The Transitivity of Special Graph Classes

[1,2]Teresa W. Haynes, [3]Jason T. Hedetniemi,
[4]Stephen T. Hedetniemi, [5]Alice McRae, and [5]Nicholas Phillips

[1]Department of Mathematics
East Tennessee State University
Johnson City, TN 37614-0002 USA
Email: haynes@etsu.edu

[2]Department of Mathematics
University of Johannesburg
Auckland Park, South Africa

[3]Department of Mathematics
Wingate University
Wingate, North Carolina 28174 USA
Email:j.hedetniemi@wingate.edu

[4]Professor Emeritus
School of Computing
Clemson University
Clemson, SC 29634 USA
Email: hedet@clemson.edu

[5]Department of Computer Science
Appalachian State University
Boone, NC 28608 USA
Email: mcraeaa@appstate.edu
Email: nichkphillips@gmail.com

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Abstract

Let $G = (V, E)$ be a graph. The *transitivity* of a graph G, denoted $Tr(G)$, equals the maximum order k of a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of V such that for every i, j, $1 \leq i < j \leq k$, V_i dominates V_j. We consider the transitivity in many special classes of graphs, including cactus graphs, coronas, Cartesian products, and joins. We also consider the effects of vertex or edge deletion and edge addition on the transivity of a graph.

We dedicate this paper to the memory of Professor Bohdan Zelinka for his pioneering work on domatic numbers of graphs.

1 Introduction

Analogous to the chromatic number with regard to vertex partitions of a graph, the domatic number was defined by Cockayne and Hedetniemi [3] in 1977 as the maximum order of a partition of the vertices of a graph into dominating sets. In 2017, J. T. Hedetniemi and S. T. Hedetniemi [4] generalized the concept of domatic number to the transitivity of a graph. In this paper, we continue the study of graph transitivity. We begin with some terminology.

Let $G = (V, E)$ be a graph of *order* $n = |V|$. The *open neighborhood* of a vertex $v \in V$, denoted $N(v)$, is the set of vertices u that are adjacent to v. That is, $N(v) = \{u \mid uv \in E\}$. The *degree* of a vertex v is $\deg(v) = |N(v)|$. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. We say that a graph G is *k-regular* if every vertex in G has $\deg(v) = k$.

The *closed neighborhood* of a vertex $v \in V$, denoted $N[v]$, is the set defined by $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is the set defined by $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set defined by $N[S] = \bigcup_{v \in S} N[v]$.

A set S is a *dominating set* of a graph G if $N[S] = V$, that is, if for every $v \in V$ either $v \in S$ or $v \in N(u)$ for some vertex $u \in S$. The minimum cardinality of a dominating set in a graph G is called the *domination number* and is denoted $\gamma(G)$. Given two disjoint sets of vertices $R, S \subset V$, we say that R *dominates* S, denoted $R \rightarrow S$, if $S \subseteq N(R)$, or equivalently, if every vertex in S is adjacent to at least one vertex in R.

As previously mentioned, the *domatic number* of a graph G, denoted $d(G)$, is the maximum order k of a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$

such that for every i, $1 \leq i \leq k$, V_i is a dominating set of G. To date over 100 papers have been published on the domatic number and its variations. For examples, cf. [1, 7, 8, 9].

The *transitivity of a graph* $Tr(G)$ was defined in [4] to be the maximum order k of a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that for all i, j, $1 \leq i < j \leq k$, $V_i \to V_j$. A vertex partition of V meeting this condition is called a *transitive partition* and a transitive partition of G having order $Tr(G)$ is called a *Tr-partition* of G.

The transitivity $Tr(G)$ of a graph G is an immediate generalization of the *Grundy number* $\Gamma(G)$ of a graph, as originally defined in 1979 by Christen and Selkow [2]. A set $S \subset V$ of vertices is called *independent* if no two vertices in S are adjacent. A *Grundy coloring* is a vertex partition $\{V_1, V_2, \ldots, V_k\}$ such that (i) for $1 \leq i \leq k$, V_i is an independent set, and (ii) for all i, j, $1 \leq i < j \leq k$, $V_i \to V_j$. Thus, a Grundy coloring is a transitive partition into independent sets. The *Grundy number* $\Gamma(G)$ equals the maximum order of a Grundy coloring of G. It follows from the definitions that for any graph G, $\Gamma(G) \leq Tr(G)$.

Preliminary results are presented in Section 2. In Section 3 we determine the transitivity of several common classes of graphs, and in Section 4 we consider transitively critical graphs. We conclude by listing some questions for future study in Section 5.

2 Preliminary Results

In this section, we first present fundamental properties that will be helpful in the remainder of the paper and next consider the complexity of transitivity.

2.1 Basics

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of a graph G. Since every vertex in V_i is dominated by each of the sets V_j for $1 \leq j \leq i-1$, we make the following straightforward, yet useful observation.

Observation 2.1 *If $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G and $v \in V_i$ for some i, then* $\deg(v) \geq i - 1$.

The following results were given in [4].

Proposition 2.2 [4] *If a graph G has a transitive partition of order k, then for every j, $1 \leq j \leq k$, G has a transitive partition of order j.*

Proposition 2.3 [4] *If H is any subgraph of a graph G, then $Tr(H) \leq Tr(G)$.*

It is straightforward to see that a Tr-partition of G is a Grundy coloring for some spanning subgraph H of G.

Observation 2.4 *If π is a Tr-partition for a graph G, then π is a Grundy coloring for some spanning subgraph of G.*

Proof. Given a Tr-partition π for G, H is the subgraph obtained from G by removing any edges uv where u and v are in the same set in π. □

Next we show that every graph with $Tr(G) \geq 3$ has a partition containing two singleton sets. This was observed in [4] for graphs having $Tr(G) \geq 4$.

Proposition 2.5 *Every connected graph G for which $Tr(G) = k \geq 3$ has a Tr-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that $|V_{k-1}| = |V_k| = 1$ and $|V_{k-i}| \leq 2^{i-1}$ for $2 \leq i \leq k-2$.*

Proof. Assume that $\pi = \{V_1, V_2, \ldots, V_k\}$ where $k \geq 3$ is a Tr-partition of G. Assume that $|V_k| \geq 2$. We can move every vertex except one, say vertex x, from the set V_k to the first set V_1 and still have a transitive partition of the same order. Thus, $\pi' = \{V_1' = V_1 \cup (V_k - \{x\}), V_2, \ldots, V_{k-1}, V_k' = \{x\}\}$ is a $Tr(G)$-partition in which $|V_k'| = 1$. Similarly, let y be a vertex in V_{k-1} that is adjacent to vertex x; there must be at least one such vertex since π' is a transitive partition, that is, $V_{k-1} \to \{x\}$. We can move every vertex in V_{k-1} except y, in the transitive partition π', to set V_1', and still have a transitive partition of the same order. Thus, $\pi'' = \{V_1' \cup (V_{k-1} - \{y\}), V_2, \ldots, V_{k-1}'' = \{y\}, \{x\}\}$ is a $Tr(G)$-partition in which $|V_{k-1}''| = 1$.

