure 2: Three uniform hyper-graph on $\mathbb{Z}_{12}$.

Example 2.2. $H_3(\mathbb{Z}_{18})$ has vertex set

$$Z(\mathbb{Z}_{18}, 3) = \{2, 3, 4, 8, 10, 14, 15, 16\}$$

and six edges, $\{2,3,15\}$,$\{4,3,15\}$,$\{8,3,15\}$, $\{10,3,15\}$,$\{14,3,15\}$ and $\{16,3,15\}$, as shown in Figure 3.

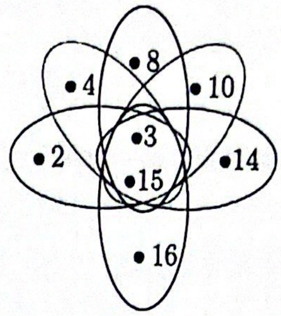

Figure 3: Three uniform hyper-graph $H_3(\mathbb{Z}_{18})$.

Diagram of this type are particularly hard to draw when there is a large number of edges. In such cases, we will describe our hyper-graph by listing vertices and counting the number of edges.

Example 2.3. The vertex set of $H_3(\mathbb{Z}_{27})$ is

$$Z(\mathbb{Z}_{27}, 3) = \{3, 6, 12, 15, 21, 24\}.$$

Any three elements of the vertex set form an edge (i.e., $Z(\mathbb{Z}_{27}, 3)$ is a *clique*). Hence, the number of edges is $\binom{6}{3} = 20$.

Definition 2.2. For any ring R, we define a **hyper-graphic constant** $HG(R)$ to be the smallest integer g such that R is a k-domain for all $k > g$. For any integral domain, we define $HG(R) = 0$. If R is not a domain but is a k-domain for all $k \geq 2$, then we say that $HG(R) = 1$. If no such integer exists, then we say that $HG(R) = \infty$.

By the definition, R is not a g-domain, that is, $Z(R, g) \neq \emptyset$. If $HG(R) = g < \infty$, then the hyper-graph $H_g(R)$ will be referred to as the **maximal** hyper-graph of R .

Example 2.4. Since $\mathbb{Z}$ is a domain, $HG(\mathbb{Z}) = 0$. $HG(\mathbb{Z}_4) = 1$ and $HG(\mathbb{Z}_{12}) = 3$.

n	$HG(\mathbb{Z}_n) = g$	$Z(\mathbb{Z}_n, g)$	Edges
$n \leq 5$	0 or 1	$\emptyset$	0
6	2	$\{2, 3, 4\}$	2
7	0	$\emptyset$	0
8	2	$\{2, 4, 6\}$	2
9	2	$\{3, 6\}$	1
10	2	$\{2,4,6,8,5\}$	4
11	0	$\emptyset$	0
12	3	$\{2,3,9,10\}$	2

Table 1: Maximal Hyper-graphs on $\mathbb{Z}_n$ for $n \leq 12$.

Maximal hyper-graphs of $\mathbb{Z}_n$ for $n \leq 12$ are described in Table 1.

Example 2.5. Let K be a field and $K[x] = K[x_1, x_2, \dots,]$ be a polynomial ring in infinitely many variables. Let I be the ideal generated by the monomials x_i^i. Then elements of I are finite sum of monomials $\Pi_1^n x_i^{\alpha_i}$ where $\alpha_i = 0$ or $\alpha_1 \geq i$. Now for any integers t, n, $1 \leq t < n$, if $x_n^t \in I$, then

$$x_n^t = \sum_j \Pi_i x_i^{\alpha_{ij}}.$$

Since for all j, $\alpha_{nj} \neq t$, the above equation gives us a nontrivial algebraic relation among polynomial variables, which is absurd. Therefore, x_n^t is not in I. We now consider the following quotient ring

$$A = K[\overline{x}_1, \overline{x}_2, \dots,] = K[x_1, x_2, \dots,]/I$$

where for each i, $\overline{x}_i$ is congruent to x_i modulo I. Ring A has only one minimal prime ideal. To see this, let P be a minimal prime ideal in A. Since $\overline{x}_i$ are nilpotent elements, P contains $\overline{x}_i$ for all i. Note that the ideal generated by $\{x_i\}$ is a maximal ideal in $K[x]$ and the maximal ideals which contain I in $K[x]$ are mapped onto maximal ideals in A (see **[4]**, Proposition 1.1 and a comment towards the end of page 9). Therefore, the ideal generated by $\{\overline{x}_i\}$ is a maximal ideal in A. Since $\{\overline{x}_i\} \subset P$, P must be equal to the ideal generated by $\{\overline{x}_i\}$. Thus A has only one minimal (or any kind of) prime ideal. Now let $k \geq 2$. Set $n = \frac{k(k+1)}{2}$ and consider $\Lambda = \{\overline{x}_n, \overline{x}_n^2, \dots, \overline{x}_n^k\}$.

We claim that Λ is a k-edge. First we observe that $\Pi_1^k(\overline{x}_n^i) = \overline{x}_n^n = 0$. Next, for any $j \geq 1$, since $\sum_{i \neq j} i = \frac{k(k+1)}{2} - j = t < \frac{k(k+1)}{2} = n$, $\Pi_{i \neq j}\overline{x}_n^i = \overline{x}_n^t \neq 0$. Finally, we note that $\overline{x}_n$ is a nilpotent element in A. Therefore, $\forall i$, $i \geq 1$, , $1 - \overline{x}_n^i$ is a unit. Therefore, $\overline{x}_n^i \neq \overline{x}_n^j$ for any $1 \leq i < j \leq k$. This shows that Λ is a nontrivial k-edge Hence, A is not a k-domain, which shows that $HG(A) = \infty$. For the future reference, we record that the Krull dimension of the total ring of fraction $T(A)$ is zero.

