The signed total Roman domatic number of a digraph

L. Volkmann

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Abstract

Let D be a finite and simple digraph with vertex set $V(D)$. A *signed total Roman dominating function* on the digraph D is a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N^-(v)} f(u) \geq 1$ for every $v \in V(D)$, where $N^-(v)$ consists of all inner neighbors of v, and every vertex $u \in V(D)$ for which $f(u) = -1$ has an inner neighbor v for which $f(v) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed total Roman dominating family* (of functions) on D. The maximum number of functions in a signed total Roman dominating family on D is the *signed total Roman domatic number* of D, denoted by $d_{stR}(D)$. In this paper we initiate the study of the signed total Roman domatic number in digraphs and we present some sharp bounds for $d_{stR}(D)$. In addition, we determine the signed total Roman domatic number of some digraphs. Some of our results are extensions of well-known properties of the signed total Roman domatic number of graphs.

Keywords: Digraph, signed total Roman dominating function, signed total Roman domination number, signed total Roman domatic number.

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1 Introduction

In this paper we continue the study of Roman dominating functions in graphs and digraphs. Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G(v) = N(v)$ be the open neighborhood of the vertex v. We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$ and C_n for the cycle of order n. A *signed total Roman dominating function* (STRDF) on a graph G is defined in [5] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$, and every vertex $u \in V(D)$ for which $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$. The *weight* of an STRDF f is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ of G is the minimum weight of an STRDF on G. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total Roman dominating family* (of functions) on G. The maximum number of functions in a signed total Roman dominating family (STRD family) on D is the *signed total Roman domatic number* of G, denoted by $d_{stR}(G)$. This parameter was introduced and investigated in [6].

Let D be a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The *order* $|V|$ of D is denoted by $n = n(D)$. For an arc $(x, y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y. We write $d_D^+(v) = d^+(v)$ for the *out-degree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$ and the *minimum* and *maximum out-degree* are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$. The sets $N_D^-(v) = N^-(v) = \{x | (x, v) \in A(D)\}$ and $N_D^+(v) = N^+(v) = \{x | (v, x) \in A(D)\}$ are called the *in-neighborhood* and *out-neighborhodd* of the vertex v. A digraph D is *r-in-regular* when $\delta^-(D) = \Delta^-(D) = r$ and *r-out-regular* when $\delta^+(D) = \Delta^+(D) = r$. If D is r-in-regular and r-out-regular, then D is called *r-regular*. We write K_n^* for the *complete digraph* of order n and $K_{p,q}^*$ for the *complete bipartite digraph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$. For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$, the *weight* of f is $\omega(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V(D))$. Consult Haynes, Hedetniemi and Slater [2] for notation and terminology which are not defined here.

A *signed total Roman dominating function* (STRDF) on a digraph D is defined in [7] as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $f(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, and such that every vertex $u \in V(D)$ for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *weight* of an STRDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed total Roman domination number* of a digraph D, denoted by $\gamma_{stR}(D)$, equals the minimum weight of an STRDF on D. The signed total Roman domination number exists when $\delta^- \geq 1$. Thus we assume throughout this paper that $\delta^-(D) \geq 1$. A *$\gamma_{stR}(D)$-function* is a signed total Roman dominating function of D with weight $\gamma_{stR}(D)$.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [1]. They have defined the domatic number of a graph by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition. The maximum number of classes of a domatic partition of G is the domatic number of G. But Rall has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. (This was mentioned by Slater and Trees [4].) Analogous to the fractional domatic number we may define the signed total Roman domatic number of digraphs.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed total Roman dominating family* (of functions) on D. The maximum number of functions in a signed total Roman dominating family (STRD family) on D is the *signed total Roman domatic number* of D, denoted by $d_{stR}(D)$. The signed total Roman domatic number is well-defined and

$$d_{stR}(D) \geq 1 \tag{1}$$

for all digraphs D with $\delta^-(D) \geq 1$, since the set consisting of the STRDF with constant value 1 forms an STRD family on D.

Our purpose in this paper is to initiate the study of signed total Roman domatic number in digraphs. We study basic properties and bounds for the signed total Roman domatic number of a digraph. In addition, we determine the signed total Roman domatic number of some classes of digraphs. Some of our results are extensions of well-known properties of the signed total Roman domatic number $d_{stR}(G)$ of graphs G.

We make use of the following results in this paper.