Continuing in this manner, we note that for set V_{k-i} to dominate the i sets following it, at most $|V_{k-i+1}| + |V_{k-i+2}| + \ldots + |V_k| = 2^{i-1}$ vertices are necessary in V_{k-i}. The remaining vertices can be moved to V_1 as in the previous arguments. Hence, $|V_{k-i}| \leq 2^{i-1}$. □

It is well-known [3] that the domatic number $d(G)$ is bounded above by $\delta(G) + 1$, and graphs attaining this bound are called *domatically full*. The following useful observation was noted in [4].

Proposition 2.6 [4] *For any graph G, $Tr(G) \leq \Delta(G) + 1$.*

Graphs G for which $Tr(G) = \Delta(G) + 1$ are called *transitively full.* The following results were given in [4].

Proposition 2.7 [4] *For any graph G of order n,*

1. *$1 \leq Tr(G) \leq n$,*
2. *$Tr(G) = n$ if and only if G is the complete graph K_n,*
3. *$Tr(G) = 1$ if and only if $G = \overline{K}_n$, and*
4. *$Tr(G) = 2$ if and only if G is a disjoint union of stars.*

Proposition 2.8 [4]

1. *For the path P_n, $n \geq 4$, $Tr(P_n) = 3$.*
2. *For the cycle C_n, $n \geq 3$, $Tr(C_n) = 3$.*
3. *For any cubic graph G, $Tr(G) = 4$.*

By Proposition 2.7, $Tr(K_n) = n = \Delta(G)+1$ and $Tr(\overline{K}_n) = 1 = \Delta(\overline{K}_n)+1$, so both the complete graph K_n and its complement are transitively full. Note that no star having three or more vertices is transitively full. Propositions 2.7 and 2.8 imply that paths, cycles, and cubic graphs are transitively full. We shall see other families of transitively full graphs in Section 3. The following proposition was also given in [4].

Proposition 2.9 [4] *For any connected graph G and $k \in \{0, 1, 2, 3\}$, if $\delta(G) \geq k$, then $Tr(G) \geq k + 1$.*

Thus, for a 4-regular graph G, Propositions 2.6 and 2.9 imply the following.

Corollary 2.10 *For a 4-regular graph G, $4 \leq Tr(G) \leq 5$.*

Both the values in Corollary 2.10 are attainable. For example, it is shown in [4] that $Tr(G) = 4$ for the complete graph K_6 minus a perfect matching

M (see Figure 1). Also, there are transitively full 4-regular graphs, including the complete graph K_5 and the complete bipartite graph $K_{4,4}$. That is, $Tr(K_5) = Tr(K_{4,4}) = 5$.

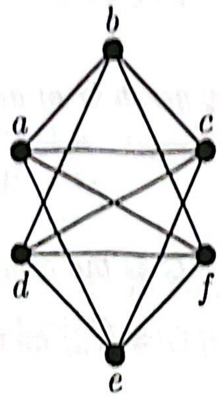

Figure 1: $Tr(K_6 - M) = 4$

Next we present a lower bound on the transitivity of a graph based on an induced subgraph.

Proposition 2.11 *If a graph G has an induced subgraph H such that every vertex of H has a neighbor in $V(G) - V(H)$, then $Tr(G) \geq Tr(H) + 1$.*

Proof. Let H be an induced subgraph of a graph G, such that every vertex of H has a neighbor in $V(G) - V(H)$, and let $\{V_1, V_2, \ldots V_k\}$ be a Tr-partition of H. It follows that $(V(G) - V(H)) \to V(H)$, and therefore $\{V(G)-V(H), V_1, V_2, \ldots, V_k\}$ is a transitive partition of G of order $Tr(H)+1$. □

The *clique number* $\omega(G)$ of a graph G is the order of a largest clique in G. By Proposition 2.11, if a graph G has clique K such that every vertex of K has a neighbor in $V(G) - V(K)$, then $Tr(G) \geq |V(K)| + 1$. Hence, we have the following corollary.

Corollary 2.12 *If $\delta(G) \geq \omega(G)$, then $Tr(G) \geq \omega(G) + 1$.*

Proof. Let K be a maximum clique of G. Since $\delta(G) \geq \omega(G)$, every vertex of K has a neighbor in $V(G)-V(K)$. By Proposition 2.11, $Tr(G) \geq |V(K)| + 1 = \omega(G) + 1$. □

Propositions 2.6 and 2.11 imply the next corollary.

Corollary 2.13 *If $\Delta(G) = \omega(G)$, then $Tr(G) = \omega(G) + 1 = \Delta(G) + 1$, and so, G is transitively full.*

2.2 Complexity

As defined in [5], the *upper iterated domination number* $\Gamma^*(G)$ equals the maximum order of a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ obtained by repeatedly removing a minimal dominating set from G, until no vertices remain. Notice that the transitivity of a graph G could equivalently be defined as the maximum order of a partition produced by repeatedly removing dominating sets from G. Hence, the only difference between $Tr(G)$ and $\Gamma^*(G)$ is that $\Gamma^*(G)$ is obtained by repeatedly removing *minimal* dominating sets, while $Tr(G)$ is obtained by repeatedly removing dominating sets. Thus, the transitivity of a graph is a relaxation of the upper iterated domination number, and so, $\Gamma^*(G) \leq Tr(G)$. In fact, it is shown in [4] that $\Gamma^*(G) = Tr(G)$. The decision problem associated with transitivity of a graph follows.

> TRANSITIVITY
> INSTANCE: Graph $G = (V, E)$, integer k
> QUESTION: Does G have a transitive partition of order at least k?