Example 2.6. Suppose R is an integral domain. For any integer $n \geq 2$ let A be the direct product of n copies of R. Elements of A are ordered n-tuples $(a_1, \ldots, a_n)$ where a_i are in R. The zero element is the element for which $a_i = 0$ for all i. Now for each i, we define

$$P_i = \{(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n\}.$$

Since A/P_i is isomorphic to R which is a domain, P_i is a prime ideal. Further, $\cap P_j = \{0\}$ implies that P_i is a minimal prime ideal and A is a reduced ring. Thus A is a reduced ring with n minimal prime ideals. Applying the proposition 3.2, we conclude that $HG(A) = n$.

Example 2.7. Suppose $R[x] = R[x_1, \ldots, x_n]$ is a polynomial ring over any integral domain R. Consider the quotient ring $A = R[x]/I$ where I is the ideal generated by $\Pi_i x_i$. Note that I is the intersection of n prime ideals (x_i) generated by x_i in $R[x]$. Now prime ideals which contain I in $R[x]$ are mapped onto prime ideals in A. If $\overline{x}_i$ is congruent to x_i modulo I, then $(\overline{x}_i)$ is a prime ideal for each i. Moreover, $\cap(\overline{x}_j) = \{0\}$. Now by proposition 1.10 (iii) on page 17 in **[4]**, we can assume that A is a sub-ring of $\Pi_j A/(\overline{x}_j)$. By our previous example, $\Pi_j A/(x_j)$ is a k-domain for any $k > n$. Therefore, A is also a k-domain for any $k > n$. But clearly $\{\overline{x}_1, \ldots, \overline{x}_n\}$ is an n -edge. Therefore, $HG(A) = n$.

Remark: In theorem 5.1, we prove that k domains have fewer than k minimal prime ideals. As we see in example 2.5, finiteness of minimal prime ideals is not sufficient for a ring to be a k domain. Only example we have of a ring which fails to be a k domain for any of k and which has a finitely many minimal prime ideals is a non-noetherian ring (nilradical is not finitely generated). We make a following conjecture:

Conjecture: Suppose R is a commutative ing with finitely many minimal prime ideals. Also assume that the nil-radical of R is a milpotent ideal. Then R is a k domain for some k. In particular, a noetherian ring is a k domain for some k.

$\mathbb{Z}_4$ provides an example of a ring which is not a domain but is a k-domain for all $k \geq 2$. Not many rings have this property, which we see from the following proposition.

Proposition 2.1. *Suppose a ring R is not a domain but is a 2-domain. Then R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

Proof. Since R is a 2-domain, it has no 2-edge. We use this to show that the square of any zero divisor is zero. Let $0 \neq x \in Z(R)$. Then there exists $y \neq 0$ such that $x \cdot y = 0$. If $x \neq y$, then $\{x, y\}$ is a 2-edge, which gives a contradiction. Therefore, $x = y$ and $x^2 = 0$.

Next we show that x is the only non-zero element in $Z(R)$. Suppose $y \neq 0$ is an element in $Z(R)$ different from x. If $x \cdot y = 0$, then $\{x, y\}$ is a 2-edge, so $x \cdot y \neq 0$. Since $y^2 = 0$, $1 - y$ is a unit. Therefore, $x \neq x \cdot y$ and $\{x, x \cdot y\}$ is a 2-edge, which is a contradiction. Therefore, no such y exist i.e. $Z(R) = \{0, x\}$. By **[9]**, R is a finite ring.

Now if $char(R) = 2$, then $R = \mathbb{Z}_2[x]/(x^2)$. If $char(R) \neq 2$ then, since $x \neq 2x$ and $x \cdot 2x = 0$, $2x$ must be equal to 0. Hence $x = 2$ and $R = \mathbb{Z}_4$. □

The following example is given in **[8]** (see page 3 example 2.3 (1)).

Example 2.8. Let n be a positive integer and $n = \Pi_1^m p_i^{\alpha_i}$ be the prime factorization of n. For any $k > \sum \alpha_i$, we will show that $\mathbb{Z}_n$ is a k-domain. Suppose $c_1, \ldots, c_k \in \mathbb{Z}_n$ are such that $\Pi c_i = 0$. We choose $a_i \in \mathbb{Z}$ such that $a_i \equiv c_i \mod n$. Then $\Pi a_i \equiv \Pi c_i \mod n$. Since $\Pi c_i = 0$ in $\mathbb{Z}_n$, $\Pi a_i \equiv 0 \mod n$. Therefore, Πa_i is divisible by $n = \Pi p_i^{\alpha_i}$ in $\mathbb{Z}$. Canceling p_i one by one, we can find a subset $\{b_1, \ldots, b_r\} < \{a_1, \ldots, a_k\}$such that $n | \Pi b_i$. Now if we write $d_i \equiv b_i \mod n$, then $\{d_1, \ldots, d_r\} < \{c_1, \ldots, c_k\}$ and $\Pi d_i = 0$. This shows that $\{c_1, \ldots, c_k\}$ is not a k-edge. Thus for all $k > \sum \alpha_i$,

$\mathbb{Z}_n$ is a k-domain, which implies that $HG(\mathbb{Z}_n) \leq \sum \alpha_i$. It will be shown in section 6 that for $n > 8$, $HG(\mathbb{Z}_n) = \sum \alpha_i$.

3 Elementary results

First we record some results for reduced rings.