Proposition A. *([6])* If $p \geq 4$ is an integer, then $d_{stR}(K_{p,p}) = p$.

Proposition B. *([7])* If $K^*_{p,p}$ is the complete bipartite digraph, then $\gamma_{stR}(K^*_{p,p}) = 2$, unless $p = 3$ in which case $\gamma_{stR}(K^*_{3,3}) = 4$.

Proposition C. *([7])* If K^*_n is the complete digraph of order $n \geq 3$, then $\gamma_{stR}(K^*_n) = 3$.

Proposition D. *([6])* If $k \geq 0$ is an integer, then $d_{stR}(K_{9k+6}) = 3k + 2$.

Proposition E. *([7])* If D is an r-out-regular digraph of order n with $r \geq 1$ and $\delta^-(D) \geq 1$, then $\gamma_{stR}(D) \geq \lceil n/r \rceil$.

Proposition F. *([7])* If D is a digraph of order n with $\delta^-(D) \geq 3$, then

$$\gamma_{stR}(D) \leq n + 1 - 2 \left\lceil \frac{\delta^-(D) - 2}{2} \right\rceil.$$

Proposition G. *([6])* If C_{4t} is the cycle of length $4t$ with an integer $t \geq 1$, then $d_{stR}(C_{4t}) = 2$.

The *associated digraph* G^* of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same end as e. Since $N^-_{G^*}(v) = N_G(v)$ for each $v \in V(G) = V(G^*)$, the following useful observation is valid.

Observation 1. If G^* is the associated digraph of a graph G, then we have $\gamma_{stR}(G^*) = \gamma_{stR}(G)$ and $d_{stR}(G^*) = d_{stR}(G)$.

Using Observation 1 and Propositions A, D and G, we obtain the next three results immediately.

Proposition H. If $p \geq 4$ is an integer, then $d_{stR}(K^*_{p,p}) = p$.

Proposition I. If $k \geq 0$ is an integer, then $d_{stR}(K^*_{9k+6}) = 3k + 2$.

Proposition J. If $t \geq 1$ is an integer, then $d_{stR}(C^*_{4t}) = 2$.

2 Properties of the total signed Roman domatic number

In this section we present basic properties of $d_{stR}(D)$ and sharp bounds on the signed total Roman domatic number of a digraph.

Theorem 2. Let D be a digraph with $\delta^-(D) \geq 1$. Then

$$d_{stR}(D) \leq \delta^-(D).$$

Moreover, if $d_{stR}(D) = \delta^-(D)$, then for each STRD family $\{f_1, f_2, \cdots, f_d\}$ on D with $d = d_{stR}(D)$ and each vertex v of minimum in-degree, we have $\sum_{u \in N^-(v)} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N^-(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STRD family on D such that $d = d_{stR}(D)$. Assume that v is a vertex of minimum in-degree $\delta^-(D)$. It follows from the definitions that

$$d \leq \sum_{i=1}^{d} \sum_{u \in N^-(v)} f_i(u) = \sum_{u \in N^-(v)} \sum_{i=1}^{d} f_i(u) \leq \sum_{u \in N^-(v)} 1 = \delta^-(D).$$

Thus $d_{stR}(D) \leq \delta^-(D)$.

If $d_{stR}(D) = \delta^-(D)$, then the two inequalities occurring in the proof become equalities. Hence for the STRD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each vertex v of minimum in-degree, $\sum_{u \in N^-(v)} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N^-(v)$. □

Propositions H and J show that Theorem 2 is sharp. Inequality (1) and Theorem 2 imply the next result immediately.

Corollary 3. If D is an oriented cycle or $D = K^*_{1,n}$, then $d_{stR}(D) = 1$.

Corollary 4. If D is a digraph with $\delta^-(D) = 3$, then $d_{stR}(D) \leq 2$.

