The Structure connectivity of Enhanced Hypercube Networks

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Abstract: The n−dimensional enhanced hypercube $Q_{n,k}(1 \le k \le n-1)$ is one of the most attractive interconnection networks for parallel and distributed computing system. Let H be a certain particular connected subgraph of graph G. The H-structure-connectivity of G, denoted by $\kappa(G;H)$, is the cardinality of minimal set of subgraphs $F = \{H_1, H_2, \cdots, H_m\}$ in G such that every $H_i \in F$ is isomorphic to H, and $G - F$ is disconnected. The H-substructure-connectivity of G, denoted by $\kappa^s(G;H)$, is the cardinality of minimal set of subgraphs $F = \{H_1, H_2, \cdots, H_m\}$ in G such that every $H_i \in F$ is isomorphic to a connected subgraph of H, and $G - F$ is disconnected. Using the structural properties of $Q_{n,k}$, the H-structure-connectivity $\kappa(Q_{n,k};H)$ and H-substructure-connectivity of enhanced hypercube $\kappa^s(Q_{n,k};H)$ were determined for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}\}$.

key words: Enhanced hypercubes; Structure connectivity; Substructure connectivity; Fault-toleranct

1 Introduction

Interconnection networks play an important role in parallel and distributed systems. The design of the interconnection network topology significantly determines the performance of the system. An interconnection network topology can be represented by an undirected graph $G = (V, E)$, where each node in V corresponds to a processor, and every edge in E corresponds to a communication link. A lot of interconnection networks have been proposed in the past decades, for example, hypercube Q_n ([1]), folded hypercube FQ_n ([2]), enhanced hypercube $Q_{n,k}$ ([3], [4],[5],[6]), crossed cube, k−ary n−cube ([7]) and Exchanged Folded Hypercubes ([8]).

The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum cardinality

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of a node set $F \subseteq V$ such that $G - F$ is disconnected or a single node. Usually, the parameter $\kappa(G)$ can be used to measure the reliability and fault tolerability of a network. However, in the event of a random node failure, it is very unlikely that all of the nodes adjacent to a single node fail simultaneously. To more accurately measure the fault tolerance of a network, Harary ([9]) introduced the concept of conditional connectivity by attaching some condition on remaining components. [10] generalized the notion of connectivity by introducing $h-$ restricted connectivity which requires each component having the minimum degree at least h. [11] proposed the concept of $g-$extra connectivity. The $g-$extra connectivity of a graph G, denoted by $\kappa_g(G)$, is the minimum number of vertex cut F of G such that $G - F$ is disconnected and each component has at least $g + 1$ vertices.

So far, most efforts on the reliability and fault-tolerance of networks have payed on the effect of individual nodes becoming faulty, that is, one often assumed that the states of a node v, with regarded to the nodes around v, is an independent event. However in reality, nodes that are connected could affect each other, and the neighbors of a faulty node might be more likely to become faulty. With the development of technology, networks and subnetworks are made into chips. When any nodes on the chip fail to work, the whole chip could be considered failure. All these motivate the study of fault-tolerance of networks from the perspective of some structures instead of considering the effect of nodes becoming faulty. Inspired by this situation, Lin et al. ([12]) brought forward the concept of structure connectivity and substructure connectivity of graphs.