Proposition 2.14 *TRANSITIVITY is NP-complete.*

Proof. In [4], it is shown that $\Gamma^*(G) = Tr(G)$. The corresponding decision problem for the parameter $\Gamma^*(G)$ is NP-complete [5]. □

Proposition 2.15 *TRANSITIVITY(K) is fixed-parameter tractable.*

Proof. There is a polynomial time algorithm for verifying that a vertex partition is transitive. From 2.5, for a fixed k, there are a polynomial number of possible configurations of $V_2, \ldots, V_k$. □

Proposition 2.16 *There is a linear algorithm for determining $Tr(T)$ for any tree T.*

Proof. It is shown in [4] that for any tree T, $\Gamma(T) = Tr(T)$. In 1982 Hedetniemi, Hedetniemi and Beyer [6] constructed a linear algorithm for computing the Grundy number of any tree. □

3 The Transitivity of Special Classes of Graphs

In this section, we will consider the transitivity of a variety of classes of graphs.

3.1 Inflated Graphs

The *inflation* G_i of a graph G is obtained from G by replacing each vertex $x \in V(G)$ with a clique K_t, where $\deg(x) = t$, and each $xy \in E(G)$ by an edge between two vertices of the corresponding cliques X and Y in such a way that the edges of G_i which correspond to the edges of G form a matching of G_i. For example, the inflation of a cycle C_n is the cycle C_{2n}. For another example, see Figure 2 for the inflation of the house graph.

Note that $\Delta(G_i) = \Delta(G)$, and for graphs G with order at least 3, $\omega(G_i) = \Delta(G) = \Delta(G_i)$. Next we show that all inflated graphs are transitively full.

Proposition 3.1 *For any graph G, $Tr(G_i) = \Delta(G_i) + 1$.*

Proof. Let G be a graph and G_i be the inflation of G. If $G \in \{K_1, K_2\}$, then $G_i = G$. Since $Tr(K_1) = 1 = \Delta(K_1)+1$ and $Tr(K_2) = 2 = \Delta(K_2)+1$, the result follows for graphs of order $n \in \{1, 2\}$. Hence we may assume that G has order $n \geq 3$. Since $\Delta(G_i) = \Delta(G) = \omega(G_i)$, Corollary 2.13 implies that $Tr(G_i) = \Delta(G_i) + 1$. □

3.2 Coronas and Cacti

For a graph G of order n with vertices $V(G) = \{v_1, v_2, ..., v_n\}$ and a graph H, the *corona* $G \circ H$ is the graph obtained from G by adding n copies of the graph H, with new edges such that the vertex v_i is adjacent to each vertex in the *ith* copy of H.

Proposition 3.2 *For any graph G, $Tr(G \circ K_p) = Tr(G) + p$.*

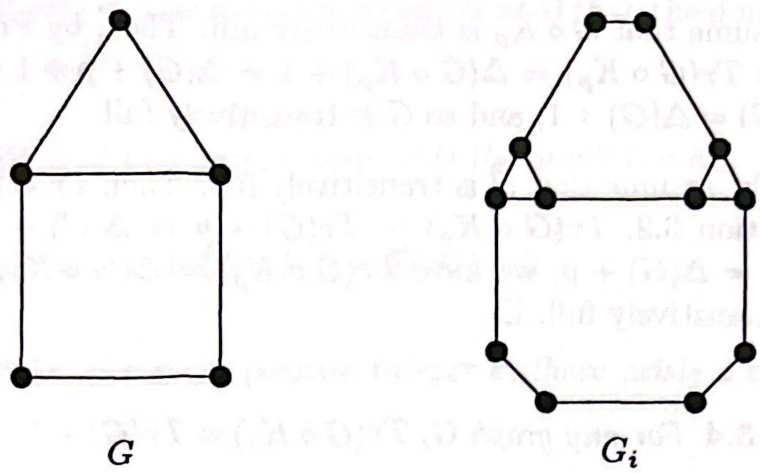

Figure 2: The house graph G and its inflation G_i.

Proof. Let G be a graph of order n, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of G. In the graph $G \circ K_p$, let $x_1^i, x_2^i, ..., x_p^i$ be the vertices in the ith copy of K_p. Let $X_j = \{x_j^i \mid 1 \leq i \leq n\}$, and let $X = \bigcup_{j=1}^{p} X_j$. Then the partition $\pi' = \{X_1, X_2, ..., X_p, V_1, V_2, \ldots, V_k\}$ is a transitive partition of order $k+p$. Thus, $Tr(G \circ K_p) \geq k+p = Tr(G)+p$.

Assume that $Tr(G) = k$, but $T(G \circ K_p) \geq k+p+1$. By Proposition 2.2, $G \circ K_p$ has a transitive partition $\pi = \{V_1, V_2, \ldots, V_s\}$, where $s = k+p+1$. Since every vertex in X has degree p, Observation 2.1 implies that $X \subseteq \bigcup_{i=1}^{p+1} V_i$. Assume that $X \cap V_{p+1} \neq \emptyset$. Since V_{p+1} dominates all sets V_j for $j \geq p+2$ and $s = k+p+1 \geq p+2$, it follows that $V_{p+1} - X \neq \emptyset$. Furthermore, no vertex in $X \cap V_{p+1}$ is necessary to dominate any vertex in any set V_j for $j \geq p+2$. Therefore, moving all of the vertices in $X \cap V_{p+1}$ to the set V_1 results in another transitive partition, say $\pi' = \{V_1', V_2', \ldots, V_p', \ldots, V_s'\}$ where $X \subseteq \bigcup_{i=1}^{p} V_i'$.

If $\bigcup_{i=1}^{p} V_i' = X$, then $\{V_{p+1}', V_{p+2}', \ldots, V_s'\}$ is a transitive partition of G of order $k+1$, a contradiction. Thus, we may assume that $X \subset \bigcup_{i=1}^{p} V_i'$. Let $U = \bigcup_{i=1}^{p} V_i' - X$. Then $\{U \cup V_{p+1}', V_{p+2}, \ldots, V_s'\}$ is a transitive partition of G of order $k+1$, again a contradiction. Thus, $Tr(G \circ K_p) \leq Tr(G) + p$, and the result holds. □

Since $\Delta(G \circ K_p) = \Delta(G) + p$, we make the following observation.

Corollary 3.3 *A graph $G \circ K_p$ is transitively full if and only if G is transitively full.*

Proof. Assume that $G \circ K_p$ is transitively full. Then, by Proposition 3.2, we see that $Tr(G \circ K_p) = \Delta(G \circ K_p) + 1 = \Delta(G) + p + 1 = Tr(G) + p$. Thus, $Tr(G) = \Delta(G) + 1$, and so G is transitively full.

