The Spectrum Problem for 3 of the Cubic Graphs of Order 10

Peter Adams[1], Chudan Chan[2], Saad I. El-Zanati[3], Emma Holdaway[4], Uğur Odabaşı[5], Jackson Ward[3]

[1]The University of Queensland, QLD 4072, Australia
[2]Illinois Wesleyan University, Bloomington, IL 61701, USA
[3]Illinois State University, Normal, IL 61790-4520, USA
[4]Brigham Young University, Provo, UT 84602, USA
[5]Istanbul University, Istanbul 34320, Turkey

Abstract

There are 19 connected cubic graphs of order 10. If G is one of a specific set 3 of the 19 graphs, we find necessary and sufficient conditions for the existence of G-decompositions of K_v.

1 Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s, and we use $K_{r\times s,t}$ to denote the complete simple multipartite graph with r parts of size s and one part of size t. If a and b are integers with $a \leq b$, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$.

A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that each edge of K appears in exactly one G_i. If each G_i in Δ is isomorphic to a given graph G, the decomposition is called a *G-decomposition* of K. Similarly, if G and H are subgraphs of K, then a $\{G, H\}$-decomposition of K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that each edge of K appears in exactly one G_i and each G_i is isomorphic to either G or H. If there exists a G-decomposition of K, then we say *G divides K* and write $G|K$. A G-decomposition of K is also known as a *(K, G)-design.*

For a positive integer v, let K_v denote the complete graph on v vertices. If $v_1, v_2, \ldots, v_t$ are positive integers, let $K_{v_1,v_2,\ldots,v_t}$ denote the complete t-partite graph with parts of size $v_1, v_2, \ldots, v_t$. A $(K_{v_1,v_2,\ldots,v_t}, K_k)$-design is also known as a *group divisible design*. For the sake of brevity, we will refrain from adding too many details on these concepts and direct the interested reader to the *Handbook of Combinatorial Designs* [9] and to the summaries within on group divisible designs [10].

Given a graph G, a classical problem in combinatorics is to find necessary and sufficient conditions for the existence of a G-decomposition of K_v. This is known as the *spectrum problem* for G. The spectrum problem has been investigated and settled for numerous classes of simple graphs G (see [2] and [6] for summaries, and the Web site maintained by Bryant and McCourt [7] for more up to date results). When G is a complete graph, the spectrum problem was settled by Kirkman [14] for $G = K_3$ and by Hanani [12] for $G \in \{K_4, K_5\}$.

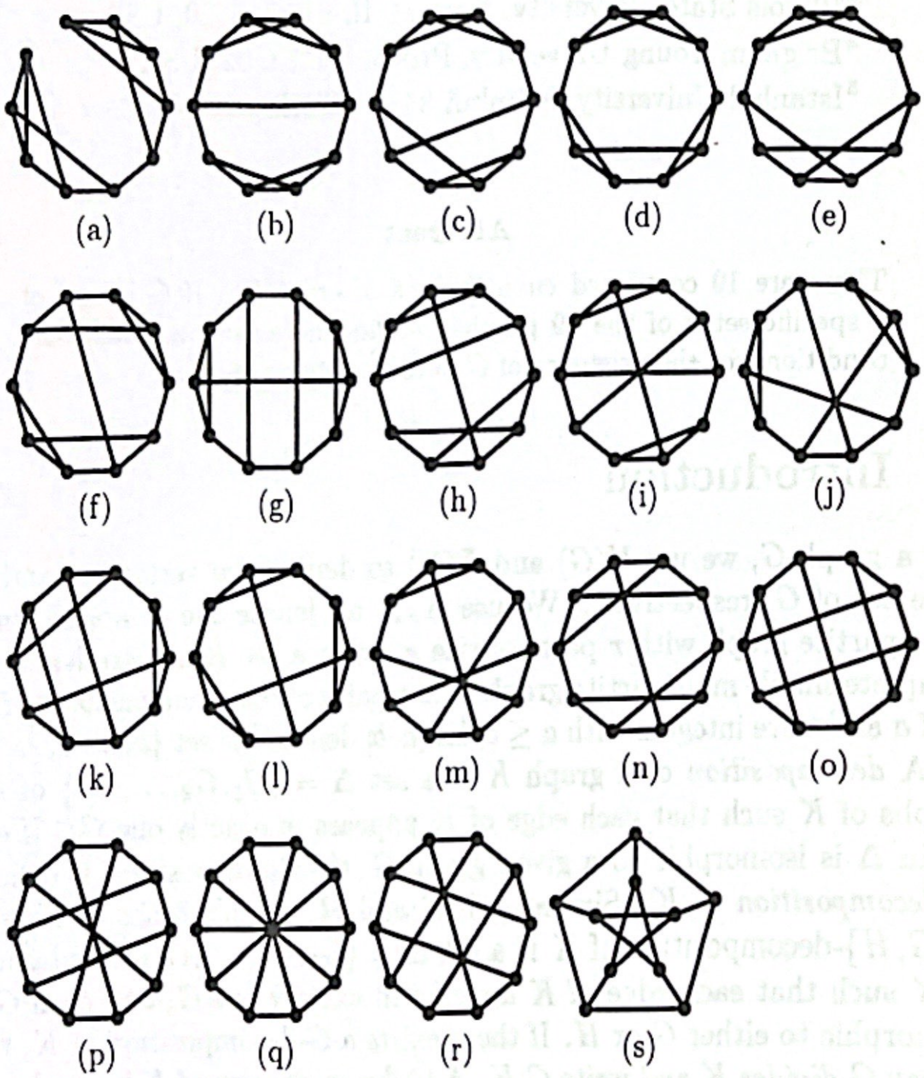

Figure 1: The 19 connected cubic graphs of order 10.