Let H be a connected subgraph of G, and F be a set of subgraph of G such that each element in F is isomorphic to H. Then F is called a $H-$structure-cut if $G - F$ is disconnected. The $H-$structure connectivity $\kappa(G; H) = \min\{|F| : F \text{ is } H\text{-structure-cut}\}$. Similarly, Let F be a set of subgraph of G such that every element of F is isomorphic to a connected subgraph of H. Then F is called a H-substructure-cut if $G - F$ is disconnected. The H-substructure connectivity, $\kappa^s(G; H) = \min\{|F| : F \text{ is } H - \text{substructure-cut}\}$. The definition implies $\kappa^s(G; H) \leq \kappa(G; H)$. Certainly, K_1-structure connectivity and K_1-substructure connectivity are the classical connectivity. Paper [12] established the $H-$structure-connectivity and $H-$substructure-connectivity for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$ in an $n-$dimensional hypercube. Note that K_1 is just a single node. $K_{m,n}$ is complete bipartite graph and C_4 is a 4-cycle. Sabir and Meng ([13]) determined the $\kappa^s(G; H)$ and $\kappa(G; H)$ with $H \in \{P_k, C_{2k}, K_{1,3}\}$ in an $n-$dimensional hypercube and folded hypercube FQ_n. Lv at al. ([7]) discussed the Hamilton cycle and path embedding problems in $k-$ary $n-$cubes based on structure faults. In this paper, we investigate structure fault tolerance for the enhanced hypercube $Q_{n,k}$.

The remainder of this paper is organized as follows. Section 2 gives the basic definitions and existing results in literature which will be used in our discussion. The structure connectivity $\kappa(Q_{n,k}; H)$ and substructure connectivity $\kappa^s(Q_{n,k}; H)$ for enhanced hypercube and $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}\}$ are presented in Section 3. Finally, some concluding remarks are given in Section 4.

2 Preliminaries

A network is usually modeled by a connected graph $G = (V, E)$. The set of vertices $N_G(v) = \{u : uv \in E\}$ is called the neighbor set of vertex v in G, that is, the set of adjacent nodes of v. $d(v) = |N_G(v)|$ is called the degree of vertex v in G when no loop occurs. Let $A \subsetneq V$, $N_G(A)$ denotes the vertex set $\bigcup_{v\in A} N_G(v) \setminus A$ and $C_G(A) = N_G(A) \bigcup A$. If each vertex is adjacent to k vertices, the graph G is called k−regular. A path is a sequence of adjacent vertices, with the original vertex v_1 and end vertex v_m, represented as $P_m = v_1v_2 \ldots v_m$, where all the vertices $v_1, v_2, \ldots, v_m$ are distinct except the case that the path is a cycle C_m where $v_1 = v_m$. The length of a path P is defined as the number of edges contained in P. The length of a shortest cycle is defined as the girth of graph G, denoted by $g(G)$. G is bipartite if the vertex set V can be partitioned into two subsets V_1 and V_2, such that every edge in G joins a vertex in V_1 with a vertex in V_2. A graph G is bipartite if and only if G contains no odd cycle. Two graphs G_1 and G_2 are isomorphic, denoted as $G_1 \cong G_2$, if there is a one to one mapping f from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$. For a subgraph H of G, $G - H$ denote the subgraph of G induced by $V(G) - V(H)$. Let $\theta_G(m)$ denote the minimum number of vertices that are adjacent to a vertex set of m vertices in graph G.

An n-dimensional hypercube, denoted by Q_n, has 2^n vertices represented by the vertex set $V(Q_n) = \{x_1x_2 \cdots x_n : x_i = 0 \text{ or } 1, 1 \le i \le n\}$, where two vertices $x_1x_2 \cdots x_n$ and $y_1y_2 \cdots y_n$ are adjacent if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$. We set $x^i = x_1x_2 \cdots x_{i-1}\bar{x}_ix_{i+1} \cdots x_n$ to denote the neighbor of $x = x_1x_2 \cdots x_{i-1}x_ix_{i+1} \cdots x_n$ in dimension i. Similarly, $x^{i,j}$ is the neighbor of x^i in dimension j. Certainly, $x^{i,j} = x^{j,i}$.