Conversely, assume that G is transitively full. Then $Tr(G) = \Delta(G) + 1$. By Proposition 3.2, $Tr(G \circ K_p) = Tr(G) + p = \Delta(G) + 1 + p$. Since $\Delta(G \circ K_p) = \Delta(G) + p$, we have $Tr(G \circ K_p) = \Delta(G \circ K_p) + 1$, and so, $G \circ K_p$ is transitively full. □

Corollary 3.4 *For any graph G, $Tr(G \circ K_1) = Tr(G) + 1$.*

Corollary 3.5 *For any integer $n \geq 1$, $Tr(K_n \circ K_1) = n + 1$.*

Note that $d(K_n \circ K_1) = 2$. Thus, we see that there can be an arbitrarily large difference between $d(G)$ and $Tr(G)$.

Corollary 3.6 *For any class of graphs G such that the corona $G \circ K_1$ is also a member of that class, and for any positive integer k, there exists a graph G in the class such that $Tr(G) \geq k$.*

Proof. Let G be any member of a specified class of graphs that is closed under the corona operation. Then consider the sequence G, $G \circ K_1$, $(G \circ K_1) \circ K_1$, $((G \circ K_1) \circ K_1) \circ K_1$, etc. In each case we create another member of the class of graphs, and in each iteration the transitivity increases. □

Corollary 3.7 *For any positive integer k, there exists a bipartite graph, a planar graph, and a chordal graph G such that $Tr(G) \geq k$.*

It is well known that for any nontrivial tree T, $d(G) = 2$. However, for trees the transitivity can be arbitrarily large.

Corollary 3.8 *For any positive integer k, there exists a tree T for which $Tr(T) = k$.*

A *cactus* is a connected graph G in which every edge appears in at most one cycle. In [9] Zelinka obtained a necessary and sufficient condition for the following upper bound to be achieved.

Proposition 3.9 [9] *For any nontrivial cactus G, $2 \leq d(G) \leq 3$.*

The transitivity of a cactus is more complicated than the domatic number of a cactus.

Lemma 3.10 *If G is a cactus, then so is the graph $G \circ K_2$.*

By Proposition 3.2, $Tr(G \circ K_2) = Tr(G) + 2$.

Corollary 3.11 *For every positive integer k, there exists a cactus G with $Tr(G) \geq k$.*

Proof. Starting with an arbitrary cactus, for example the cycle C_4, form the sequence of graphs $C_4 \circ K_2$, $(C_4 \circ K_2) \circ K_2$, $((C_4 \circ K_2) \circ K_2) \circ K_2$, etc. With each iteration of this process, the transitivity increases by two and the resulting graph, according to Lemma 3.10 is another cactus.

Notice that an even simpler construction is possible, namely, starting with a cycle such as C_4 form the sequence of coronas $C_4 \circ K_1$, $(C_4 \circ K_1) \circ K_1$, $((C_4 \circ K_1) \circ K_1) \circ K_1$, etc. Each succeeding graph so formed is also a cactus, and each iteration of this process increases the transitivity by exactly one. □

3.3 Maximal Outerplanar Graphs and 2-Trees

An *outerplanar graph* is a graph having a plane embedding (no edges meet except at a vertex) such that every vertex lies on the exterior face. A *maximal outerplanar graph* is an outerplanar graph having the property that the addition of any edge between two non-adjacent vertices results in a graph that is not outerplanar. The family of non-trivial maximal outerplanar graphs can be defined recursively, as follows. Let $G_1 = K_3$. Then let G_{i+1} be the graph obtained from the graph G_i by adding a new vertex and joining it to any two adjacent vertices on the exterior face of G_i.

The following result for maximal outerplanar graphs is well known and easy to prove.

Proposition 3.12 *For any maximal outerplanar graph G, $d(G) = 3$.*

For any maximal outerplanar graph G, one can define the graph $G\Delta K_1$ to be the graph obtained from G by adding a new vertex and joining it to two adjacent vertices on the exterior face of G, and performing this operation for every edge on the exterior face of G. Notice that if G is a maximal outerplanar graph, so is $G\Delta K_1$.

Proposition 3.13 *For any maximal outerplanar graph G, $Tr(G\Delta K_1) = Tr(G) + 1$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of a maximal outerplanar graph G, and consider the maximal outerplanar graph $G\Delta K_1$. Let W be the set of new vertices added to G to form $G\Delta K_1$. Then the new partition $\pi' = \{W, V_1, V_2, \ldots, V_k\}$ is a transitive partition of $G\Delta K_1$, since the set W is a dominating set of $G\Delta K_1$. Thus, $Tr(G\Delta K_1) \geq 1 + Tr(G)$.

Assume that $Tr(G) = k \geq 2$, but $T(G\Delta K_1) \geq Tr(G) + 2$. By Proposition 2.2, $G\Delta K_1$ has a transitive partition of order $s = Tr(G) + 2 = k + 2$. Among all such transitive partitions, select $\pi = \{V_1, V_2, \ldots, V_s\}$ be one that maximizes $|V_1 \cap W|$. Since every vertex in W has degree two, Observation 2.1 implies that $W \subseteq V_1 \cup V_2 \cup V_3$. If $W \subseteq V_1$, then $\{V_2, V_3, \ldots, V_s\}$ is a transitive partition of some subgraph of G of order $k+1$, a contradiction to Proposition 2.3.

Hence, we may assume that $W \cap (V_2 \cup V_3) \neq \emptyset$. Since V_3 dominates all sets V_j for $j \geq 4$ and $k + 2 \geq 4$, it follows that $V_3 - W \neq \emptyset$. Furthermore, no vertex in W is necessary to dominate any vertex in any set V_j for $j \geq 4$. Therefore, moving all of the vertices in $V_3 \cap W$ to the set V_1 results in another transitive partition of $G\Delta K_1$ having more vertices of W in V_1 than π has, a contradiction. Thus, we may assume that $W \subseteq V_1 \cup V_2$ and $W \cap V_2 \neq \emptyset$. Let $w \in W \cap V_2$. Then w has a neighbor, say x, in V_1. If w is not necessary in V_2 to dominate a vertex in V_j for some $j \geq 3$, then w can be moved to V_1 resulting in another transitive partition having more vertices of W in V_1 than π has, again a contradiction. Let $y \in V_j$ for some $j \geq 3$ such that $N(y) \cap V_2 = \{w\}$. By the construction of $G\Delta K_1$, it follows that $N(w) = \{x, y\}$ and $xy \in E(G)$ is an edge on the boundary of G. That is, $x, y \notin W$. Consider swapping x and $N(x) \cap W \cap V_2$, that is, putting x in V_2 and $N(x) \cap W \cap V_2$ in V_1, forming a new partition π'. If π' is a transitive partition of $G\Delta K_1$, then π' has more vertices of W in V_1 than π has, a contradiciton. Thus, we may assume that π' is not a transitive partition of $G\Delta K_1$. This implies that x is necessary in V_1 of π' to dominate a vertex, say z, in $V_2 - W$. Now $xz \in E(G)$ is an edge on the boundary of G. Hence, $w' \in W$ such that $N(w') = \{x, z\}$. But then w' is in V_1 of π' and w' dominates z, a contradiction. Thus, $Tr(G\Delta K_1) \leq Tr(G) + 1$, and the result holds. □