In this work, we are concerned with the spectrum problem for the 3-regular graphs (i.e., *cubic* graphs) of order 10. There are 19 non-isomorphic connected such graphs (see Figure 1).

If G is a cubic graph of order 10 and if there exists a (K_v, G)-design, then we must have $15|\binom{v}{2}$ (since the number of edges in G is 15 and the number of edges in K_v is $\binom{v}{2}$)) and $3|v-1$ (since G is 3-regular and K_v is $(v-1)$-regular). Thus we must have $v \equiv 1$ or 10 (mod 15).

The spectrum problem was settled for the Petersen graph (Graph (s) in Figure 1) in [1]. More recently, the spectrum problem for the 5-Prism and 5-Mobius (Graphs (o) and (q) in Figure 1, respectively) was settled in [16]. It is also known that 5 of the connected cubic graphs of order 10, namely Graphs (b), (n), (o), (q) and (s) in Figure 1, do not decompose K_{10} [3].

In [17], it is shown that if G is a cubic tripartite graph of order 10, then there exists a G-design of order v for all $v \equiv 1$ (mod 30). Thus to settle the spectrum problem for the three graphs in Figure 2, it suffices to settle the cases $v \equiv 10, 16$ and 25 (mod 30).

Several authors have considered (K_v, G)-designs for various cubic graphs G. In [12], Hanani settled the spectrum problem for K_4-designs by showing that there exists a (K_v, K_4)-design if and only if $v \equiv 1$ or 4 (mod 12). The spectrum problem for $K_{3,3}$-designs was settled by Guy and Beineke in [11]. The spectrum problem for the 3-prism was settled by Carter [8]. The spectrum problem for the 3-dimensional cube was settled by Maheo in [15]. More recently, the spectrum problem for the remaining 4 connected cubic graphs of order 8 was settled in [5].

In this work, we will settle the spectrum problem for 3 of the cubic graphs of order 10 by showing that if $v \equiv 1$ or 10 (mod 15) and if $G \in \{G_1, G_{16}, G_{18}\}$ (from Figure 2), then there exists a G-design of order v. Henceforth, each of the graphs in Figure 2, with vertices labeled as in the figure, will be represented by $G_i[v_0, v_1, \ldots, v_9]$. A G-decomposition of a graph with vertex set V may be written as a pair (V, B), where B is a collection of copies of G that partitions the edge-set of the graph.

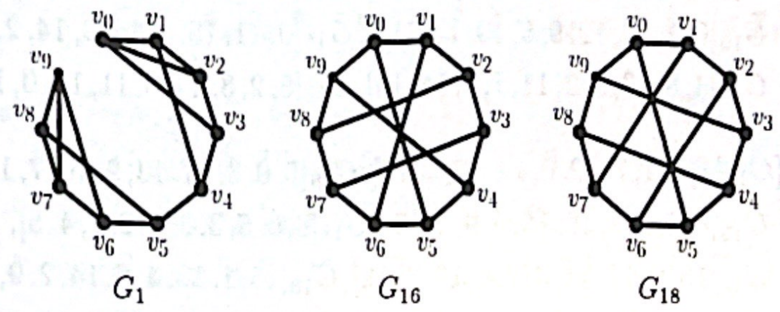

Figure 2: The 3 cubic graphs of order 10 of interest in this work with their vertices labeled.

We first state a well known result on K_4-decompositions of certain complete multipartite graphs as well as a result on $\{K_3, K_5\}$-decompositions of

K_n. We make use of these results in the next section. All of these results can be found in the *Handbook of Combinatorial Designs* [9].

Theorem 1.1 *There exists a K_4-decomposition of $K_{n\times 2}$ if and only if $n \equiv 1 \pmod 3$ and $n \geq 4$.*

Theorem 1.2 *There exists a K_4-decomposition of $K_{n\times 2,5}$ if and only if $n \equiv 0 \pmod 3$ and $n \geq 9$.*

Theorem 1.3 *There exists a $\{K_3, K_5\}$-decomposition of K_n if n is odd.*

2 Main Results

In this section we show that if G is one of the three graphs of order 10 in Figure 2, then there exists a G-decomposition of K_v for all $v \equiv 10, 16$ or $25 \pmod{30}$. Combined with previously known results, this will show that for these 3 graphs, there exists a G-decomposition of K_v if and only if $v \equiv 1$ or $10 \pmod{15}$. We first present several G-decompositions of various graphs. These decompositions are needed for the main construction.