Enhanced hypercube $Q_{n,k} = (V, E)$ for $1 \le k \le n - 1$ is an undirected simple graph with the vertices set V (or $V(Q_{n,k})$)and the edge set of E (or $E(Q_{n,k})$). $V = \{x_1x_2 \cdots x_n : x_i = 0, \text{or } 1, 1 \le i \le n\}$, in fact, $V(Q_n) = V(Q_{n,k})$. Two vertices $x = x_1x_2 \cdots x_n$ and y are connected by an edge of E if and only if y satisfies one of the following two conditions:

(1) $y = x^i = x_1x_2 \cdots x_{i-1}\bar{x}_ix_{i+1} \ldots x_n$, $1 \le i \le n$, or

(2) $y = \bar{x} = x_1x_2 \cdots x_{k-1}\bar{x}_k\bar{x}_{k+1} \ldots \bar{x}_n$, where $\bar{x}_i = 1 - x_i$.

$x\bar{x}$ is called a complementary edge of $Q_{n,k}$. $E_c = \{x\bar{x} : x \in V\}$ the set of complementary edges, $E_i = \{xx^i : x \in V\}$ the set of all i–dimensional edges. Thus we have $E(Q_{n,k}) = E(Q_n) \cup E_c$.

According to the above definition, $Q_{n,k}$ is $(n + 1)$-regular and has 2^n vertices and $(n + 1)2^{n-1}$ edges, and it contains Q_n as its subgraph. It has 2^{n-1} more links than Q_n. If $k = 1$, $Q_{n,k}$ is the well-known folded hypercube denoted by FQ_n. If $k = n$, $Q_{n,k}$ reduce to Q_n. This paper mainly consider $2 \le k \le n - 1$.

When n and k have the same parity, $Q_{n,k}$ is a bipartite graph with containing no odd cycle , for example, $Q_{4,2}$ is a bipartite graph with bipartition $V_1 = \{x = x_1x_2x_3x_4 : \sum_{i=1}^{4} |x_i| \text{ is even}, x \in V\}$ and $V_2 = \{x = x_1x_2x_3x_4 : \sum_{i=1}^{4} |x_i| \text{ is odd}, x \in V\}$. When n and k have different parity, $Q_{n,k}$ is not bipartite graph.

With the definition, we can partition $Q_{n,k}$ into two subgraphs along some component $i(1 \leq i \leq n)$. We use $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ to denote the two subgraphs respectively. For convenience, $Q_{n,k}$ can be expressed as $Q^{i0}_{n-1,k} \uplus Q^{i1}_{n-1,k}$. From the definition of the partition, we can conclude that when $i < k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are isomorphic to $(n-1)$-dimensional enhanced hypercube; when $i \geq k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are isomorphic to $(n-1)$-dimensional hypercube. A vertex $x = x_1x_2\cdots x_n$ belong to $Q^{i0}_{n-1,k}$ if and only if the i-th position $x_i = 0$; Similarly, x belongs to $Q^{i1}_{n-1,k}$ if and only if the i-th position $x_i = 1$.

3 Main Results

The following lemmas are benefit for us.

Lemma 1

(i) ([14]) If $n \geq 4$, then

$$\kappa_g(Q_n) = \begin{cases} (g+1)n - 2g - \binom{g}{2}, & 0 \leq g \leq n-4, \\ \dfrac{n(n-1)}{2}, & n-3 \leq g \leq n. \end{cases}$$

(ii) ([15], [16]) Let W be a vertex of $V(Q_n)$ with $|W| = m$, then

$$\theta_{Q_n}(m) = \begin{cases} -\dfrac{1}{2}m^2 + (n - \dfrac{1}{2})m + 1, & 1 \leq m \leq n+1, \\ -\dfrac{1}{2}m^2 + (2n - \dfrac{3}{2})m - n^2 + 2, & n+2 \leq m \leq 2n. \end{cases}$$

Lemma 2 Let x, y be any two vertices in $Q_{n,k}$ for $n \geq 4$, then one of the following holds.

(i) $x, y \in V(Q_{n,k})$ for $2 \leq k \leq n-4$, then x and y have exact two common neighbors if they have.

(ii) $x, y \in V(Q_{n,n-3})$

If $\overline{x} \in N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x, y have exact two common neighbors.