Proposition 3.1. *Let R be a reduced ring. Then*

$$Z(R, k+1) \subset Z(R, k).$$

Proof. If $Z(R, k+1) = \emptyset$, then the result is vacuously true. Suppose $v \in Z(R, k+1)$. Then there exists a $(k+1)$-edge $\Lambda = \{v_1, \ldots, v_{k+1}\}$ with $v = v_1$. Set

$$v_i' = \begin{cases} v_i & \text{for } 1 \leq i \leq k-1 \\ v_k v_{k+1} & \text{for } i = k \end{cases}$$

Then $\Pi_1^k v_i' = 0$ and $\Pi_{i \neq j} v_i' \neq 0$ for all j. Also since $v_1, v_2, \ldots, v_{k-1}$ are vertices in a $(k+1)$-edge Λ, $v_i' \neq v_j'$ for $1 \leq i < j \leq k-1$. If for some $j < k$, $v_j' = v_k'$, then

$$(\Pi_{i \neq j} v_i')^2 = (\Pi_{i \neq j,k} v_i')(\Pi_i v_i') = 0.$$

Since R is a reduced ring, $\Pi_{i \neq j} v_i' = 0$ which is a contradiction because Λ is a $(k+1)$-edge. Therefore, $\{v_1', \ldots, v_k'\}$ is a k-edge and $v = v_1' \in Z(R, k)$. This shows that $Z(R, k+1) \subset Z(R, k)$. □

By induction, we can derive the following corollaries;

Corollary. *Let R be a reduced ring. If R is a k- domain then R is a t-domain for any $t \geq k$.*

Corollary. *Let R be a reduced ring. If R is a k- domain then $HG(R) < k$.*

The prime avoidance theorem (proposition 1.11 in **[4]**) states that for any prime ideal P and prime ideals Q_i, $1 \leq i \leq r$, $P \subset \cup Q_i$ if and only if $P \subset Q_i$ for some i. Thus, if none of the Q_i contain P, then P contains an element which is not in $\cup Q_i$.

Proposition 3.2. *If R is a reduced ring with a finite number of minimal prime ideals, then $HG(R)$ is equal to the number of minimal prime ideals.*

Proof. Suppose there are n minimal prime ideals $P_1, P_2, \ldots, P_n$ in R. By proposition 3.2 (1) in **[11]**, R is a k-domain for any $k > n$. We will produce an n-edge to complete the proof. Let v_i be an element of P_i which is not in P_j for any $j \neq i$ (prime avoidance). Clearly all v_i are distinct. Further, $\Pi v_i \in \cap P_i = N = \{0\}$ and since $\Pi_{i \neq j} v_i \notin P_j \, \forall j$, $\Pi_{i \neq j} v_i \neq 0$. Thus, $\{v_1, v_2, \ldots v_n\}$ is an n-edge. □

Ring R and it's polynomial ring $R[x]$ have equal number of minimal prime ideals. Also, if R is reduced, then so is $R[x]$. Hence we have the following corollary.

Corollary. *If R is a reduced ring with a finite number of minimal prime ideals, then $HG(R) = HG(R[x])$.*

In general, $HG(R)$ may not be equal to $HG(R[x])$. For example, $HG(\mathbb{Z}_4) = 1$ whereas $HG(\mathbb{Z}_4[x]) = 2$.

We now turn our attention to the nilradical N of a ring R. We say that N has a finite exponent e if $a^e = 0$ for all $a \in N$. This does not imply that $N^t = \{0\}$ for some t. But if N is finitely generated (and has a finite exponent), then we can find such a t. We will say that N is **nilpotent ideal** if we can find a positive integer t such that $N^t = \{0\}$. If R is a k-domain, then N has an exponent less than $k(k+1)/2$. We record a result from **[11]** which states a little more.

Lemma 3.3. *Let R be a ring with finitely many k-zero divisors. Then N has a finite exponent. Further, if N is finitely generated, then N is nilpotent ideal.*

Proof. Proof of the statement can be found in proposition 2.2 in **[11]**. □

Corollary. *Suppose $R[x]$ is a k-domain. Then the nilradical N of R has an exponent $r < k$.*

Proof. Suppose $a \in N_R$ is such that $d_{nil}(a) = t \geq k$. Define

$$v_i = \begin{cases} ax^{i-1} & \text{for } 1 \leq i \leq k-1 \\ a^{t-k+1}x^{k-1} & \text{for } i = k \end{cases}$$

We claim that $\Lambda = \{v_1, v_2, \ldots, v_k\}$ is a k-edge. Clearly

$$\Pi_{i=1}^{k} v_i = a^t x^{\frac{k(k-1)}{2}} = 0.$$

Now if $ax^i = ax^j$ for some $i < j$, then $x^i(a - ax^{j-i}) = 0$. Now x^i is not a zero divisor, therefore, $a = ax^{j-i}$ which is a contradiction. Finally, since $d_{nil}(a) = t$, for any s, $1 \leq s < t$, $a^s \neq 0$. Therefore, $\Pi_{i\neq j} v_i = a^{t-j} x^{\alpha} \neq 0$ where $\alpha = \frac{k(k-1)}{2} - j$. Thus, Λ is a k-edge, which is a contradiction as $R[x]$ is a k-domain. Therefore, for all $a \in N$, $d_{nil}(a) < k$. This completes the proof. □

4 Localization and quotient of a k-domain

Subrings inherit the hyper-graph substructure. That is, if A is a subring of B, then $Z(A, k) \subset Z(B, k)$. Further, any edge in $H_k(A)$ is an edge in $H_k(B)$. In this sense, $H_k(A)$ is a sub-hyper-graph of $H_k(B)$. Moreover, if B is a k-domain then so is A. Hence, $HG(A) \leq HG(B)$.