Example 2.1 Let $V = \mathbb{Z}_{16}$ and let

$$\begin{aligned}B_1 = \{&G_1[11,6,0,12,3,15,4,1,9,13], G_1[11,8,13,14,4,3,1,10,5,12],\\ &G_1[14,8,0,9,7,4,6,13,10,15], G_1[0,2,1,5,4,9,3,8,6,10],\\ &G_1[1,7,6,14,10,0,13,2,15,12], G_1[1,8,15,11,5,13,3,2,7,14],\\ &G_1[2,6,5,9,12,4,0,3,11,7], G_1[5,7,15,14,12,8,2,10,9,11]\},\end{aligned}$$

$$\begin{aligned}B_{16} = \{&G_{16}[5,0,1,4,2,12,15,3,7,13], G_{16}[13,11,1,9,7,6,12,3,5,4],\\ &G_{16}[6,1,15,9,11,8,12,4,10,3], G_{16}[0,3,8,5,6,2,1,10,14,4],\\ &G_{16}[0,7,14,12,9,6,10,13,1,8], G_{16}[0,11,15,8,10,9,14,2,7,12],\\ &G_{16}[4,8,13,9,2,11,7,5,14,15], G_{16}[5,2,3,14,6,11,10,0,13,15]\},\end{aligned}$$

$$\begin{aligned}B_{18} = \{&G_{18}[4,8,1,7,12,0,14,13,5,11], G_{18}[5,0,3,11,13,9,10,7,15,2],\\ &G_{18}[6,8,2,3,12,15,0,9,11,1], G_{18}[8,0,6,3,9,7,2,1,4,5],\\ &G_{18}[13,0,11,14,4,7,6,10,2,12], G_{18}[15,1,13,4,6,14,2,9,12,10],\\ &G_{18}[15,4,12,1,10,11,8,3,14,5], G_{18}[7,5,10,8,15,3,13,6,9,14]\}.\end{aligned}$$

Then it can be verified that (V, B_i) is a G_i-decomposition of K_{16} for $i \in \{1, 16, 18\}$.

Example 2.2 Let $V = \{a_b : a \in \mathbb{Z}_5 \text{ and } b \in \{1,2,3,4,5\}\}$. If i is a nonnegative integer, $j \in [0,9]$, $b_j \in \{1,2,3,4,5\}$, and $G \in \{G_1, G_{16}, G_{18}\}$, then $G[(a_0)_{b_0}, (a_1)_{b_1}, \ldots, (a_9)_{b_9}] + i$ means $G[(a_0+i)_{b_0}, (a_1+i)_{b_1}, \ldots, (a_9+i)_{b_9}]$. Let

$$\begin{aligned} B_1 = \{&G_1[0_1,0_2,1_1,2_2,3_2,0_3,2_1,2_3,3_1,0_4]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_1[0_1,1_4,2_1,0_5,1_1,4_2,0_3,3_3,2_3,2_4]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_1[0_1,1_5,1_3,2_5,0_2,0_3,1_2,0_4,1_4,0_5]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_1[0_3,3_5,2_5,4_5,2_4,4_4,1_2,3_4,3_2,1_5]+i : i \in \mathbb{Z}_5\}, \end{aligned}$$

$$\begin{aligned} B_{16} = \{&G_{16}[0_1,2_1,1_2,0_4,4_2,1_1,3_2,1_3,0_3,0_2]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{16}[0_1,1_3,0_2,0_5,0_4,0_3,2_1,1_4,3_4,2_3]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{16}[0_1,1_4,4_3,4_5,3_2,0_4,3_1,1_5,2_3,0_5]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{16}[0_2,2_2,4_3,1_5,2_1,3_5,2_4,3_4,0_5,4_5]+i : i \in \mathbb{Z}_5\}, \end{aligned}$$

$$\begin{aligned} B_{18} = \{&G_{18}[0_2,0_1,2_1,1_2,0_4,2_2,3_2,1_1,4_2,0_3]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{18}[2_3,0_1,1_3,4_2,0_5,0_4,2_1,0_3,2_2,4_4]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{18}[2_4,0_1,1_4,0_5,3_4,4_4,0_3,0_4,1_5,3_1]+i : i \in \mathbb{Z}_5\} \\ &\cup \{G_{18}[4_5,0_1,1_5,3_3,2_3,3_5,1_2,0_5,0_3,2_5]+i : i \in \mathbb{Z}_5\}. \end{aligned}$$

Then it can be verified that (V, B_i) is a G_i-decomposition of K_{25} for $i \in \{1, 16, 18\}$.