Proof. Suppose to the contrary that $d_{stR}(D) = 3$. Let $\{f_1, f_2, f_3\}$ be an STRD family on D, and let v be a vertex of minimum in-degree. By Theorem 2, we deduce that $\sum_{x \in N^-(v)} f_i(x) = 1$ for each function f_i and $\sum_{i=1}^{3} f_i(x) = 1$ for all $x \in N^-(v)$. It follows that $f_i(x) \leq 1$ for all $x \in N^-(v)$ and $1 \leq i \leq 3$. This implies $f_i(v) \geq 1$ for $1 \leq i \leq 3$, and we obtain the contradiction $\sum_{i=1}^{3} f_i(v) \geq 3$. Thus $d_{stR}(D) \leq 2$. □

Theorem 5. If D is a digraph of order n with $\delta^-(D) \geq 1$, then

$$\gamma_{stR}(D) \cdot d_{stR}(D) \leq n.$$

Moreover, if we have $\gamma_{stR}(D) \cdot d_{stR}(D) = n$, then for each STRD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_{stR}(D)$, each function f_i is a $\gamma_{stR}(D)$-function and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STRD family on D such that $d = d_{stR}(D)$ and let $v \in V(D)$. Then

$$\begin{aligned} d \cdot \gamma_{stR}(D) &= \sum_{i=1}^{d} \gamma_{stR}(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 1 = n. \end{aligned}$$

If $\gamma_{stR}(D) \cdot d_{stR}(D) = n$, then the two inequalities occurring in the proof become equalities. Hence for the STRD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} f_i(v) = \gamma_{stR}(D)$. Thus each function f_i is a $\gamma_{stR}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V(D)$. □

For $p \geq 4$, we conclude from Propositions B and H that $\gamma_{stR}(K^*_{p,p}) \cdot d_{stR}(K^*_{p,p}) = 2p$ and therefore equality in Theorem 5. Using Propositions C and I, we obtain $\gamma_{stR}(K^*_{9k+6}) \cdot d_{stR}(K^*_{9k+6}) = 9k + 6$. This is a further example that demonstrates the sharpness of Theorem 5. For some out-regular digraphs we will improve the upper bound given in Theorem 2.

Corollary 6. Let D be an r-out-regular digraph of order n such that $r \geq 2$ and $\delta^-(D) \geq 1$. If n is not a multiple of r, then $d_{stR}(D) \leq r - 1$.

Proof. Let $n = pr + t$ with integers $p \geq 1$ and $1 \leq t \leq r - 1$. According to Proposition E, we have

$$\gamma_{stR}(G) \geq \left\lceil \frac{n}{r} \right\rceil = \left\lceil \frac{pr + t}{r} \right\rceil = p + 1.$$

Now Theorem 5 yields to

$$d_{stR}(D) \leq \frac{n}{\gamma_{stR}(D)} \leq \frac{n}{p+1} < r,$$

and therefore $d_{stR}(D) \leq r - 1$. □

Propositions H and J demonstrate that Corollary 6 is not valid in general when n is a multiple of r.

A digraph without directed cycles of length 2 is called an *oriented graph*. An oriented graph is called a *tournament* when either $(u, v) \in A(D)$ or $(v, u) \in A(D)$ for each pair of distinct vertices $u, v \in V(D)$. By D^{-1} we denote the digraph obtained by reversing all arcs of D.

Corollary 7. If T is an r-regular tournament of order n with $r \geq 2$, then $d_{stR}(T) \leq r - 1$.

Proof. Since T is an r-regular tournament, we observe that $n = 2r + 1$. Therefore it follows from Corollary 6 that $d_{stR}(T) \leq r - 1$. □

Corollary 8. If D is an oriented graph of order n with $\delta^-(D), \delta^-(D^{-1}) \geq 1$, then $d_{stR}(D) + d_{stR}(D^{-1}) \leq n - 2$.

Proof. If D is not a tournament or D is a non-regular tournament, then $\delta^-(D) + \delta^-(D^{-1}) \leq n - 2$, and hence we deduce from Theorem 2 that

$$d_{stR}(D) + d_{stR}(D^{-1}) \leq \delta^-(D) + \delta^-(D^{-1}) \leq n - 2.$$

Let now D be an r-regular tournament. Then D^{-1} is also an r-regular tournament such that $n = 2r + 1$. Now Corollary 7 implies

$$d_{stR}(D) + d_{stR}(D^{-1}) \leq (r-1) + (r-1) = 2r - 2 = n - 3.$$

This completes the proof. □

For the next theorem we use the following three digraphs of order 4. Let K_4^* be the complete digraph with vertex set $\{u_1, u_2, u_3, u_4\}$. Define the digraphs of order 4 by

$$A_1 = K_4^* - \{(u_1, u_2), (u_2, u_1), (u_1, u_4), (u_2, u_3)\},$$

$$A_2 = K_4^* - \{(u_1, u_2), (u_2, u_1), (u_1, u_4), (u_4, u_3)\}$$

and

$$A_3 = K_4^* - \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1)\}.$$

Note that A_1, A_2 and A_3 are 2-in-regular.