Proposition 4.1. *Suppose $S \subset R^o$ is a multiplicative set where R^o is the set of regular elements of a ring R. Then R is a k- domain if and only if $S^{-1}(R)$ is k-domain.*

Proof. If $S^{-1}(R)$ is a k-domain, then so is R since R is a subring of $S^{-1}(R)$. For the reverse implication, suppose $\{w_1, w_2, \ldots, w_k\}$ is a k-edge over $S^{-1}(R)$. Now we choose $s \in S$ such that $v_i = sw_i \in R$. Clearly all v_i are distinct and $\Pi v_i = 0$. Since R is a k-domain, $\Pi_{i\neq j} v_i = 0$ for some j which implies that $s^{k-1}(\Pi_{i\neq j} w_i) = 0$. Since s is a regular element, $\Pi_{i\neq j} w_i = 0$ which is a contradiction. Therefore, $S^{-1}(R)$ is a k-domain. □

Corollary. *Suppose $S \subset R^o$. Then $HG(R) = HG(S^{-1}(R))$. In particular, if $T(R)$ is a total ring of fractions, then $HG(R) = HG(T(R))$.*

Proposition 4.2. *If R is a k domain, then R/N is also a k domain.*

Proof. We prove by contradiction. Suppose that R is a k-domain but R/N is not. Suppose for $1 \leq i \leq k$, $v_i \in R$ are such that

$\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is a k-edge where $\bar{v}$ is the image of v under the canonical map $R \to R/N$. Now $\overline{\Pi v_i} = \Pi \bar{v}_i = 0$ implies that $\Pi v_i \in N$. On the other hand, $\overline{\Pi_{i \neq j} v_i} = \Pi_{i \neq j} \bar{v}_i \neq 0$ implies that $\Pi_{i \neq j} v_i \notin N$ for any j. Suppose $d_{nil}(\Pi v_i) = n$.

We claim that $\{v_1^n, v_2^n, \ldots, v_k^n\}$ forms an edge in $H_k(R)$. If for some $i \neq j$, $v_i^n = v_j^n$, then $(\Pi_{h \neq j} v_i)^{2n} = (\Pi_{h \neq j} v_i)^n (\Pi v_h)^n = 0$ contradicts the fact that $\Pi_{h \neq j} v_h \notin N$. Thus all the v_i^n are distinct. Further, since $\Pi_{i \neq j} v_i \notin N$, $(\Pi_{i \neq j} v_i)^n$ cannot be zero. This shows that $v_1^n, v_2^n, \ldots, v_k^n$ is a k-edge, a contradiction. □

The converse of the above proposition is not true.

Example 4.1. The nilradical of $R = \mathbb{Z}_{36}$ is $N = 6\mathbb{Z}_{36}$. Therefore, $R/N = \mathbb{Z}_6$. Now $\mathbb{Z}_6$ is a 3-domain because it is a reduced ring with two minimal prime ideals, but $\{2, 3, 6\}$ is a 3-edge, showing that $\mathbb{Z}_{36}$ is not a 3-domain.

An interesting consequence of the proposition 4.2 is the following corollary.

Corollary. *Let R be a ring with $HG(R) < \infty$. Then $HG(R/N) \leq HG(R)$.*

Proof. Suppose $HG(R) = g$. Then R is a $g + 1$-domain and so is R/N. Since R/N is a reduced ring, by the second corollary of the proposition 3.1, $HG(R/N) < g + 1$. □

Proposition 4.3. *Suppose R is a k domain with finitely generated nilradical, then $HG(R) < \infty$. Further if $N^t = 0$, then $HG(R) \leq t(k - 1)$.*

Proof. Since N is finitely generated, N is nilpotent ideal. So we can assume that $N^t = \{0\}$ for some t. Let k_1 be an integer and $k_1 > t(k - 1)$. We claim that R is a k_1-domain. Assume that R is not a k_1-domain and suppose $\Lambda = \{v_1, v_2, \ldots, v_{k_1}\}$ is a k_1-edge. Now for any

$$\{w_1, w_2, \ldots, w_{k-1}\} \subset \{v_1, v_2, \ldots, v_{k_1}\}.$$

set $w_k = \Pi_{v_i \neq w_j} v_i$ to be the product of all the $v_i \neq w_j$ for $1 \leq j \leq k - 1$. Clearly $\Pi w_i = 0$ and $\forall j$, $\Pi_{i \neq j} w_i \neq 0$. Since R is a k-domain, $w_i = w_j$ for some $i \neq j$. But $w_i, w_2, \ldots, w_{k-1}$ are pairwise distinct by the hypothesis. Therefore, $w_i = w_k$ for some i, $1 \leq i \leq k-1$. This shows that $(\Pi_{i=1}^{k-1} w_i)^2 = 0$. Thus the product of any $k-1$ elements of the set Λ is in N. Since $k_1 > t(k-1)$, $\Pi_{i=1}^{t(k-1)} v_i \in N^t = \{0\}$. Therefore, Λ is not a k_1-edge, or R is a k_1-domain which completes the proof. □

5 Main result

The Krull dimension of a ring is the length of a longest chain of prime ideals (starting from zero). Thus a zero dimensional ring has only minimal prime ideals. Kaplansky's theorem states that in a zero dimensional reduced ring, every principal ideal is an idempotent. That is for any element a which is not a unit, there exists y such that $a = a^2 y$ (Theorem 3.1 on page 5 in **[1]**). Thus $aR = (ay)R$ with $(ay)^2 = ay$. We will use this result in our main theorem.