Example 2.3 Let $V = \{a_b : a \in \mathbb{Z}_{13} \text{ and } b \in \{1,2,3\}\} \cup \{\infty\}$, and let

$$\begin{aligned} B_1 = \{&G_1[0_1,3_1,1_1,5_1,0_2,2_1,8_1,2_2,1_2,4_2]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_1[0_1,4_2,0_2,5_2,2_1,3_2,10_1,0_3,9_2,1_3]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_1[0_1,2_3,0_3,5_3,4_1,1_3,2_1,8_3,0_2,11_3]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_1[0_2,6_3,0_3,9_3,12_2,11_3,3_1,10_3,8_2,\infty]+i : i \in \mathbb{Z}_{13}\}, \end{aligned}$$

$$\begin{aligned} B_{16} = \{&G_{16}[0_1,2_1,8_1,8_2,0_2,1_1,6_1,2_2,1_2,3_1]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{16}[0_1,2_2,10_1,1_3,5_2,1_2,7_1,11_2,0_3,3_2]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{16}[0_1,1_3,0_2,11_3,5_1,0_3,2_1,7_3,3_2,2_3]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{16}[0_2,5_3,9_1,\infty,1_3,0_3,2_3,9_2,3_3,8_3]+i : i \in \mathbb{Z}_{13}\}, \end{aligned}$$

$$\begin{aligned} B_{18} = \{&G_{18}[3_1,0_1,2_1,0_2,7_2,2_2,6_1,1_1,7_1,4_2]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{18}[4_2,0_1,3_2,11_1,5_3,0_3,8_1,2_2,9_1,2_3]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{18}[2_3,0_1,1_3,2_2,4_3,5_3,2_1,0_3,0_2,12_2]+i : i \in \mathbb{Z}_{13}\} \\ &\cup \{G_{18}[1_3,0_2,6_3,11_2,\infty,4_1,12_3,2_2,7_3,5_3]+i : i \in \mathbb{Z}_{13}\}. \end{aligned}$$

Then it can be verified that (V, B_i) is a G_i-decomposition of K_{40} for $i \in \{1, 16, 18\}$.

Example 2.4 Let $V = \{a_b : a \in \mathbb{Z}_{11} \text{ and } b \in \{1,2,3,4,5\}\}$, and let

$$\begin{aligned}
B_1 = &\{G_1[0_1,3_1,1_1,5_1,0_2,2_1,1_2,8_1,2_2,4_2]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_1,2_2,1_2,0_3,10_1,1_3,2_1,6_3,3_1,8_3]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_1,8_3,7_3,0_4,0_2,4_2,9_2,3_3,0_3,1_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_1,2_4,1_4,3_4,4_1,0_4,3_1,0_5,5_1,1_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_1,0_5,4_4,1_5,7_1,1_4,0_2,0_3,2_2,4_3]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_1,2_5,10_5,3_5,0_2,1_3,6_2,0_4,9_2,4_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_2,7_4,4_4,4_5,10_2,0_5,1_2,1_5,4_2,10_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_3,0_4,5_3,2_4,6_3,5_4,0_5,3_3,7_5,6_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_1[0_4,3_5,5_4,4_5,1_3,10_5,0_3,0_5,4_3,5_5]+i : i \in \mathbb{Z}_{11}\},
\end{aligned}$$

$$\begin{aligned}
B_{16} = &\{G_{16}[0_1,2_1,2_2,4_2,0_2,1_1,6_1,7_2,1_2,3_1]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_1,3_2,9_1,5_3,0_3,2_2,7_1,2_3,1_3,4_2]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_1,1_3,4_1,1_4,0_2,0_3,2_1,0_4,9_3,2_3]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_1,0_4,1_1,6_4,3_2,4_3,0_2,3_3,3_4,1_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_1,4_4,8_1,1_5,1_2,3_4,7_2,1_3,0_5,6_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_1,1_5,0_2,10_5,5_1,0_5,2_1,9_5,6_3,2_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_2,4_4,5_2,2_5,0_4,10_3,8_3,1_4,1_5,6_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_2,3_5,2_3,8_4,0_3,2_5,8_2,6_5,7_4,4_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{16}[0_3,10_4,6_4,2_5,10_5,5_5,3_5,5_3,3_4,6_5]+i : i \in \mathbb{Z}_{11}\},
\end{aligned}$$

$$\begin{aligned}
B_{18} = &\{G_{18}[3_1,0_1,2_1,2_2,8_1,1_2,6_1,1_1,0_2,4_2]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[8_2,0_1,7_2,0_3,4_3,2_3,1_2,2_2,5_2,3_3]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[2_3,0_1,1_3,5_1,1_4,7_3,2_1,0_3,3_1,0_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[6_3,0_1,4_3,7_2,0_4,1_4,1_2,3_3,3_2,2_4]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[2_4,0_1,1_4,9_1,3_5,0_5,7_1,0_4,1_1,1_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[7_5,0_1,6_5,1_2,10_5,7_3,0_2,1_5,2_1,0_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[3_4,0_2,2_4,6_2,3_5,10_3,0_3,1_4,3_2,10_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[3_5,0_2,2_5,0_3,7_5,2_4,5_3,5_4,2_3,1_5]+i : i \in \mathbb{Z}_{11}\} \\
&\cup \{G_{18}[9_5,0_3,6_5,7_5,4_4,4_5,2_4,10_4,6_4,1_4]+i : i \in \mathbb{Z}_{11}\}.
\end{aligned}$$

Then it can be verified that (V, B_i) is an G_i-decomposition of K_{55} for $i \in \{1, 16, 18\}$.