Lemma 9. For $i = 1, 2, 3$, we have $d_{stR}(A_i) = 2$.

Proof. Theorem 2 implies that $d_{stR}(A_i) \leq 2$ for $i = 1, 2, 3$. Now define the functions $f_1, f_2 : V(A_i) \longrightarrow \{-1, 1, 2\}$ by

$$f_1(u_1) = f_1(u_3) = -1, \quad f_1(u_2) = f_1(u_4) = 2$$

and

$$f_2(u_1) = f_2(u_3) = 2, \quad f_2(u_2) = f_2(u_4) = -1.$$

It is easy to see that f_1 and f_2 are signed total Roman dominating functions, and that $\{f_1, f_2\}$ is a signed total Roman dominating family on A_1, A_2 and A_3. Therefore $d_{stR}(A_i) \geq 2$ and so $d_{stR}(A_i) = 2$ for $i = 1, 2, 3$. □

Theorem 10. Let D be a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$. Then $d_{stR}(D) \leq n - 2$, with equality if and only if $n = 3$ or $n = 4$ and D is isomorphic to A_1, A_2, A_3 or C_4^*.

Proof. For $n = 3$ Theorem 10 is valid. Let now $n \geq 4$ and let $\delta^- = \delta^-(D)$. If $\delta^- \leq n - 2$, then Theorem 2 implies that $d_{stR}(D) \leq \delta^- \leq n - 2$. If $\delta^- = n - 1$, then D is isomorphic to the complete digraph K_n^*, and we deduce from Proposition C that $\gamma_{stR}(D) = 3$. Hence it follows from Theorem 5 that

$$d_{stR}(D) \leq \frac{n}{\gamma_{stR}(D)} = \frac{n}{3} \leq n - 2.$$

If $n = 4$ and D is isomorphic to A_1, A_2, A_3 or C_4^*, then it follows from Lemma 9 and Proposition J that $d_{stR}(D) = 2 = n - 2$.

Coversely, assume that $d_{stR}(D) = n-2$. If $\delta^- \leq n-3$, then Theorem 2 leads to the contradiction $n-2 = d_{stR}(D) \leq \delta^- \leq n-3$. Thus there remain the cases $\delta^- = n-1$ or $\delta^- = n-2$. If $\delta^- = n-1$, then we observe as above that $n-2 = d_{stR}(D) \leq n/3$, a contradiction to $n \geq 4$.

Next assume that $\delta^- = n-2$. If $n = 4$, then it is straightforward to verify that D is isomorphic to A_1, A_2, A_3 or C_4^*. Let now $n \geq 5$, and let f be a $\gamma_{stR}(D)$-function. If $f(x) = 1$ for all $x \in V(D)$, then $\gamma_{stR}(D) = n > 2$. If $f(v) = -1$ for any vertex $v \in V(D)$, then there exixts a vertex $w \in V(D)$ with $f(w) = 2$. If $d^-(w) = n-1$, then it follows that $\gamma_{stR}(D) = f(w) + \sum_{x \in N^-(w)} f(x) \geq 2+1 = 3 > 2$. If $d^-(w) = n-2$, then let $u \neq w$ be a vertex such that $u \notin N^-(w)$. We observe that $\gamma_{stR}(D) = f(w) + \sum_{x \in N^-(w)} f(x) + f(u) \geq 2+1-1 = 2$. Altogether, we have shown that $\gamma_{stR}(D) \geq 2$. Using again Theorem 5, we obtain

$$n-2 = d_{stR}(D) \leq \frac{n}{\gamma_{stR}(D)} \leq \frac{n}{2},$$

a contradiction to $n \geq 5$. This completes the proof. □

Corollary 11. If D is a digraph of order $n \geq 5$ with $\delta^-(D) \geq 1$, then $d_{stR}(D) \leq n-3$.

For r-out-regular digraphs we can improve Theorem 10.

Corollary 12. If D is an r-out-regular digraph of order n with $r \geq 1$ and $\delta^-(D) \geq 1$, then $d_{stR}(D) \leq n/2$.

Proof. It follows from Proposition E that $\gamma_{stR}(D) \geq \lceil n/r \rceil \geq 2$. Thus Theorem 5 implies that $d_{stR}(D) \leq n/\gamma_{stR}(D) \leq n/2$. □

Proposition H demonstrates that Corollary 12 is sharp. Using Observation 1 and Theorems 2 and 5, we obtain the corresponding results for graphs given in [6].