Corollary 3.14 *For any positive integer $k \geq 3$, there exists a maximal outerplanar graph G for which $Tr(G) = k$.*

Closely related to the class of maximal outerplanar graphs is the class of 2-trees, that can be defined recursively as follows. Let $G_1 = K_3$. Then let G_{i+1} be the graph obtained from the graph G_i by adding a new vertex and joining it to any two adjacent vertices in G_i. It is easy to see that every 2-tree is a planar graph, although not necessarily an outerplanar graph.

Corollary 3.15 *For any positive integer k, there exists a 2-tree G for which $Tr(G) = k$.*

Note that the complete graph K_3 is a maximal outerplanar graph that is transitively full. However, the next corollary to Proposition 3.13 shows that for any maximal outerplanar graph G, the maximal outerplanar graph $G\Delta K_1$ is not transitively full.

Corollary 3.16 *For any maximal outerplanar graph G, $G\Delta K_1$ is not transitively full, that is, $Tr(G\Delta K_1) \leq \Delta(G\Delta K_1)$.*

Proof. Suppose, to the contrary, that $G\Delta K_1$ is transitively full for some maximal outerplanar graph G. Then $Tr(G\Delta K_1) = \Delta(G\Delta K_1) + 1$. Since every vertex of the maximal outerplanar graph G lies on its exterior face and is incident to exactly two new edges on the exterior face of $G\Delta K_1$, it follows that $\Delta(G\Delta K_1) = \Delta(G) + 2$. By Proposition 3.13, $Tr(G\Delta K_1) = Tr(G) + 1$, so we have $Tr(G\Delta K_1) = \Delta(G\Delta K_1) + 1 = (\Delta(G) + 2) + 1 = \Delta(G) + 3 = Tr(G) + 1$. But this implies that $Tr(G) > \Delta(G) + 1$, contradicting Proposition 2.6. □

3.4 Cartesian Products

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edges such that two vertices (u, v) and (w, x) are adjacent in $E(G \square H)$ if and only if either $u = w$ and v is adjacent to x in H, or u is adjacent to w in G and $v = x$. The Cartesian product $G \square H$ can be visualized as follows. The vertices of $G \square H$ are arranged in rows and columns. The vertices in each column induce a subgraph isomorphic to G, while the vertices in each row induce a subgraph isomorphic to H.

For Cartesian products involving paths P_n and cycles C_n, the following is shown in [4].

Proposition 3.17 [4]

1. *For* $m, n \geq 4$, $Tr(P_m \square P_n) = 5$.
2. *For* $m, n \geq 3$, $Tr(C_m \square C_n) = 5$.

We give a lower bound for $Tr(G \square H)$ for arbitrary graphs G and H.

Proposition 3.18 *For any graphs G and H,*

$$Tr(G \square H) \geq Tr(G) + Tr(H) - 1.$$

Proof. If $Tr(G) = 1$, then $G \cong \overline{K}_n$ for some n. Hence, $G \square H$ is a disjoint union of n copies of H. Thus, it is clear that $Tr(G \square H) = Tr(H) = Tr(G) + Tr(H) - 1$.

If $Tr(G) \geq 2$, let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition for G such that $V_k = \{u\}$ is a singleton set, and let $\pi_2 = \{W_1, W_2, \ldots, W_{k_2}\}$ be a Tr-partition for H.

Form the partition $Z = \{Z_1, Z_2, \ldots, Z_{k+k_2-1}\}$ where for $1 \leq i \leq k-1$, $Z_i = \{(x, y) \mid x \in V_i \text{ and } y \in V(H)\}$, and for $1 \leq j \leq k_2$, $Z_{j+k-1} = \{(u, v) \mid v \in W_j\}$. By construction, Z is a transitive partition for $G \square H$. □

Since $Tr(K_n) = n$, we have the following corollary to Proposition 3.18.

Corollary 3.19 *For any graph G, $Tr(G \square K_n) \geq Tr(G) + n - 1$.*

Proposition 3.20 *If a graph G is transitively full, then $G \square K_n$ is transitively full.*

Proof. Suppose G is transitively full, that is, $Tr(G) = \Delta(G) + 1$. By Corollary 3.19, $Tr(G \square K_n) \geq Tr(G) + n - 1 = \Delta(G) + n = \Delta(G \square K_n) + 1$. By Proposition 2.6, $Tr(G \square K_n) \leq \Delta(G \square K_n) + 1$, and so, $Tr(G \square K_n) = \Delta(G \square K_n) + 1$. Hence, $G \square K_n$ is transitively full. □

Since $Tr(P_3) = 2$, by Proposition 3.18, $Tr(G \square P_3) \geq Tr(G) + 1$. Our next result shows that this lower bound can be improved slightly.

Proposition 3.21 *For any graph G with at least one edge, $Tr(G \square P_3) \geq Tr(G) + 2$.*

Proof. Let $V(P_3) = \{a, b, c\}$ and $E(P_3) = \{ab, bc\}$.

If $Tr(G) \geq 3$, let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition for G such that $V_{k-1} = \{u\}$ and $V_k = \{v\}$ are singleton sets. We note that this is possible by Proposition 2.5. Define $\pi_2 = \{W_1, W_2, \cdots, W_{k+2}\}$ where for $1 \leq i \leq k-2$, $W_i = \{(x, a), (x, b), (x, c) \mid x \in V_i\}$, $W_{k-1} = \{(u, a), (v, c)\}$, $W_k = \{(u, c), (v, a)\}$, $W_{k+1} = \{(u, b)\}$, and $W_{k+2} = \{(v, b)\}$. It is not hard to see that π_2 is a transitive partition for $Tr(G \square P_3)$.