Example 2.5 Let $V = \{a_b : a \in \mathbb{Z}_{17} \text{ and } b \in \{1,2,3,4,5\}\}$, and let

$$\begin{aligned}
B_1 = &\{G_1[0_1,3_1,1_1,5_1,11_1,4_1,0_2,2_1,1_2,2_2]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_1,4_2,1_2,5_2,2_1,8_2,1_1,0_3,13_1,1_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_1,1_3,8_2,2_3,4_1,13_2,2_1,9_3,3_1,11_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_1,10_3,3_3,11_3,15_1,0_4,1_1,2_4,2_1,5_4]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_1,6_4,5_4,7_4,10_1,1_4,6_1,16_4,7_1,0_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_1,1_5,0_5,2_5,3_1,6_5,1_1,8_5,2_1,14_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_2,0_3,5_2,1_3,2_2,8_2,1_2,3_3,16_2,6_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_2,8_3,3_3,9_3,0_4,1_2,7_3,1_4,15_3,5_4]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_2,5_4,0_4,6_4,2_2,3_4,1_2,12_4,5_2,0_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_2,0_5,8_4,1_5,9_1,6_5,1_2,11_4,2_2,8_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_2,2_5,12_4,3_5,5_2,1_4,4_2,0_5,0_3,12_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_3,0_4,4_3,6_4,1_3,5_4,8_3,0_5,10_3,4_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_3,4_5,10_4,5_5,11_2,3_5,1_3,2_5,0_4,16_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_1[0_3,14_5,6_5,16_5,6_3,9_5,9_4,2_4,11_4,15_5]+i : i \in \mathbb{Z}_{17}\},
\end{aligned}$$

$$\begin{aligned}
B_{16} = &\{G_{16}[0_1,2_1,12_1,3_2,9_1,1_1,6_1,1_2,0_2,3_1]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,1_2,5_1,0_3,10_1,0_2,2_1,12_2,8_2,2_2]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,6_2,14_2,6_3,11_2,4_2,1_2,3_3,1_3,0_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,2_3,5_1,0_4,7_1,1_3,3_1,13_3,9_3,3_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,6_3,14_1,1_4,9_2,5_3,5_2,2_3,0_4,8_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,1_4,4_1,13_4,6_1,0_4,2_1,8_4,9_4,2_4]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,0_5,1_1,4_5,2_2,8_4,0_2,3_3,3_5,1_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_1,5_5,12_1,2_5,10_1,4_5,7_1,15_5,7_5,6_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_2,8_3,10_2,3_4,0_4,7_3,0_3,6_2,4_4,10_3]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_2,1_4,2_2,0_5,12_2,0_4,5_2,8_4,6_4,2_4]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_2,3_5,0_3,14_5,8_2,1_5,6_2,0_5,5_3,4_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_2,8_5,6_3,3_4,3_3,7_5,0_3,5_5,1_4,9_5]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_3,2_4,6_3,2_5,9_4,1_4,13_3,4_4,0_5,3_4]+i : i \in \mathbb{Z}_{17}\}\\
&\cup \{G_{16}[0_3,10_5,11_4,11_5,8_4,9_5,4_4,16_5,5_5,15_5]+i : i \in \mathbb{Z}_{17}\},
\end{aligned}$$

$$
\begin{aligned}
B_{18} = & \{G_{18}[3_1,0_1,2_1,9_1,5_2,0_2,6_1,1_1,7_1,2_2]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[1_2,0_1,0_2,9_1,13_2,2_2,10_2,8_1,11_2,14_2]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[9_2,0_1,7_2,3_3,1_1,6_3,0_3,6_2,1_3,8_3]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[4_3,0_1,3_3,7_1,13_3,12_3,4_1,1_3,3_1,0_4]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[11_3,0_1,9_3,1_2,1_4,15_3,0_2,7_3,12_1,0_4]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[2_4,0_1,1_4,6_1,0_5,5_4,2_1,0_4,4_1,13_4]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[11_4,0_1,8_4,2_2,12_4,0_3,0_2,4_4,1_2,3_3]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[2_5,0_1,1_5,6_1,11_5,5_5,2_1,0_5,3_1,10_5]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[10_5,0_1,9_5,2_2,3_4,3_3,0_2,6_5,1_2,6_3]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[7_4,0_2,5_4,7_2,0_5,0_4,3_3,6_3,4_3,2_4]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[14_4,0_2,13_4,3_3,2_5,5_4,4_3,9_4,2_3,0_5]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[2_5,0_2,1_5,6_3,16_5,6_5,2_2,0_5,3_2,14_5]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[12_5,0_2,8_5,0_4,5_5,6_4,2_3,3_5,0_3,16_4]+i : i \in \mathbb{Z}_{17}\} \\
& \cup \{G_{18}[9_5,0_3,11_5,4_4,8_5,10_5,0_4,2_5,10_4,8_4]+i : i \in \mathbb{Z}_{17}\}.
\end{aligned}
$$

Then it can be verified that (V, B_i) is a G_i-decomposition of K_{85} for $i \in \{1, 16, 18\}$.