Theorem 5.1. *Let R be a ring such that the total ring of fraction $T(R)$ is zero dimensional. If R is a k-domain, then R has finitely many minimal prime ideals. In particular, the number of minimal prime ideals is less than k.*

Proof. If $k = 2$, then by the proposition 2.1, R has only one minimal prime ideal. Thus we can assume that $k \geq 3$. Now if R is a k domain then by proposition 4.2 R/N is also a k-domain which has the same number of minimal prime ideals as R has. So we can assume that R is a reduced ring. Further, by proposition 4.1, if R is a k-domain then so is the total ring of fractions $T(R)$ and both R and $T(R)$ have the same number of minimal prime ideals. Thus we can assume that R is a zero dimensional reduced ring. We claim that R has less than k minimal prime ideals.

Suppose $P_1, P_2, \ldots, P_k$ are minimal prime ideals in R. For each i, choose an element $e_i \in P_i$ which is not in P_j for any $j \neq i$. By Kaplansky's theorem, we can assume that the e_i are idempotents. Now $\Pi_1^{k-1} e_i \cdot (1 - \Pi_1^{k-1} e_i) = 0$.

Observe that for any distinct i and j, $e_i \neq e_j$. Further, since $\Pi_1^{k-1} e_i \in P_j$ for $1 \leq j \leq k-1$, $e_j \neq 1 - \Pi_1^{k-1} e_i$ for any such j.

Thus, $\Lambda = \{e_1, e_2, \ldots, e_{k-1}, \Pi_1^{k-1} e_i\}$ is a set of distinct elements with zero product. We now show that no proper subset has a zero product. First observe that $\Pi_1^{k-1} e_i \neq 0$ because it does not belong to P_k. Next, P_j does not contain either of $\Pi_{i \neq j}^{k-1} e_i$ or $1 - \Pi_1^{k-1} e_i$. Since, P_j is a prime ideal, $\Pi_{i \neq j}^{k-1} e_i(1 - \Pi_1^{k-1} e_i)$ is not in P_j. In particular, $\Pi_{i \neq j}^{k-1} e_i(1 - \Pi_1^{k-1} e_i) \neq 0$. This shows that Λ is a k-edge which is a contradiction. Therefore R has fewer than k minimal prime ideals. □

Example 2.5 provides a counter example to the converse of the theorem. This result combined with the following lemmas extends theorem 3.4 of **[11]**. For completeness, we will state the theorem here.

Theorem 3.4. **[11]** Suppose R is not a k-integral domain and is such that (1) the nilradical N is finitely generated and (2) R has finitely many and more than k minimal prime ideals. If R has finitely many k-zero divisors, then R is finite.

Lemma 5.2. *Suppose $k \geq 2$ and R is a reduced ring such that $T(R)$ has a zero dimension. If $|Z(R,k)| < \infty$, then R has finitely many minimal prime ideals.*

Proof. By proposition 3.1, $Z(R, k+1) \subset Z(R, k)$. Inductively, we can derive that $Z(R, n) \subset Z(R, k)$ for any $n \geq k$. Suppose $n > |Z(R, k)|$. If $H_n(R) \neq \emptyset$, then there is an n-edge containing n distinct element of $Z(R, n)$. But $Z(R, n) \subset Z(R, k)$. Therefore,

$$n \leq |Z(R, n)| \leq |\, Z(R, k)| < n$$

which is a contradiction. Hence $Z(R, n) = \emptyset$ and R is an n-domain. We apply now theorem 5.1 to conclude that R has finitely many minimal prime ideals. □

Lemma 5.3. *Suppose in a ring R, $Z(R, k)$ is nonempty and finite. Then the nilradical N is finite.*

Proof. Suppose N is infinite. Let $\{a_1, a_2, \ldots\}$ be an infinite subset of N and $\Lambda = \{v_1, v_2, \ldots, v_k\}$ be a k-edge. Now for all i, define $v_1(i) = (1 + a_i) \cdot v_1$. Since $1 + a_i$ is a unit, the elements $v_1(i)$,

$v_2, \ldots, v_k$ satisfy conditions (2) and (3) of the definition 2.1 of k-zero divisors. But there are only finitely many k-zero divisors. Therefore, there exists i_0 such that $(1 + a_{i_0}) \cdot v_1 = (1 + a_i) \cdot v_1$, or $a_{i_0} \cdot v_1 = a_i \cdot v_1$ for infinitely many values of i. Since N is an ideal, $a_i - a_{i_0} \in N$ for all i. Let $b'_i = a_i - a_{i_0}$, then $b'_i \cdot a_i = 0$ for infinitely many values of i.. Hence, the annihilator $\mathrm{Ann}(v_1)$ contains infinitely many nilpotent elements, namely $\{b_1, b_2, \ldots\}$. Now $\mathrm{Ann}(v_1) \cap N$ is an infinite ideal, so we can repeat the same process for v_2 to find infinite subset of $\{b_i\}$ in $\mathrm{Ann}(v_2)$. By induction, we can find an infinite subset $\{c_1, c_2, \ldots\}$ of N, which is contained in $\cap_i^k \mathrm{Ann}(v_i)$.