If $Tr(G) = 2$, let uv be an edge in G. Define $W_1 = \{(x, a), (x, b), (x, c) \mid x \in V(G) - \{u, v\}\} \cup \{(u, a), (v, c)\}$, $W_2 = \{(u, c), (v, a)\}$, $W_3 = \{(u, b)\}$, and $W_4 = \{(v, c)\}$. □

Of special interest in the study of Cartesian products of graphs are the *prisms*, that is, graphs of the form $G \square P_2$, consisting of two copies G_1 and G_2 of a graph G, with a matching between the corresponding vertices of G_1 and G_2. Since $Tr(P_2) = 2$, the following is a corollary to Proposition 3.18.

Corollary 3.22 *For any prism* $G \square P_2$, $Tr(G \square P_2) \geq Tr(G) + 1$.

Proposition 3.20 applied to prisms gives the following corollary.

Corollary 3.23 *If G is transitively full, then $G \square P_2$ is transitively full.*

We note that the converse of Corollary 3.23 is not necessarily true. For example, the prism $P_3 \square P_2$ is transitively full as $\Delta(P_3 \square P_2) = 3$ and $Tr(P_3 \square P_2) = 4 = \Delta(P_3 \square P_2) + 1$, while P_3 is not transitively full as $Tr(P_3) = 2 < \Delta(P_3) + 1 = 3$.

The class of graphs called *n-cubes* is defined recursively as follows. The 0-cube, denoted Q_0 is the graph K_1. The n-cube Q_n is the graph $Q_n = Q_{n-1} \square P_2$. Thus, n-cubes are a sub-family of the family of prisms.

Proposition 3.24 *For any positive integer n, $Tr(Q_n) = n+1$ and so, the n-cube Q_n is transitively full.*

Proof. Note the n-cube is a n-regular graph. Since, Q_0 is transitively full, applying Corollary 3.23 recursively yields that Q_n is transitively full. That is, $Tr(Q_n) = \Delta(Q_n) + 1 = n + 1$. □

Our final result in this subsection strengthens Proposition 3.18.

Proposition 3.25 *For any graph G, if $Tr(G) \geq 2$, then $Tr(G \square H) \geq Tr(G) + Tr(P_2 \square H) - 2$.*

Proof. If $Tr(G) = 2$, select an edge $uv \in E(G)$. It follows that $\pi = \{V(G) - \{v\}, \{v\}\}$ is a Tr-partition for G; otherwise, if $Tr(G) > 2$, let $\pi = \{V_1, \ldots, V_k\}$ be a Tr-partition for G such that $V_{k-1} = \{u\}$ and $V_k = \{v\}$ are singleton sets. This is possible by Proposition 2.5.

Let G_1 be the P_2 subgraph of G induced by the vertices $\{u, v\}$. The graph $G_1 \square H$ is a subgraph of $G \square H$. Let $W = \{W_1, W_2, \ldots, W_{k_2}\}$ be a Tr-partition for $G_1 \square H$. Define a function $f : V(G) \to \{1, \ldots, k\}$, such that for $y \in V_x$, we have $f(y) = x$ for $1 \leq x \leq k$. Define a function $g : V(G_1 \square H) \to \{1, \ldots, |W|\}$, such that for $v \in W_x$, we have $g(v) = x$, for $1 \leq x \leq |W|$.

A transitive partition $Z = \{Z_1, Z_2, \ldots, Z_{k+|W|-2}\}$ can be formed using the function $h : V(G) \times V(H) \to \{1, 2, \ldots, k + |W| - 2\}$ to define the sets, where $h(x, y) = f(x)$, if $x \in V(G) - \{u, v\}$ and $h(x, y) = k - 2 + g(y)$, if $x \in \{u, v\}$. The sets $Z_i = \{(x, y) \mid h(x, y) = i\}, 1 \leq i \leq k + |W| - 2\}$ form the partition Z. □

3.5 The Join $G + H$ of Two Graphs

The *join* $G + H$ of two graphs G and H is the graph obtained from the disjoint union of G and H by adding all possible edges between the vertices of G and the vertices of H. We give both lower and upper bounds on $Tr(G + H)$.

Proposition 3.26 *Let G and H be graphs with order m and n. Then $Tr(G + H) \leq min\{m + Tr(H), n + Tr(G)\}$.*

Proof. Let G be a graph of order m, and let π be a Tr-partition for $G + H$. At most m sets in π contain a vertex in $V(G)$ and the remaining sets form a transitive partition for a subgraph of H. Hence, $Tr(G + H) \leq m + Tr(H)$. A similar argument holds for H. □

Proposition 3.27 *Let G and H be graphs with order m and n, respectively, such that $m \leq n$. Then $m + 1 \leq Tr(G + H)$.*

Proof. Assume that $|V(G)| = m \leq n = |V(H)|$. Construct $m - 1$ sets of order two consisting of one vertex from G and one vertex from H. And let the remaining vertex in G be a singleton set. Finally, place into one set all remaining vertices in H. This collection of $m+1$ sets forms a transitive partition of $G+H$. □

The next result follows directly from Propositions 3.26 and 3.27.

Corollary 3.28 *Let G and H be graphs with order m and n, respectively, such that $m \leq n$. Then $m+1 \leq Tr(G+H) \leq m+Tr(H)$.*

The *complete bipartite graph* $K_{m,n}$ is the graph $\overline{K}_m + \overline{K}_n$. The domatic number of $K_{m,n}$ is easy to determine.

Proposition 3.29 *For any positive integers $1 \leq m \leq n$, $d(K_{m,n}) = m$.*

The transitivity of $K_{m,n}$ is one larger than the domatic number.

Corollary 3.30 *For any positive integers $1 \leq m \leq n$, $Tr(K_{m,n}) = m+1$.*

Proof. Note that $Tr(\overline{K}_n) = 1$. Thus, Proposition 3.26 implies that $Tr(K_{m,n}) \leq m+1$. Since $m \leq n$, Proposition 3.27 implies that $Tr(K_{m,n}) \geq m+1$. □

We give another lower bound.