If $\overline{x} \notin N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x and y have exact two common neighbors if they have.

(iii) $x, y \in V(Q_{n,n-2})$

$$\begin{cases} x = x_1 \cdots x_i \cdots x_j \cdots x_n, \\ y \neq x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n, \end{cases}$$

where $\{i, j\} \subseteq \{n-2, n-1, n\}$, then x and y have exact two common neighbors if they have.

(iv) $x, y \in V(Q_{n,n-2})$

$$\begin{cases} x = x_1 \cdots x_i \cdots x_j \cdots x_n, \\ y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n, \end{cases}$$

where $\{i, j\} \subseteq \{n-2, n-1, n\}$, then x and y have exact four common neighbors $\{x^{n-2}, x^{n-1}, x^n, \overline{x}\}$.

(v) $x, y \in V(Q_{n,n-1})$

If $y \notin \{x^{n-1}, x^n\}$, then x, y have exact two common neighbors if they have.

If $y \in \{x^{n-1}, x^n\}$, then x, y have exact two common neighbors $\{\overline{x}, \overline{y}\}$.

Proof: Let $x = x_1 \cdots x_i \cdots x_j \cdots x_n$, $y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$. Then x, y have some common neighbors.

(i) x and y have exact two common neighbors $x^i = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_j \cdots x_n$ and $x^j = x_1 \cdots x_i \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$.

(ii) $x, y \in V(Q_{n,n-3})$

If $\overline{x} \in N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, by lemma 4, there exists a smallest odd cycle with length 5 passing through vertices x and y. Now x, y have exact two common neighbors $\overline{x}$ and $\overline{y}$.

If $\overline{x} \notin N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x and y have exact two common neighbors if they have, the proof is similar to (i).

(iii) The proof is similar to (i).

(iv) $x, y \in Q_{n,n-2}$, then $\overline{x} = x_1x_2 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n$. x and y have exact four common neighbors x^{n-2}, x^{n-1}, x^n and $\overline{x}$.

If $y = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n$, there are four internal disjoint paths of length two between x and y as follows.

$$x \to x^{n-2} = x_1 \cdots \overline{x}_{n-2}x_{n-1}x_n \to x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$
$$x \to x^{n-1} = x_1 \cdots x_{n-2}\overline{x}_{n-1}x_n \to x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$
$$x \to x^n = x_1 \cdots x_{n-2}x_{n-1}\overline{x}_n \to \overline{x^n} = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$
$$x \to \overline{x} = x_1 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n \to (\overline{x})^n = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$

If $y = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n$, the four paths of length two as:

$$x \to x^{n-2} = x_1 \cdots \overline{x}_{n-2}x_{n-1}x_n \to \overline{x^{n-2}} = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$
$$x \to x^{n-1} = x_1 \cdots x_{n-2}\overline{x}_{n-1}x_n \to x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$
$$x \to x^n = x_1 \cdots x_{n-2}x_{n-1}\overline{x}_n \to x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$
$$x \to \overline{x} = x_1 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n \to (\overline{x})^{n-2} = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$

If $y = x_1 \cdots \overline{x}_{n-2}x_{n-1}\overline{x}_n$, the four paths with length two as:

$$x \to x^{n-2} = x_1 \cdots \overline{x}_{n-2}x_{n-1}x_n \to x_1 \cdots \overline{x}_{n-2}x_{n-1}\overline{x}_n = y$$
$$x \to x^{n-1} = x_1 \cdots x_{n-2}\overline{x}_{n-1}x_n \to \overline{x^{n-1}} = x_1 \cdots \overline{x}_{n-2}x_{n-1}\overline{x}_n = y$$
$$x \to x^n = x_1 \cdots x_{n-2}x_{n-1}\overline{x}_n \to x_1 \cdots \overline{x}_{n-2}x_{n-1}\overline{x}_n = y$$
$$x \to \overline{x} = x_1 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n \to (\overline{x})^{n-1} = x_1 \cdots \overline{x}_{n-2}x_{n-1}\overline{x}_n = y$$

(v) $x, y \in V(Q_{n,n-1})$, then $\overline{x} = x_1x_2\cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n$.