Now for each j, define

$$w_i(j) = \begin{cases} v_i + c_j & \text{for } i = 1 \\ v_i & \text{otherwise} \end{cases}$$

Clearly $\Pi w_i(j) = \Pi v_i == 0$. Further,

$$\Pi_{i \neq h} w_i(j) = \Pi_{i \neq h} v_i \neq 0. \ \forall h$$

Now for any integers s and t, $s \neq t$, $v_1 + c_s \neq v_1 + c_t$. Since there are infinitely many c_j, and only finitely many v_i we obtain infinitely many k-zero divisors $w_i(j) = v_1 + c_j$, which is a contradiction. ☐

Lemma 5.4. *For $k \geq 2$, if R is a ring with $|Z(R,k)| < \infty$ in which N is finitely generated, then R is a t-domain for some t.*

Proof. By lemma 3.3, the nilradical N is a nilpotent ideal. Assume that $N^d = \{0\}$ for some positive integer d. Let $n = |Z(R,k)|$ and $r = \lceil \frac{n}{k-1} \rceil$ and $t = (r + d)(k - 1) + 1$. We claim that R is a t-domain. If not, then there exists a t-edge Λ on R. Now we partition Λ into $r+d$ disjoint subsets Λ_j of size $k-1$ and one singleton subset Λ_{r+d+1}. Now for any $\Lambda_j = \{v_1, , v_2, \ldots, v_{k-1}\}$, define v_k to be the product of rest of the elements in Λ.

Clearly, $\Pi_{i=1}^k v_i = 0$ and since Λ is a t-edge and $\Lambda_j \subset \Lambda$, $\Pi_{i \neq h} v_i \neq 0$. If all v_i are pairwise distinct, then the set Λ_j consists of k-zero divisors. Otherwise, $v_i = v_h$ for some $i < h$. Since $v_1, v_2, \ldots, v_{k-1}$ are part of a t-edge Λ, h must be equal to k. But then,

$$(\Pi_{h=1}^{k-1} v_h)^2 = (\Pi_1^k v_h)(\Pi_{h \neq i,k} v_h) = 0,$$

which implies that $\Pi\Lambda_j = \Pi_1^{k-1} v_i \in N$. Thus for any j, either Λ_j contains k-zero divisors or $\Pi\Lambda_j \in N$. If there are more than r subsets Λ_j containing k-zero divisors, then $|Z(R,k)| > r(k-1) \geq n$ which is not true. Therefore at least d subsets Λ_j have product belonging to N. By reordering, we can assume that first d subsets are such that $\Pi\Lambda_i \in N$. But then $\Pi_1^d \Lambda_j \in N^d = \{0\}$, showing that Λ is not a t-edge. This proves that R is a t-domain. □

From the above lemmas and theorem 3.4 from **[11]**, we conclude the following theorem:

Theorem 5.5. *For $k \geq 2$, if R is not a k-domain and has more than k minimal prime ideals. Further, suppose that $T(R)$ is a zero dimensional ring. Then $Z(R,k)$ is finite if and only if R is finite.*

Proof. By the lemma 5.3, N is finite and hence has an exponent. Hence by the lemma 5.4, R is a t domain for some t. Next we apply theorem 5.1 to conclude that R has finitely many and less than t minimal prime ideals. Now theorem 3.4 from **[11]** completes the proof. □

6 Maximal hyper-graphs on $\mathbb{Z}_n$

In this section, we present a way to construct a maximal k-uniform hyper-graph on $\mathbb{Z}_n$. Recall that $H_k(R)$ is maximal if $k = HG(R)$. For $n \leq 5$, $HG(\mathbb{Z}_n)$ is either zero or one and hence, $\mathbb{Z}_n$ has a null maximal hyper-graph. For $n = 6$, 8 and 9, $HG(\mathbb{Z}_n) = 2$. We have described the maximal hyper-graphs of $\mathbb{Z}_n$ for $n \leq 12$ in the table 2 of section 2.

When n is a prime, $HG(\mathbb{Z}_n) = 0$ and hence $\mathbb{Z}_n$ has a null maximal hyper-graph. Therefore, for the rest of the paper we will assume that n is a composite integer. We will also assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ where $p_1, p_2, \dots, p_m$ are distinct primes. By example 2.8, $HG(\mathbb{Z}_n) \leq \sum \alpha_i$. For $n > 8$,we will produce a k-edge of length $\sum \alpha_i$ to show that $HG(\mathbb{Z}_n) = \sum \alpha_i$. For any positive integer n, let $U(n) = \{a \in \mathbb{Z} | 1 \leq a \leq n-1, GCD(a,n) = 1\}$ be the set of units modulo n and let $\phi(n)$ be the cardinality of $U(n)$.

Now for any prime p, we define,

$$U_p(n) = \begin{cases} U(n/p) \text{ if } p|n \\ U(n) \text{ otherwise} \end{cases}$$

and $\phi_p(n) = |U_p(n)| = \phi(n/p)$, ϕ is well known as the Euler phi function in literature (see page 107 in [10]). The Euler function is multiplicative, that is, for two relatively prime integers m, n, $\phi(mn) = \phi(m) \cdot \phi(n)$ (see theorem 4.3.3 on page 108 in [10]). Now for such m and n, if p divides mn, then p divides m or p divides n, but not both. Therefore, ϕ_p is also multiplicative. In the following lemma we will employ this property.

Lemma 6.1. *Suppose $p^\alpha|n$ for some prime p and $n > 8$. Then $\phi_p(n) \geq \alpha$.*

Proof. Without loss of generality, we may assume that α is a maximum integer for which $p^\alpha|n$. Then $n = n'p^\alpha$ where n' is not a multiple of p. If $\alpha = 1$, then since $1 \in U_p(n)$, the result is true. Suppose $\alpha > 1$. Then

$$\phi_p(n) = \phi_p(n')\phi_p(p^\alpha) = \phi(n')\phi(p^{\alpha-1}) = \phi(n')p^{\alpha-2}(p-1).$$

Now an easy induction argument can verify that for $p > 2$, $p^\alpha \geq \alpha + 1$. Therefore, if $p > 2$, then $p^{\alpha-2}(p-1) \geq (\alpha-1)(2) \geq \alpha$. Hence $\phi_p(n) > \alpha$.