Example 2.6 Let $V = \{a_b : a \in \mathbb{Z}_5 \text{ and } b \in \{1,2,3,4\}\}$ and let $\{\cup_{j\in[1,4]}\{a_j : a \in \mathbb{Z}_5\}\}$ be a partition of V. Let

$$
\begin{aligned}
B_1 = & \{G_1[0_1,0_3,0_2,1_3,2_1,1_2,3_3,1_1,0_4,4_2]+i : i \in \mathbb{Z}_5\} \\
& \cup \{G_1[0_1,1_4,1_2,3_4,0_3,4_4,2_1,4_2,2_3,2_4]+i : i \in \mathbb{Z}_5\},
\end{aligned}
$$

$$
\begin{aligned}
B_{16} = & \{G_{16}[0_1,1_2,2_3,4_1,0_3,0_2,2_1,1_3,0_4,2_2]+i : i \in \mathbb{Z}_5\} \\
& \cup \{G_{16}[0_1,0_3,2_4,2_3,0_2,0_4,1_2,3_4,4_1,1_4]+i : i \in \mathbb{Z}_5\},
\end{aligned}
$$

$$
\begin{aligned}
B_{18} = & \{G_{18}[2_2,0_1,1_2,2_3,4_1,1_3,2_1,0_2,0_3,0_4]+i : i \in \mathbb{Z}_5\} \\
& \cup \{G_{18}[0_3,0_1,1_4,4_2,3_4,2_2,4_3,0_4,1_1,4_4]+i : i \in \mathbb{Z}_5\}.
\end{aligned}
$$

Then it can be verified that (V, B_i) is a G_i-decomposition of $K_{4\times 5}$ for $i \in \{1, 16, 18\}$.

Example 2.7 Let $V = \{a_b : a \in \mathbb{Z}_{15} \text{ and } b \in \{1,2,3\}\}$ and let $\{\cup_{j\in[1,3]}\{a_j : a \in \mathbb{Z}_{15}\}\}$ be a partition of V. Let

$$\begin{aligned} B_1 = &\{G_1[0_1, 0_3, 0_2, 1_3, 2_1, 1_2, 3_1, 5_2, 4_1, 7_3] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 2_3, 3_2, 6_3, 9_1, 0_2, 6_1, 1_2, 8_1, 13_3] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 10_3, 4_2, 11_3, 2_2, 12_3, 1_2, 11_1, 7_2, 5_3] + i : i \in \mathbb{Z}_{15}\}, \end{aligned}$$

$$\begin{aligned} B_{16} = &\{G_{16}[2_2, 0_1, 13_2, 13_3, 8_2, 0_3, 14_2, 6_1, 7_3, 13_1] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_{16}[0_1, 1_2, 4_3, 2_1, 2_3, 0_2, 4_1, 0_3, 9_1, 3_2] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_{16}[0_2, 6_3, 13_1, 5_2, 0_1, 4_3, 1_1, 0_3, 4_2, 12_3] + i : i \in \mathbb{Z}_{15}\}, \end{aligned}$$

$$\begin{aligned} B_{18} = &\{G_{18}[0_2, 11_3, 12_1, 14_3, 13_2, 11_1, 3_3, 14_2, 5_3, 5_1] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_{18}[3_2, 0_1, 1_2, 5_1, 10_2, 3_3, 2_1, 0_2, 3_1, 8_3] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_{18}[4_3, 0_1, 9_2, 8_3, 10_1, 1_2, 3_3, 8_2, 6_3, 11_1] + i : i \in \mathbb{Z}_{15}\}. \end{aligned}$$

Then it can be verified that (V, B_i) is a G_i-decomposition of $K_{3\times15}$ for $i \in \{1, 16, 18\}$.

Example 2.8 Let $V = \{a_b : a \in \mathbb{Z}_{15} \text{ and } b \in \{1,2,3,4,5\}\}$ and let $\{\cup_{j\in[1,5]}\{a_j : a \in \mathbb{Z}_{15}\}\}$ be a partition of V. Let

$$\begin{aligned} B_1 = &\{G_1[9_5, 11_2, 14_3, 2_4, 5_5, 1_3, 12_4, 6_1, 14_4, 11_5] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[1_2, 11_4, 12_5, 12_4, 8_2, 3_5, 0_2, 6_1, 7_3, 8_5] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[3_4, 13_5, 13_1, 14_5, 5_4, 0_3, 14_2, 7_1, 1_4, 6_5] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[2_1, 7_3, 14_5, 12_3, 7_2, 8_4, 11_3, 4_2, 14_3, 14_1] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[5_4, 6_1, 5_2, 1_3, 14_2, 1_5, 10_2, 4_1, 2_3, 10_5] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 4_3, 0_2, 6_3, 10_1, 5_2, 1_1, 3_2, 2_1, 2_4] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 8_3, 11_2, 2_4, 5_1, 2_2, 4_1, 0_4, 0_3, 13_5] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 4_4, 2_3, 10_4, 12_1, 8_5, 3_2, 12_3, 4_2, 11_4] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_1, 10_5, 7_3, 13_5, 13_3, 5_4, 7_5, 3_1, 11_5, 2_3] + i : i \in \mathbb{Z}_{15}\} \\ &\cup \{G_1[0_2, 5_4, 14_3, 9_4, 7_2, 6_5, 3_1, 6_4, 9_2, 10_5] + i : i \in \mathbb{Z}_{15}\}, \end{aligned}$$