If $y \notin \{x^{n-1}, x^n\}$, the proof is similar to (i) and x, y have exact two common neighbors.

If $y = x^{n-1} = x_1\cdots x_{n-2}\overline{x}_{n-1}x_n$ (or $y = x^n = x_1\cdots x_{n-1}\overline{x}_n$), then $\overline{y} = x^n$ (or $\overline{y} = x^{n-1}$), Therefore x, y have exact two common neighbors $\{\overline{x}, \overline{y}\}$.

The proof is finished.

Lemma 2 means that if two vertices x, y have some common neighbors, then number of common neighbors is two or four when $k = n-2$, otherwise number of common neighbors is two.

Lemma 3 Let P_m be a path in $Q_{n,k}$ for $n \geq 5, 1 \leq k \leq n-2, 1 \leq m \leq n$. If $v \in Q_{n,k} - P_m$, then $|N_{Q_{n,k}}(v) \bigcap V(P_m)| \leq \lceil \frac{m}{2} \rceil$.

Proof: By the definition of $Q_{n,k}$, when $k \neq n-1$, the girth $g(Q_{n,k}) = 4$, then $Q_{n,k}$ cannot contain triangle. Therefore, v can be adjacent to at most one vertex of any two consecutive vertices on P_m. The lemma is true.

Next, we consider the structure connectivity and substructure connectivity.

From the definition, we have $\kappa(Q_{n,k}; K_1) = n+1$ and $\kappa^s(Q_{n,k}; K_1) = n+1$ for $n \geq 3, 1 \leq k \leq n-1$. Hence, we just discuss $H \in \{K_{1,1}, K_{1,2}, K_{1,3}\}$

Lemma 4 $\kappa(Q_{n,k}; K_{1,1}) \leq n$ and $\kappa^s(Q_{n,k}; K_{1,1}) \leq n$ for $n \geq 5, 2 \leq k \leq n-3$.

Proof: We set $u = 00\cdots0, v = u^1, uv \in E(Q_{n,k})$ and $F = \{\{u^i, v^i\} : 2 \leq i \leq n\} \bigcup \{\overline{u}, \overline{v}\}$. Then F forms a $K_{1,1}$-structure-cut of $Q_{n,k}$ and $|F| = n$. Thus, $\kappa(Q_{n,k}; K_{1,1}) \leq n$ and $\kappa(^sQ_{n,k}; K_{1,1}) \leq n$

Lemma 5 $\kappa^s(Q_{n,k}; K_{1,1}) \geq n$ and $\kappa(Q_{n,k}; K_{1,1}) \geq n$ for $n \geq 5, 2 \leq k \leq n-3$.

Proof: By contradiction. Let $F = \{\overbrace{K_1, K_1, \cdots, K_1}^{l}, \overbrace{K_{1,1}, K_{1,1}, \cdots, K_{1,1}}^{t}\}$ and $|F| = l + t < n$. Suppose that $Q_{n,k} - F$ is not connected and has at least two components. Let W be a smallest component of $Q_{n,k} - F$. According to the order of W, we consider two cases.

Case 1. $|V(W)| = 1$

Let $V(W) = \{u\}$, then $|N_{Q_{n,k}}(u)| = n+1$. Therefore, $Q_{n,k} - F$ cannot isolate the vertex u. This implies that $|F| \leq n-1$ is impossible .

Case 2. $|V(W)| \geq 2$

Because of Q_n being a spanning subgraph of $Q_{n,k}$ and lemma 1, we have $\kappa_1(Q_{n,k}) \geq \kappa_1(Q_n) = 2n-2$. This means that we have to delete more than $2n-2$ nodes to isolate the component W in $Q_{n,k}$. But $|F| < n$. It is impossible. Therefore, $\kappa^s(Q_{n,k}; K_{1,1}) \geq n$ and $\kappa(Q_{n,k}; K_{1,1}) \geq n$.