Proposition 3.31 *For any graphs G and H, $Tr(G)+Tr(H) \leq Tr(G+H)$.*

Proof. Let $\pi = \{U_1, U_2, \ldots, U_r\}$ be a Tr-partition of G, and let $\pi = \{V_1, V_2, \ldots, V_s\}$ be a Tr-partition of H. Then $\pi' = \{U_1, U_2, \ldots, U_r, V_1, V_2, \ldots, V_s\}$ is a transitive partition of $G+H$. Therefore, $Tr(G+H) \geq Tr(G)+Tr(H)$. □

Thus, for graphs G and H with order m and n, respectively, such that $m \leq n$, we have that $Tr(G+H) \geq \max\{m+1, Tr(G)+Tr(H)\}$. The next two corollaries follow from Propositions 3.26 and 3.31.

Corollary 3.32 *For any graph G, $Tr(G+K_1) = Tr(G)+1$.*

Corollary 3.33 *For any graph G, $Tr(G+K_n) = Tr(G)+n$.*

An improved lower bound for $Tr(G+H)$ can be obtained as follows. Recall that $\omega(G)$ is the clique number of G.

Proposition 3.34 *Let G be a graph of order m, H a graph of order n, where $m \leq n$, and let $\omega(H) = t$. Then if $\omega(H) = t \leq n - m$, then $Tr(G+H) \geq m + \omega(H)$, else $Tr(G+H) \geq n$.*

Proof. Let the vertices in graph G be $U = \{u_1, u_2, \ldots, u_m\}$ and the vertices in graph H be $V = \{v_1, v_2, \ldots, v_n\}$. Assume that the maximum size of a clique in H is t, where $\omega(H) = t \leq n - m$, and further assume that the t vertices in this clique appear in the list of vertices of V as $v_m, v_{m+1}, \ldots, v_{m+t-1}$. As in the proof of Proposition 3.27, construct the following transitive partition of $G+H$:

$\pi = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \{v_m\}, \{v_{m+1}\}, \{v_{m+2}\}, \ldots, \{v_{m+t-1}\}\}$.

Finally, place any remaining vertices not in any set of π into set $\{u_1, v_1\}$. It can be seen that the resulting partition is a transitive partition of order $m + t = m + \omega(H)$.

Assume that $\omega(H) = t > n - m$. In this case, order the vertices in the clique so that they are the last t vertices in V. It can be seen that the partition $\pi = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \{v_m\}, \{v_{m+1}\}, \ldots, \{v_{m+t-1}\}\}$ is a transitive partition of $G+H$ of order $m + n - m = n$. □

4 Transitively Critical Graphs

As observed in [4], deleting a vertex v or an edge e, or adding an edge e can change the transitivity of a graph by at most one.

Proposition 4.1 [4] *For any graph G with a vertex $v \in V(G)$, edge $e \in E(G)$, and edge $f \in E(\overline{G})$,*
(i) $Tr(G) - 1 \leq Tr(G - v) \leq Tr(G)$,
(ii) $Tr(G) - 1 \leq Tr(G - e) \leq Tr(G)$;
(iii) $Tr(G) \leq Tr(G + f) \leq Tr(G) + 1$.

We begin with the following definition.

Definition 1 *A graph $G = (V, E)$ is called transitively vertex-critical if $Tr(G) \geq 1$ and deleting any vertex $v \in V$ results in a graph whose transitivity is less than that of G. If G is transitively vertex-critical and $Tr(G) = k$, we say that G is Tr_k-critical.*

By Proposition 2.7, $Tr(G) = n$ if and only if $G = K_n$, and so we make the following observation.

Observation 4.2 *A graph G of order n is Tr_n-critical if and only if G is the complete graph K_n.*

Lemma 4.3 *If G is Tr_k-critical, then G is connected.*

Proof. Suppose that G is disconnected. Let $C_1, C_2, \ldots, C_j$ be the components of G. Notice that the transitivity of G is equal to the maximum transitivity among its components, so $Tr(C_i) = Tr(G)$ for some i. But then deleting a vertex from any component C_j for $j \neq i$ does not change the transitivity of G, so G is not transitively vertex-critical. □

Consider Proposition 2.5 once again. If G is transitively vertex critical, then the conditions in Proposition 2.5 must hold in all Tr-partitions of G.

Lemma 4.4 *If G is Tr_k-critical, and $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G, then $|V_k| = 1$.*

Proof. Assume that G is Tr_k-critical, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of G. For the sake of contradiction, suppose that $|V_k| \geq 2$ and that $u, v \in V_k$. We see that $\pi' = \{V_1, V_2, \ldots, V_{k-1}, V_k' = \{u\}\}$ is a transitive partition of $G - v$, a contradiction. □

Corollary 4.5 *The only Tr_1-critical graph is K_1.*

Lemma 4.6 *If G is Tr_k-critical with $k \geq 2$, and $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G, then $|V_{k-1}| = 1$.*

Proof. Assume that G is Tr_k-critical for $k \geq 2$. Let $\pi = \{V_1, V_2, \ldots, V_{k-1}, V_k\}$ be a Tr-partition of G. By Lemma 4.4, we know that $|V_k| = 1$. Let $x \in V_k$. For the sake of contradiction, suppose that $|V_{k-1}| \geq 2$ and let

$u, v \in V_{k-1}$. Without loss of generality, suppose that $ux \in E(G)$. We see that $\pi' = \{V_1, V_2, \ldots, V_{k-2}, V'_{k-1} = \{u\}, V_k\}$ is a transitive partition of $G - v$, a contradiction. □

Corollary 4.7 *The only Tr_2-critical graph is K_2.*

Lemma 4.8 *If G is Tr_k-critical with $k \geq 3$, and $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G, then $|V_{k-2}| \leq 2$.*

Proof. Assume that G is Tr_k-critical with $k \geq 3$. Let $\pi = \{V_1, \ldots, V_{k-1}, V_k\}$ be a Tr-partition of G. By Lemma 4.4 and Lemma 4.6, we have that $|V_k| = 1$ and $|V_{k-1}| = 1$. Let $x \in V_k$ and $y \in V_{k-1}$. For the sake of contradiction, suppose that $|V_{k-2}| \geq 3$. Let u be a vertex in V_{k-2} that dominates y, let v be a vertex in V_{k-2} that dominates x, and let w in V_{k-2} be distinct from u and v. Note that $u = v$ is possible. We see that $\pi' = \{V_1, \ldots, V'_{k-2} = V_{k-2} - \{u, v\}, V_{k-1}, V_k\}$ is a transitive partition of $G - w$, a contradiction. □

Corollary 4.9 *If G is Tr_3-critical, then $3 \leq |G| \leq 4$.*

Simple enumeration will show that the only Tr_3-critical graph on 3 vertices is K_3, and the only two of order four are P_4 and C_4.