$$B_{16} = \{G_{16}[9_1, 0_5, 13_2, 0_3, 10_4, 4_5, 3_3, 1_4, 6_1, 12_2] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[6_1, 10_5, 14_3, 10_1, 3_2, 9_3, 3_1, 12_2, 0_1, 11_4] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[12_5, 0_1, 5_5, 0_3, 8_1, 14_4, 12_3, 0_2, 4_1, 2_3] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[13_4, 11_2, 12_3, 0_5, 9_2, 2_5, 0_3, 5_2, 11_4, 8_3] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_2, 12_4, 8_5, 14_2, 10_3, 10_1, 13_5, 14_4, 11_2, 1_5] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_1, 1_2, 6_1, 5_5, 7_3, 0_2, 2_1, 2_4, 2_3, 4_2] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_1, 2_3, 7_1, 0_3, 9_4, 1_3, 8_2, 7_5, 1_4, 5_3] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_1, 2_4, 5_1, 5_5, 2_2, 1_4, 3_1, 10_4, 3_5, 3_4] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_2, 7_5, 3_3, 2_5, 0_1, 4_4, 7_3, 2_2, 9_4, 11_5] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{16}[0_3, 7_4, 13_2, 8_4, 0_1, 9_5, 11_2, 0_5, 3_4, 8_5] + i : i \in \mathbb{Z}_{15}\},$$

$$B_{18} = \{G_{18}[11_5, 11_4, 7_5, 12_3, 6_1, 4_3, 3_2, 10_5, 1_2, 5_1] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[11_4, 9_2, 7_3, 13_2, 11_1, 1_5, 3_4, 5_1, 13_3, 0_5] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[1_4, 14_3, 13_5, 8_2, 0_4, 1_1, 14_2, 4_4, 10_5, 11_1] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[7_1, 6_2, 13_3, 10_1, 2_2, 1_4, 8_1, 2_5, 14_1, 7_5] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[12_2, 12_1, 3_4, 2_3, 1_4, 9_5, 9_2, 7_4, 1_3, 5_4] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[6_2, 0_1, 5_2, 12_1, 8_5, 7_4, 0_3, 1_2, 7_1, 6_3] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[4_3, 0_1, 1_3, 4_2, 8_4, 2_5, 4_1, 0_3, 5_1, 1_4] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[4_4, 0_1, 2_4, 11_2, 14_5, 9_3, 6_2, 1_4, 4_1, 6_5] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[7_4, 0_1, 7_5, 9_2, 0_5, 7_2, 11_3, 4_5, 12_4, 14_3] + i : i \in \mathbb{Z}_{15}\}$$
$$\cup \{G_{18}[9_4, 0_3, 4_2, 6_3, 7_5, 6_2, 12_3, 9_5, 3_3, 6_5] + i : i \in \mathbb{Z}_{15}\}.$$

Then it can be verified that (V, B_i) is a G_i-decomposition of $K_{5\times 15}$ for $i \in \{1, 16, 18\}$.

We now give our main constructions.

Lemma 2.9 *Let $G \in \{G_1, G_{16}, G_{18}\}$. There exists a G-decomposition of K_v for all $v \equiv 10 \pmod{30}$.*

Proof. Let x be a nonnegative integer and let $v = 30x+10$. For $v = 10$, the results follow from the paper by Adams, Bryant and Khodkar [3]. For $v = 40$, the results follow from Example 2.3. So we may assume $x \geq 2$.

By Theorem 1.1, there exists a K_4-decomposition of $K_{(3x+1)\times 2}$. Replacing each vertex of $K_{(3x+1)\times 2}$ by a set of 5 vertices and each edge of $K_{(3x+1)\times 2}$ by a copy of $K_{5,5}$ gives a $K_{4\times 5}$-decomposition of $K_{(3x+1)\times 10}$. By Examples 2.6 and 2.1, there exist G-decompositions of $K_{4\times 5}$ and K_{16}. Thus the result now follows. ■

Lemma 2.10 *Let $G \in \{G_1, G_{16}, G_{18}\}$. There exists a G-decomposition of K_v for all $v \equiv 16 \pmod{30}$.*

Proof. Let x be a nonnegative integer and let $v = 30x+16$. For $v = 16$, the result follows from Example 2.1. So we may assume $x \geq 1$.