In words, lemma 5 is true.

Lemma 4 and lemma 5 lead to the theorem 6.

Theorem 6 $\kappa(Q_{n,k}; K_{1,1}) = n$ and $\kappa^s(Q_{n,k}; K_{1,1}) = n$ for $n \geq 5$.

Lemma 7 $\kappa(Q_{n,k}; K_{1,2}) \leq \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,2}) \leq \lceil \frac{n+1}{2} \rceil$ for $n \geq 5, 2 \leq k \leq n-3$.

Proof: Let $n + 1 = 2t + r, 0 \leq r \leq 1$. Set $u = 00\cdots0$. According to the value of r, we distinguish two cases.

Case 1. $r = 0$.

$$F_1 = \{u^1, u^{1,2}, u^2\}$$
$$F_2 = \{u^3, u^{3,4}, u^4\}$$
$$\vdots$$
$$F_t = \{u^n, \bar{u}^n, \bar{u}\}$$

Then $F = \{F_1, F_2, \cdots F_t\}$ forms a $K_{1,2}$-structure-cut of $Q_{n,k}$, and $|F| = \lceil \frac{n+1}{2} \rceil$.

Case 2. $r = 1$.

$$F_1 = \{u^1, u^{1,2}, u^2\}$$
$$F_2 = \{u^3, u^{3,4}, u^4\}$$
$$\vdots$$
$$F_t = \{u^{n-1}, u^{n-1,n}, u^n\}$$
$$F_{t+1} = \{\bar{u}, \bar{u}^n, \bar{u}^1\}$$

Then $F = \{F_1, F_2, \cdots F_t, F_{t+1}\}$ forms a $K_{1,2}$-structure-cut of $Q_{n,k}$, and $|F| = \lceil \frac{n+1}{2} \rceil$.

Consequently, $\kappa(Q_{n,k}; K_{1,2}) \le \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,2}) \le \lceil \frac{n+1}{2} \rceil$.

Lemma 8 $\kappa^s(Q_{n,k}; K_{1,2}) \ge \lceil \frac{n+1}{2} \rceil$ and $\kappa(Q_{n,k}; K_{1,2}) \ge \lceil \frac{n+1}{2} \rceil$ for $n \ge 5, 2 \le k \le n-3$.

Proof: By contradiction.

Let $F = \{\overbrace{K_1, K_1, \cdots, K_1}^{l}, \overbrace{K_{1,1}, K_{1,1}, \cdots, K_{1,1}}^{t}, \overbrace{K_{1,2}, K_{1,2}, \cdots, K_{1,2}}^{m}\}$ and $|F| = l + t + m \le \lceil \frac{n+1}{2} \rceil - 1$. Suppose that $Q_{n,k} - F$ is not connected and has at least two components. Let W be a smallest component of $Q_{n,k} - F$. According to the order of W, we consider two cases.

Case 1. $|V(W)| = 1$

Let $V(W) = \{u\}$, then $N_{Q_{n,k}}(u) = n + 1$. With lemma 3, every element in F contains at most 2 neighbors of u. Thus we have t delete at least $\lceil \frac{n+1}{2} \rceil$ elements of F to isolate W. It is impossible since $|F| \le \lceil \frac{n+1}{2} \rceil - 1 < \lceil \frac{n+1}{2} \rceil$.