For $p = 2$, we consider two cases.

Case 1: Suppose $n' = 1$. Since $n > 8$, $\alpha > 3$. Again, using induction we can show that $2^{\alpha-2} \geq \alpha$. Hence $\phi(n') \cdot 2^{\alpha-2} = 2^{\alpha-2} \geq \alpha$.

Case 2: Suppose $n' > 1$. Since n' is not divisible by $p = 2$, $n' \geq 3$. Now $1, n'-1 \in U(n')$, hence, $\phi(n') \geq 2$. Therefore,

$$\phi(n')p^{\alpha-2}(p-1) \geq 2^{\alpha-1} \geq \alpha \text{ for } \alpha > 1.$$

□

Suppose $u \in U_{p_i}(n)$. Then $GCD(u, n/p_i) = 1$. If $\alpha_i > 1$, then $p_i|(n/p_i)$ and therefore, $u \in U(n)$. Further, $u < n/p_i$. Hence,

$$U_{p_i}(n) = \{u \in U(n) | 1 \leq u < n/p_i\} \text{ when } \alpha_i > 1.$$

Example 6.1. Suppose $n = 18 = 2 \cdot 3^2$. Then

$$U(18) = \{1, 5, 7, 11, 13, 17\},$$

$U_2(18) = U(9) = \{1, 2, 4, 5, 7, 8\}$ and $\phi_2(18) = 6$. Similarly $U_3(18) = U(6) = \{1, 5\}$ and $\phi_3(18) = 2$.

Theorem 6.2. *For $n > 8$, $HG(\mathbb{Z}_n) = \sum \alpha_i$.*

Proof. Let $u_1 < u_2 \in U_{p_i}(n)$. Then $u_1 p_i < u_2 p_i < n$. Therefore, $u_1 p_i$ and $u_2 p_i$ are distinct elements in $\mathbb{Z}_n$. Since there are at least α_i elements in $U_{p_i}(n)$, we can select α_i distinct elements $w_{1i}, w_{2i}, \ldots, w_{\alpha_i i}$ of the form $u \cdot p_i$ with $u \in U_{p_i}(n)$. It is easy to show that $\Lambda = \{w_{ij}\}$ forms a k-edge with $k = \sum \alpha_i$. Therefore $\mathbb{Z}_n$ is not a k-domain. Since $HG(\mathbb{Z}_n) \leq k$, the proof is complete. □

In the following theorem, elements of the quotient ring $\mathbb{Z}_n$ will be denoted by their representatives in $\{1, 2, \ldots, n-1\}$.

Theorem 6.3. *Suppose $n > 8$ has a prime factorization $n = \Pi_1^m p_i^{\alpha_i}$ and Λ is a k-edge in $H_k(\mathbb{Z}_n)$ where $k = \sum \alpha_i$. Then for each vertex $x \in \Lambda$, there is exactly one p_i such that $p_i | x$. Further, for each i, there are α_i vertices in Λ whose representatives are of the form up_i with $u \in U_{p_i}(n)$.*

Proof. Suppose $\Lambda = \{x_1, \ldots, x_k\}$. First we note that $Z(\mathbb{Z}_n, k) \subset Z(\mathbb{Z}_n)$. Therefore, for all l, p_j divides x_l for some j. Therefore, $x_l = up_j$. Now we will prove that only one prime p_j divides x_l. Suppose there is an element, say x_1, which is divisible by more than one p_j, that is, $x_1 = up_j \cdot p_h$, $j \neq h$ for some $h \in \mathbb{Z}$. Let

$$A_i = \{x \in \{x_l, x_2 \ldots, x_k\}; \, p_i \text{ divides } x\}.$$

We will write ΠA_i for the product of elements of A_i. Now $\Pi_1^k x_l$ is divisible by n. Therefore, the exponent of p_i in the prime factorization of Πx_l must be at least α_i. Hence $p_i^{\alpha_i}$ divides $\Pi_{p_i | x_l} x_l = \Pi A_i$, that is, $\Pi A_i = ap_i^{\alpha_i}$ for some integer a.

Now $x_1 \in A_j \cap A_h$. For each i, We will chose a subset B_i of A_i with the following properties:

1. $x_1 \notin B_j$ and $x_1 \in B_h$.

2. Cardinality of B_j is at most $\min(\alpha_j - 1, |A_j|)$.

3. For $i \neq j$, cardinality of B_i is at most $\min(\alpha_i, |A_i|)$.

Set $B = \cup B_i$. First we observe that $|B| \leq \sum \alpha_i - 1$. Hence, $B < \{x_l\}$. Further, $p_i^{\alpha_i}$ divides ΠB_i for all i. Therefore, n divides ΠB. But then we have $\overline{B} = \{\overline{x}_l | x \in B\} < \Lambda$ with $\Pi \overline{B} = 0$, which contradicts the hypothesis that Λ is a k-edge. Thus, $\forall l$, x_l is of the form up_j where u is not a multiple of any other prime p_h where $h \neq j$.

Thus for each $x_i \in \Lambda$, we have a unique j such that $x_i \in A_j$. In other words, sets A_j partitions Λ. Therefore, $\sum_j |A_j| = |\Lambda| = \sum_i |\alpha_i|$.

Now suppose for some j, $|A_j| > \alpha_j$. Choose a subset B_j of A_j of cardinality equal to α_j. Then $B = \cup_{i \neq j} A_i \cup B_j < \Lambda$ and $\Pi B = 0$ which is a contradiction. Therefore, $|A_j| \leq \alpha_j$. Finally, since $\sum_i |A_i| = \sum_i |\alpha_i|$, $|A_j| = \alpha_j$. for all j, this completes the proof.