3 Upper bounds on $\gamma_{stR}(D) + d_{stR}(D)$

The upper bound on the product $\gamma_{stR}(D) \cdot d_{stR}(D)$ leads to an upper bound on the sum of these two parameters.

Theorem 13. Let D be a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$, then

$$\gamma_{stR}(D) + d_{stR}(D) \leq n+1,$$

with equality if and only if $\gamma_{stR}(D) = n$.

Proof. It follows from Theorem 5 that

$$\gamma_{stR}(D) + d_{stR}(D) \leq \frac{n}{d_{stR}(D)} + d_{stR}(D).$$

According to inequality (1) and Theorem 2, we conclude that $1 \leq d_{sR}(G) \leq n-1$. Using these bounds, and the fact that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n-1$, the last inequality leads to

$$\begin{aligned}\gamma_{stR}(D) + d_{stR}(D) &\leq \frac{n}{d_{stR}(D)} + d_{stR}(D) \\ &\leq \max\left\{n+1, \frac{n}{n-1} + n - 1\right\} = n+1.\end{aligned}$$

If $\gamma_{stR}(D) = n$, then we have $\gamma_{stR}(D) + d_{stR}(D) = n+1$. Conversely, if $\gamma_{stR}(D) + d_{stR}(D) = n+1$, then the above inequality chain leads to $d_{stR}(D) = 1$ and thus $\gamma_{stR}(D) = n$. □

For example, if D is the disjoint union of oriented cycles, then D is 1-regular, and it follows from Proposition E that $\gamma_{stR}(D) = n(D)$ and thus $\gamma_{stR}(D) + d_{stR}(D) = n+1$. Theorem 13 and Proposition F lead to the next corollary.

Corollary 14. If D is a digraph of order n with $\delta^-(D) \geq 3$, then

$$\gamma_{stR}(D) + d_{stR}(D) \leq n.$$

For digraphs D with $\delta^-(D) \geq 5$, we can improve Corollary 14.

Theorem 15. If D is a digraph of order n with $\delta^-(D) \geq 5$, then

$$\gamma_{stR}(D) + d_{stR}(D) \leq n-2,$$

unless $n = 6$, in which case $D = K_6^*$ with $\gamma_{stR}(K_6^*) + d_{stR}(K_6^*) = 5 = n-1$.

Proof. If $\gamma_{stR}(D) > n/2$, then Theorem 5 implies $d_{stR}(D) = 1$. Using the hypothesis $\delta^-(D) \geq 5$ and Proposition F, we arrive at $\gamma_{stR}(D) + d_{stR}(D) \leq (n-3)+1 = n-2$. Let now $\gamma_{stR}(D) \leq n/2$. If $\gamma_{stR}(D) \leq 1$, then it follows from Corollary 11 that $\gamma_{stR}(D) + d_{stR}(D) \leq 1 + (n-3) = n-2$. So assume that $\gamma_{stR}(D) \geq 2$. Then Theorem 5 leads to

$$\gamma_{stR}(D) + d_{stR}(D) \leq \gamma_{stR}(D) + \frac{n}{\gamma_{stR}(D)} \leq 2 + \frac{n}{2}. \quad (2)$$

If $n \geq 8$, then we deduce from (2) that $\gamma_{stR}(D) + d_{stR}(D) \leq 2 + n/2 \leq n-2$.

Assume next that $n = 7$. Then we observe that $2 \leq \gamma_{stR}(D) \leq 3$. If $\gamma_{stR}(D) = 2$, then Theorem 5 leads to $d_{stR}(D) \leq 3$ and so $\gamma_{stR}(D) +$

$d_{stR}(D) \leq 5 = n-2$. If $\gamma_{stR}(D) = 3$, then $d_{stR}(D) \leq 2$ by Theorem 5 and hence $\gamma_{stR}(D) + d_{stR}(D) \leq 5 = n-2$ again.

Finally, assume that $n = 6$. The hypothesis $\delta^-(D) \geq 5$ yields to $D = K_6^*$. According to Propositions C and I, we have $\gamma_{stR}(D) = 3$ and $d_{stR}(D) = 2$ and thus $\gamma_{stR}(D) + d_{stR}(D) = 5 = n-1$. This completes the proof. □

4 Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph or digraph and its complement. In their classical paper [3], Nordhaus and Gaddum discussed this problem for the chomatic number of graphs. We present such inequalities for the signed total Roman domatic number of digraphs.