Case 2. $|V(W)| \ge 2$

By lemma 1, $\kappa_1(Q_n) = 2n - 2$ and Q_n being a spanning subgraph of $Q_{n,k}$, then $\kappa_1(Q_{n,k}) \ge \kappa_1(Q_n) = 2n - 2$. This means that we have to delete more than $2n - 2$ nodes to isolate W. But $|F| \le \lceil \frac{n+1}{2} \rceil - 1$ leads to $|V(F)| \le 3(\lceil \frac{n+1}{2} \rceil - 1) \le \frac{3n}{2} < 2n-2$. Therefore, $\kappa^s(Q_{n,k}; K_{1,2}) \ge \lceil \frac{n+1}{2} \rceil$ and $\kappa(Q_{n,k}; K_{1,2}) \ge \lceil \frac{n+1}{2} \rceil$.

Lemma 7 and lemma 8 guarantee the following theorem.

Theorem 9 $\kappa(Q_{n,k}; K_{1,2}) = \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,2}) = \lceil \frac{n+1}{2} \rceil$ for $n \ge 5, 2 \le k \le n-3$.

Lemma 10 $\kappa(Q_{n,k}; K_{1,3}) \le \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,3}) \le \lceil \frac{n+1}{2} \rceil$ for $n \ge 5, 2 \le k \le n-2$.

Proof: We set $u = 00\cdots 0$ being a node of $Q_{n,k}$.

When $n = 2t$ is even, then we set

$$
\begin{aligned}
F_1 &= \{u^1, u^2, u^{1,2}, u^{1,2,3}\} \\
F_2 &= \{u^3, u^4, u^{3,4}, u^{3,4,5}\} \\
&\vdots \\
F_{t-1} &= \{u^{n-3}, u^{n-2}, u^{n-3,n-2}, u^{n-3,n-2,n-1}\} \\
F_t &= \{u^{n-1}, u^n, u^{n-1,n}, u^{n-1,n,1}\} \\
F_{t+1} &= \{\bar{u}, \bar{u}^1, \bar{u}^2, \bar{u}^3\}
\end{aligned}
$$

When $n = 2t+1$ is odd, for $1 \le i \le t$ and $k \ne n-2$, we set

$$
\begin{aligned}
F_i &= \{u^{2i-1}, u^{2i}, u^{2i-1,2i}, u^{2i-1,2i,2i+1}\} \\
F_{t+1} &= \{u^n, \bar{u}, \bar{u}^n, \bar{u}^{n,1}\}
\end{aligned}
$$

If $k = n-2$, in $Q_{n,n-2}$, $\bar{u} = u^{2t+1,2t,2t+1}$, then select $F_{t+1} = \{u^n, u^{n,1}, u^{n,2}, u^{n,3}\}$.

In words, $|F| = t+1 = \lceil \frac{n+1}{2} \rceil$ and each element of F is isomorphic to $K_{1,3}$. Since $N(u) \subseteq V(F)$, and $|V(F)| = 4\lceil \frac{n+1}{2} \rceil < 2^n - 1$. $Q_{n,k} - F$ is disconnected and one component of it is $\{u\}$. Hence $\kappa(Q_{n,k}; K_{1,3}) \le \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,3}) \le \lceil \frac{n+1}{2} \rceil$.

Lemma 11 $\kappa^s(Q_{n,k}; K_{1,3}) \ge \lceil \frac{n+1}{2} \rceil$ and $\kappa(Q_{n,k}; K_{1,3}) \ge \lceil \frac{n+1}{2} \rceil$ for $n \ge 5, 2 \le k \le n-2$.

Proof: By contradiction. Let

$$F = \{\overbrace{K_1, K_1, \cdots, K_1}^{l}, \overbrace{K_{1,1}, K_{1,1}, \cdots, K_{1,1}}^{t}, \overbrace{K_{1,2}, K_{1,2}, \cdots, K_{1,2}}^{m}, \overbrace{K_{1,3}, K_{1,3}, \cdots, K_{1,3}}^{h}\}$$

and $|F| = l + t + m + h \le \lceil \frac{n+1}{2} \rceil - 1$. Suppose that $Q_{n,k} - F$ is not connected and has at least two components. Let W be a smallest component of $Q_{n,k} - F$. We consider the following cases.