Corollary 4.10 *A graph G is Tr_3-critical if and only if $G \in \{K_3, P_4, C_4\}$.*

In general, we have the following.

Lemma 4.11 *If G is Tr_k-critical with $1 \leq j \leq k$, and $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G, then $|V_{k-j}| \leq \sum_{i=0}^{j-1} |V_{k-i}|$.*

Proof. Suppose G is Tr_k-critical with $k \geq j$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of G. Each vertex in $V_{k-j+1}, V_{k-j+2}, \ldots, V_k$ must be dominated by a vertex in V_{k-j}. If $|V_{k-j}| > \sum_{i=0}^{j-1} |V_{k-i}|$, then there is at least one vertex in V_{k-j} whose removal will not decrease the transitivity, a contradiction. □

Thus, by combining our lemmas, we have the following result.

Proposition 4.12 *If G is Tr_k-critical, then $|V(G)| \leq 2^{k-1}$.*

By Proposition 4.12, for any Tr_4-critical graph G of order n, then $4 \leq n \leq 8$. By Observation 4.2, if $G \neq K_4$, then $5 \leq n \leq 8$. The corona $P_4 \circ K_1$ is an example of a Tr_4-critical graph of order $n = 8$.

Transitively edge-critical graphs are defined in a similar manner.

Definition 2 *A graph $G = (V, E)$ is called a transitively edge-critical graph if $Tr(G) \geq 1$ and deleting any edge $e \in E$ results in a graph whose transitivity is less than that of G. If G is transitively edge-critical and $Tr(G) = k$, we denote this by Tr_k^e-critical.*

Clearly, the complete graph K_n is both vertex-critical and edge-critical. Since $C_n - v = P_{n-1}$ for any vertex $v \in V(C_n)$ and $C_n - e = P_n$ for any edge $e \in E(C_n)$, Proposition 2.8 implies that no cycle is edge-critical and that the only vertex-critical cycles are C_3 and C_4.

Our first lemma concerning transitively edge-critical graphs is in distinction to Lemma 4.4. In addition, note that transitively edge-critical graphs may be disconnected (see $K_3 \cup K_1$ for example).

Lemma 4.13 *If G is Tr_k^e-critical and $\pi = \{V_1, V_2, \ldots, V_k\}$ is a Tr-partition of G, then V_i is an independent set for $1 \leq i \leq k$.*

Proof. Suppose G is a Tr_k^e-critical graph, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of G. If e is an edge joining two vertices in V_j, for some j, then π is a transitive partition of $G - e$. Our result follows. □

Corollary 4.14 *If G is Tr_k^e-critical, then $Tr(G) = \Gamma(G)$.*

Proof. Let G be a Tr_k^e-critical graph, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a Tr-partition of G. By Lemma 4.13, V_i is independent for $1 \leq i \leq k$. Thus, π is a Grundy coloring, in which case $\Gamma(G) \geq Tr(G)$. As previously noted, $\Gamma(G) \leq Tr(G)$ for all graphs G. □

The converse of Corollary 4.14 does not hold in general. For instance, the path P_5 has $Tr(P_5) = 3 = \Gamma(P_5)$, but P_5 is not transitively edge-critical.

Finally, we consider transitively critical graphs where the operation under consideration is edge addition.

Definition 3 *A graph $G = (V, E)$ is called a transitively edge$^+$-critical graph if adding any edge $e \in E(\overline{G})$ results in a graph whose transitivity is more than that of G. If G is transitively edge$^+$-critical and $Tr(G) = k$, we denote this by Tr_k^{+e}-critical.*

We note that vacuously the complete graph K_n is edge$^+$-critcal making it vertex-critical, edge-critical and edge$^+$-critical. Since $Tr(G) = 1$ if and only if $G = \overline{K}_n$, it is straightforward to characterize the Tr_1^{+e}-critical graphs.

Observation 4.15 *A graph G is Tr_1^{+e}-critical if and only if $G = \overline{K}_n$.*

Moreover, Proposition 2.7 implies the following.

Observation 4.16 *A graph G is Tr_2^{+e}-critical if and only if G is a disjoint union of stars.*

We conclude this section by noting that the graph $K_n - e$ is Tr_{n-1}^{+e}-critical. In fact, the graph $K_n - M$, where M is any matching with an odd number of edges is edge$^+$-critical.

5 Open Problems

1. What is a necessary and sufficient condition for a graph G to have $Tr(G) \geq 4$? We can show that if $Tr(G) \geq 4$, then G must contain one of 14 (not necessarily induced) subgraphs. Note that by Part 3 of Proposition 2.8, this implies that every cubic graph contains one of 13 subgraphs, since one of these subgraphs contains a vertex of degree greater than three.

2. What is a necessary and sufficient condition for a graph G to have $Tr(G) = 3$?

3. The independent domination number of a graph, denoted $i(G)$, is the minimum order of an independent dominating set. If G is a graph of order n, is $i(G) + Tr(G) \leq n + 1$?

4. For what classes of graphs is $\Gamma(G) = Tr(G)$?

5. For what classes of graphs is $d(G) = Tr(G)$?

6. When does equality hold in Proposition 3.18? That is, for what classes of graphs is $Tr(G \square H) = Tr(G) + Tr(H) - 1$?

7. Let $L(G)$, $T(G)$, and $S(G)$ denote the line graph, total graph, and subdivision graph of G, respectively. For any graph G, what is the relationship between $Tr(G)$ and $Tr(H_G)$ where $H_G \in \{L(G), T(G), S(G)\}$.

8. Recall the operation $G\Delta K_1$ that we applied only to the exterior edges of a maximal outerplanar graph. Let $G\Delta_e K_1$ denote the graph obtained by applying this operation to every edge of a graph G. What is the relation between $Tr(G)$ and $Tr(G\Delta_e K_1)$?

9. Define a *tournament transitive partition* as a transitive partition with the additional constraint that $V_i \not\leftarrow V_j$, for $i < j$, and define the tournament transitive partition number $TTr(G)$ as the maximum order of a tournament transitive partition. For the graph K_n, the only transitive partition is $\{V(K_n)\}$. Investigate tournament transitive partitions of graphs.

10. In Section 4, we briefly introduced transitively critical graphs, a rich avenue for future research. What is a necessary and sufficient condition for a a graph to be Tr_k-critical for $k \geq 4$. Answer similar questions for edge-critical and edge$^+$-critical graphs.

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