By Theorem 1.3, there exists a $\{K_3, K_5\}$-decomposition of K_{2x+1}. Replacing each vertex of K_{2x+1} by a set of 15 vertices and each edge of $K_{(2x+1)\times 2}$ by a copy of $K_{15,15}$ gives a $\{K_{3\times 15}, K_{5\times 15}\}$-decomposition of $K_{(2x+1)\times 15}$. Moreover, each group of 15 points along with an additional ∞-vertex constitute a K_{16}. By Examples 2.7, 2.8 and 2.1, there exist G-decompositions of $K_{3\times 15}$, $K_{5\times 15}$ and K_{16}. Thus, we have a G-decomposition of K_{30x+16}. ∎

Lemma 2.11 *Let $G \in \{G_1, G_{16}, G_{18}\}$. There exists a G-decomposition of K_v for all $v \equiv 25 \pmod{30}$.*

Proof. Let x be a nonnegative integer and let $v = 30x + 25$. There exist G-decompositions of K_{16}, K_{55} and K_{85} by Examples 2.1, 2.4 and 2.5, respectively. Thus it remains to consider the case $x \geq 3$.

By Theorem 1.2, there exists a K_4-decomposition of $K_{(3x)\times 2,5}$ for $x \geq 3$. Replacing each vertex of $K_{(3x)\times 2,5}$ by a set of 5 vertices and each edge of $K_{(3x)\times 2,5}$ by a copy of $K_{5,5}$ gives a $K_{4\times 5}$-decomposition of $K_{(3x)\times 10,25}$ for $x \geq 3$. Thus it suffices to find G-decompositions of K_{10}, K_{25}, K_{55}, K_{85} and $K_{4\times 5}$. By [3], there exists a G-decomposition of K_{10}. Moreover, G-decompositions of K_{25}, K_{55}, K_{85} and $K_{4\times 5}$ are given by Examples 2.2, 2.4, 2.5 and 2.6. Thus, we have a G-decomposition of K_{30x+25}. ∎

Combining the previously known results and the results from Lemmas 2.9, 2.10 and 2.11 we obtain the following.

Theorem 2.12 *Let $G \in \{G_1, G_{16}, G_{18}\}$. There exists a G-decomposition of K_v if and only if $v \equiv 1$ or $10 \pmod{15}$.*

As stated previously, there are 19 connected cubic graphs of order 10. The spectrum problem was settled already for the three named ones among them; those being the Petersen graph [1] and both the 5-prism and the 5-Mobius [16]. This manuscript settles for the problem for 3 additional graphs. In a forthcoming manuscript [4], we settled the spectrum for the remaining 13 graphs.

3 Acknowledgements

This research is supported in part by grant number A1659815 from the Division of Mathematical Sciences at the National Science Foundation. Part of this work was done while the second, fourth, and sixth authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

U. Odabaşı acknowledges a grant from the Scientific and Technical Research Council of Turkey (TUBITAK, 2219-International Postdoctoral Research Fellowship Program) for a postdoctoral fellowship.

References

[1] P. Adams and D. Bryant, The spectrum problem for the Petersen graph, *J. Graph Theory* **22** (1996), 175–180.

[2] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[3] P. Adams, D. Bryant, and A. Khodkar, Uniform 3-factorisations of K_{10}, *Congr. Numer.* **127** (1997), 23–32.

[4] P. Adams, S. I. El-Zanati, U. Odabasi, and W. Wannasit, The spectrum problem for the cubic graphs of order 10, forthcoming.

[5] P. Adams, S. I. El-Zanati, and W. Wannasit, The spectrum problem for the cubic graphs of order 8, *Ars Combin.*, to appear.

[6] D. Bryant and S. El-Zanati, "Graph decompositions," *in* Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 477–485.

[7] D. E. Bryant and T. A. McCourt. Existence results for G-designs, http://wiki.smp.uq.edu.au/G-designs/.

[8] J. E. Carter, Designs on Cubic Multigraphs, Ph.D. thesis, McMaster University, Canada (1989)

[9] C. J. Colbourn and J. H. Dinitz, Handbook of Combinatorial Designs, Chapman/CRC Press, Boca Raton, FL (2007).

[10] G. Ge, Group divisible designs, *in* Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 255–260.

[11] R. K. Guy and L. W. Beineke, The coarseness of the complete graph, *Can. J. Math.* **20** (1968), 88–894.

[12] H. Hanani, The existence and construction of balanced incomplete block designs, *Ann. Math. Statist.* **32** (1961), 361–386.

[13] C. Huang, On the existence of balanced bipartite designs, II, Discrete Math. 9 (1974), 147–159.

[14] T. P. Kirkman, On a Problem in Combinatorics, *Cambridge Dublin Math. J.* **2** (1847), 191–204.

[15] M. Maheo, Strongly graceful graphs, *Discrete Math.* **29** (1980), 39–46.

[16] M. Meszka, R. Nedela, A. Rosa, and M. Skoviera, Decompositions of complete graphs into circulants, *Discrete Math.* **339** (2016), 2471–2480.

[17] W. Wannasit and S. I. El-Zanati, On cyclic G-designs where G is a cubic tripartite graph, *Discrete Math.* **312** (2012), 293–305.