Case 1. $|V(W)| = 1$

Let $V(W) = \{u\}$. Because $N_{Q_{n,k}}(u) = n+1$, the girth of $Q_{n,k}$ is 4 and lemma 3, then each element of F contains at most 2 neighbors of u. This leads that we have to delete at least $\lceil \frac{n+1}{2} \rceil$ elements of F to isolate W. It is impossible since $|F| \le \lceil \frac{n+1}{2} \rceil - 1 < \lceil \frac{n+1}{2} \rceil$.

Case 2. $|V(W)| = 2$

Since W is a connected component and $|V(W)| = 2$ and $Q_{n,k}$ is $n+1$-regular, we have to delete at least $2n$ nodes to isolate W. However from the assumption $|F| \le \lceil \frac{n+1}{2} \rceil - 1$, which infers to $|V(F)| \le 4(\lceil \frac{n+1}{2} \rceil - 1) \le 4(\frac{n+1}{2} - 1) = 2n - 2 < 2n$ is wrong when n is even.

Case 3. $|V(W)| \ge 3$

Without loss of generality, we assume that $\{x, y, z\} \subseteq V(W)$ and xy, yz are edges of $Q_{n,k}$, then $N_{Q_{n,k}}(\{x, y, z\}) \subseteq V(F)$. Thus $|V(F)| \geq N_{Q_{n,k}}(\{x, y, z\}) \geq 3n - 1 > 4(\lceil \frac{n+1}{2} \rceil - 1)$. It is a contradiction since $|V(F)| \leq \lceil \frac{n+1}{2} \rceil - 1$.

Lemma 10 and lemma 11 conduct the following theorem 12.

Theorem 12 $\kappa(Q_{n,k}; K_{1,3}) = \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(Q_{n,k}; K_{1,3}) = \lceil \frac{n+1}{2} \rceil$ for $n \geq 5, 2 \leq k \leq n-2$.

The following examples provide some $K_{1,3}$-structure-cut in $Q_{7,3}$ and $Q_{7,5}$, respectively. We select $u = 0000000$ and $F = \{F_1, F_2, F_3, F_4\}$ as follows.

In $Q_{7,3}, t = 3$:

$F_1 = \{u^1 = 1000000, u^2 = 0100000, u^{1,2} = 1100000, u^{1,2,3} = 1110000\}$

$F_2 = \{u^3 = 0010000, u^4 = 0001000, u^{3,4} = 0011000, u^{3,4,5} = 0011100\}$

$F_3 = \{u^5 = 0000100, u^6 = 0000010, u^{5,6} = 0000110, u^{5,6,7} = 0000111\}$

$F_4 = \{u^7 = 0000001, \bar{u} = 0011111, \bar{u}^7 = 0011110, \bar{u}^{7,1} = 1011110\}$

In $Q_{7,5}, t = 3$, the F_1, F_2, F_3 are same as that in $Q_{7,3}$, but $F_4 = \{u^n, u^{n,1}, u^{n,2}, u^{n,3}\}$ is different from that one in $Q_{7,3}$ which is $F_4 = \{u^n, \bar{u}, \bar{u}^n, \bar{u}^{n,1}\}$, otherwise some node will occur repeat.

$F_4 = \{u^7 = 0000001, u^{7,1} = 1000001, u^{7,2} = 0100001, u^{7,3} = 0010001\}$

4 Conclusion

This paper has performed the structural analysis of $Q_{n,k}$ and identified a number of good features, which then established the basis block of structural fault tolerance. For the enhanced hypercube $Q_{n,k}$ and $H \in \{K_{1,1}, K_{1,2}, K_{1,3}\}$ we determined $\kappa(Q_{n,k}; H)$ and $\kappa^s(Q_{n,k}; H)$ for $n \geq 5, 2 \leq k \leq n-3$. One may investigate the structure connectivity and substructure connectivity of other networks.

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