Now if $\alpha_j = 1$, then $u < n/p_j$, is a unit modulo n/p_j. Hence $u \in U_{p_j}(n)$. For $\alpha_j > 1$, we will show that $x \neq ap_j^2$ to prove that $u \in U_{p_j}(n)$. To the contrary, let us assume that $x_1 = ap_j^2$. Once again, we will chose subsets B_i of A_i with the following properties:

1. $x_1 \notin B_j$

2. Cardinality of B_j is at most $\min(\alpha_j - 2, |A_j|)$.

3. For $i \neq j$, cardinality of B_i is at most $\min(\alpha_i, |A_i|)$.

Set $B = (\cup B_i) \cup \{x_1\}$. Observe that $|B| \leq \sum \alpha_i - 1$. Hence, $B < \{x_l\}$. Further, $p_i^{\alpha_i}$ divides ΠB for all i. Therefore, n divides ΠB. Thus we have $\overline{B} = \{\overline{x}_l | x_l \in B\} < \Lambda$ and $\Pi \overline{B} = 0$, which is a contradiction. This shows that p_j^2 does not divides x_1. Thus, $x_1 = up_j$ and u is not divisible by the prime p_j. Therefore, $u \in U_{p_j}(n)$.

To complete the proof, we need to show that $|A_i| = \alpha_i$. Suppose $\alpha_i = 1$. If $|A_i| > 1$, then the set B consisting of one element from A_i and all elements of A_j, $i \neq j$ has less than k elements. Also $\Pi B \equiv 0 \mod n$, which is a contradiction. Therefore, $|A_i| = 1$. When $\alpha_i > 1$, we note the following properties of A_i, namely, (1)

$p_i^{\alpha_i}$ divides ΠA_i (2) p_i^2 does not divide any element of A_i and (3) no other prime divides any element of A_i. It is clear that $|A_i| \geq \alpha_i$. Since $\sum \alpha_i = k = |\cup A_i|$, $|A_i| = \alpha_i$. This completes the proof. □

Remark: Let $v \in Z(\mathbb{Z}_n, k)$. By the theorem 6.3, v is a multiple of p_i for exactly one i. If $\alpha_i > 1$, then v is a multiple of p_i but not of p_i^2. This implies that $v = p_i \cdot u$ for some unit u. Suppose $u = r(n/p_i)+u'$. Then, $v = p_i \cdot u = p_i \cdot u'$. Therefore, $v \in p_i \cdot U_{p_i}(n)$.

When $\alpha_i = 1$, then $v = p_i^{\alpha} \cdot u$ for some unit u and $\alpha > 0$. Set $u' = p_i^{\alpha-1} \cdot u$, then we can write $v = p_i \cdot u'$ where $u' \in U(n/p_i)$. Once again we get, $v \in p_i \cdot U_{p_i}(n)$. Thus, $Z(\mathbb{Z}_n, k)$ is a disjoint union of $p_i \cdot U_{p_i}(n)$ for $1 \leq i \leq m$. We summarize this in the following formulas:

- Vertex set $Z(\mathbb{Z}_n, k) = \dot{\bigcup}_1^m p_i \cdot U_{p_i}(n)$.
- $|Z(R,k)| = \sum_1^m \phi_{p_i}(n) = \sum_1^m \phi(\frac{n}{p_i})$
- Number of edges $= |E| = \Pi_{i=1}^m \binom{\phi_{p_i}(n)}{\alpha_i}$.

To illustrate the procedure, we will redo the example 2.2.

Example 6.2. Suppose $R = \mathbb{Z}_{18}$. Then

$$U_2(18) = U(9) = \{1, 2, 4, 5.7.8\} \text{ and } U_3(18) = U(6) = \{1, 5\}.$$

Therefore,

$$Z(\mathbb{Z}_{18}, 3) = 2 \cdot U_2(18) \cup 3 \cdot U_3(18) = \{2, 4, 8, 10, 14, 16, 3, 15\}.$$

Number of edges $= \binom{\phi_2(18)}{1} \cdot \binom{\phi_3(18)}{2} = \binom{6}{1} \cdot \binom{2}{2} = 6$. We can write the edges explicitly by selecting one element from $2 \cdot U_2(18)$ and two (both) elements from $3 \cdot U_3(18)$. The edges are $e_1 = \{2, 3, 15\}$, $e_2 = \{4, 3, 15\}$,$e_3 = \{8, 3, 15\}$,$e_4 = \{10, 3, 15\}$,$e_5 = \{14, 3, 15\}$ and $e_6 = \{16, 3, 15\}$.

Example 6.3. Suppose $n = 180 = 2^2 \cdot 3^2 \cdot 5$. We will compute $\phi_p(n)$ and $|E|$.

$$\phi_2(180) = \phi(90) = \phi(2 \cdot 3^2 \cdot 5) = 24$$
$$\phi_3(180) = \phi(60) = \phi(2^2 \cdot 3 \cdot 5) = 16$$
$$\phi_2(180) = \phi(36) = \phi(2^2 \cdot 3^2) = 12$$

Thus there are $24 + 16 + 12 = 52$ 5-zero divisors in $\mathbb{Z}_{180}$ and the number of 5-edges of $H_5(\mathbb{Z}_{180})$ equals

$$|E| = \binom{24}{2} \cdot \binom{16}{2} \cdot \binom{12}{1}$$
$$= (276)(120)(12) = 397,440.$$

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