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc (u, v) belongs to $\overline{D}$ if and only if (u, v) does not belong to D. As an application of Theorem 2, we will prove the following Nordhaus-Gaddum type result.

Theorem 16. If D is a digraph of order n such that $\delta^-(D), \delta^-(\overline{D}) \geq 1$, then

$$d_{stR}(D) + d_{stR}(\overline{D}) \leq n-1.$$

Furthermore, if $d_{stR}(D) + d_{stR}(\overline{D}) = n-1$, then D is in-regular.

Proof. Since $\delta^-(\overline{D}) = n-1-\Delta^-(D)$, it follows from Theorem 2 that

$$d_{stR}(D) + d_{stR}(\overline{D}) \leq \delta^-(D) + \delta^-(\overline{D}) = \delta^-(D) + (n-1-\Delta^-(D)) \leq n-1.$$

If D is not in-regular, then $\Delta^-(D) - \delta^-(D) \geq 1$, and hence the above inequality chain implies the better bound $d_{stR}(D) + d_{stR}(\overline{D}) \leq n-2$. □

As an application of Corollary 6, we improve Theorem 16 for r-regular digraphs.

Theorem 17. Let D be an r-regular digraph of order $n \geq 5$ with $1 \leq r \leq n-2$. Then

$$d_{stR}(D) + d_{stR}(\overline{D}) \leq n-2. \tag{3}$$

Proof. Since D is r-regular, $\overline{D}$ is $(n-r-1)$-regular. We assume, without loss of generality, that $n-r-1 \leq r$.

If $n = pr + t$ with integers $p \geq 1$ and $1 \leq t \leq r-1$, then it follows from Theorem 2 and Corollary 6 that

$$d_{stR}(D) + d_{stR}(\overline{D}) \leq r-1 + (n-r-1) = n-2.$$

Thus assume now that $n = pr$ with an integer $p \geq 2$. As $n - r - 1 \leq r$, we observe that $pr = n = r + (n - r - 1) + 1 \leq 2r + 1$ and so $p = 2$. Therefore $n = 2r$ and hence $n - r - 1 = 2r - r - 1 = r - 1$ and $r \geq 3$.

If $n = q(n - r - 1) + t$ with integers $q \geq 1$ and $1 \leq t \leq n - r - 2$, then it follows from Theorem 2 and Corollary 6 that

$$d_{stR}(D) + d_{stR}(\overline{D}) \leq r + (n - r - 1) - 1 = n - 2.$$

Thus assume next that $n = q(n - r - 1)$ with an integer $q \geq 2$. Altogether, we have

$$n = 2r = q(n - r - 1) = q(r - 1)$$

with $r \geq 3$. It is easy to see that the last identity is only possible for $q = 3$ and $r = 3$ and so $n = 6$. However, in this case we conclude from Corollary 4 and Theorem 2 that $d_{stR}(D) + d_{stR}(\overline{D}) \leq 2 + 2 = 4 = n - 2$. This completes the proof. □

Clearly, since $d_{stR}(D) + d_{stR}(\overline{D}) \geq 2$ by (1), Theorem 17 is not valid for $n = 3$. In addition, the digraphs C_4^* and A_3 show that Theorem 17 is not valid for $n = 4$ in general.

Corollary 18. If T is a tournament of odd order $n \geq 5$ such that $\delta^-(T) \geq 1$ and $\delta^-(\overline{T}) \geq 1$, then $d_{stR}(T) + d_{stR}(\overline{T}) \leq n - 3$.

Proof. If T is an r-regular tournament, then $\overline{T}$ is also an r-regular tournament such that $n = 2r + 1$. Therefore it follows from Corollay 7 that $d_{stR}(T) + d_{stR}(\overline{T}) \leq (r - 1) + (r - 1) = n - 3$.

Assume now that T is not regular. Then $\delta^-(T) \leq (n-3)/2$ and $\delta^-(\overline{T}) \leq (n - 3)/2$, and we deduce from Theorem 2 that

$$d_{stR}(T) + d_{stR}(\overline{T}) \leq \delta^-(T) + \delta^-(\overline{T}) \leq \frac{n-3}{2} + \frac{n-3}{2} = n - 3.$$

□

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