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Article

A Generalisation of Maximal (k,b)-Linear-Free Sets of Integers

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Abstract: Fix integers k, b, q with $k \geq 2, b \geq 0, q \geq 2$. Define the function p to be: $p(x) = kx + b$. We call a set S of integers (k, b, q)*-linear-free* if $x \in S$ implies $p^i(x) \notin S$ for all $i = 1, 2, \ldots, q-1$, where $p^i(x) = p\left(p^{i-1}(x)\right)$ and $p^0(x) = x$. Such a set S is maximal in $[n] := \{1, 2, \ldots, n\}$ if $S \cup \{t\}, \forall t \in [n] \backslash S$ is not (k, b, q)-linear-free. Let $M_{k,b,q}(n)$ be the set of all maximal (k, b, q)-linear-free subsets of $[n]$, and define $g_{k,b,q}(n) = \min_{S \in M_{k,b,q}(n)} |S|$ and $f_{k,b,q}(n) = \max_{S \in M_{k,b,q}(n)} |S|$. In this paper, formulae for $g_{k,b,q}(n)$ and $f_{k,b,q}(n)$ are proposed. Also, it is proven that there is at least one maximal (k, b, q)-linear-free subset of $[n]$ with exactly x elements for any integer x between $g_{k,b,q}(n)$ and $f_{k,b,q}(n)$, inclusively.

Keywords: Integer, Maximal

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1. Introduction

A set of integers S is said to be k*-multiple-free* if $x \neq ky$ for all $x, y \in S$. Let $g_k(n) = \max\{|A| : A \subseteq [n] := \{1, 2, \ldots, n\}$ is k-multiple-free$\}$. E.T.H Wang [1], when working on *double-free* sets, showed that

$$g_2(n) = \left\lceil \frac{n}{2} \right\rceil + g_2\left(\left\lfloor \frac{n}{4} \right\rfloor\right),$$

with $g_2(0) = 0$, and an asymptotic formula for $g_2(n)$ is given by $g_2(n) = \frac{2}{3}n + O(\log n)$. In a similar vein, Leung and Wei [2] proved that for every integer $r > 1$, $g_r(n) = \frac{r}{r+1}n + O(\log n)$.

Instead of k-multiple-free subsets, one can ask more general questions about a subset S of $[n]$ such that: for any two elements x and y of S, $\frac{y}{x}$ does not belong to C, where C is a fixed set of rational numbers. For example, when $C = \{2, 3\}$, such a subset S is called *strongly triple-free*, which is a subject of great interest in, for example, [3] and [4]. In fact, the genesis of this paper is the case when $C = \{k, k^2, \ldots, k^q\}$, where k and q are positive integers.

On the other hand, You, Yu, and Liu [5] studied an alternative, natural generalisation of multiple-free sets, which are translation-free, or linear-free sets. In particular, a set X of integers is called (k, b)*-linear-free* if $x \in X$ implies $kx + b \notin X$. Obviously, when $b = 0$, X is k-multiple-free. Define $g_{k,b}(n) = \min\{|X| : X \subseteq [n]$ is a maximal (k, b)-linear-free subset$\}$ and $f_{k,b}(n) = \max\{|X| : X \subseteq [n]$ is a maximal (k, b)-linear-free subset$\}$. You, Yu, and Liu proposed explicit formulae for the two functions and showed that there is a maximal (k, b)-linear-free subset of $[n]$ with exactly x elements for any integer x between the minimum and maximum possible orders. Moreover, Liu and Zhou [6] also dealt with the same generalisation, albeit having a different approach.

This paper extends the aforementioned results by combining those two concepts. Henceforth, k, b, and q will denote integers with $k \geq 2, b \geq 0$, and $q \geq 2$. Define the function p to be: $p(x) = kx + b$. A set S of integers is said to be (k, b, q)*-linear-free* if $x \in S$ implies $p^i(x) \notin S, \forall i = 1, 2, \ldots, q-1$, where

$p^i(x) = p\left(p^{i-1}(x)\right)$ and $p^0(x) = x$. Such a set S is *maximal* in $[n]$ if $S \cup \{t\}$ is not (k,b,q)-linear-free for any $t \in [n] \backslash S$. Let $M_{k,b,q}(n)$ be the set of all maximal (k,b,q)-linear-free subsets of $[n]$, and define $g_{k,b,q}(n) = \min_{S \in M_{k,b,q}(n)} |S|$ and $f_{k,b,q}(n) = \max_{S \in M_{k,b,q}(n)} |S|$. In this paper, formulae for $g_{k,b,q}(n)$ and $f_{k,b,q}(n)$ are derived. Also, it is proven that there is at least one maximal (k,b,q)-linear-free subset of $[n]$ of cardinality x for any integer x between $g_{k,b,q}(n)$ and $f_{k,b,q}(n)$, inclusively.

Notice that when $q = 2$ we get back the notion of (k,b)-linear-free subsets: $f_{k,b,2}(n) = f_{k,b}(n)$ and $g_{k,b,2}(n) = g_{k,b}(n)$.

There is yet another classical way to look at the topic. For a fixed $r \times s$ integer matrix $\mathbf{A}$, let us call a subset $S_{\mathbf{A}}(n) \subseteq [n]$ ***A**-missing* if for every vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{pmatrix}$ with $x_i \in S_{\mathbf{A}}$ for all i, the column vector $\mathbf{Ax}$ has none of its entries being equal to 0. With this definition, take $\mathbf{A} = \begin{pmatrix} k & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$, then $\mathbf{A}$-missing subsets and $\mathbf{B}$-missing subsets are, respectively, k-multiple-free subsets and strongly triple-free subsets. Naturally, we can further fix a $r \times 1$ matrix $\mathbf{b}$ and look at the entries of $\mathbf{Ax} + \mathbf{b}$. We specify the $(q-1) \times q$ matrix $\mathbf{A}$ and the $(q-1) \times 1$ matrix $\mathbf{b}$ to be

$$\mathbf{A} = \begin{pmatrix} k & -1 & 0 & \dots & 0 \\ k^2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k^{q-1} & 0 & 0 & \dots & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} p^1(0) \\ p^2(0) \\ \vdots \\ p^{q-1}(0) \end{pmatrix}.$$

Interpreted in terms of matrices, a (k,b,q)-linear-free subset $S_{\mathbf{A},\mathbf{b}}(n) \subseteq [n]$ is equivalently a subset which satisfies: for every vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}$ with $x_i \in S_{\mathbf{A},\mathbf{b}}$ for all i, the column vector $\mathbf{Ax} + \mathbf{b}$ has none of its entries being equal to 0. For instance, the case of (k,b)-linear-free subsets corresponds to $\mathbf{A} = \begin{pmatrix} k & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b \end{pmatrix}$, and none of the vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1, x_2 \in S_{\mathbf{A},\mathbf{b}}$ is a solution to the equation $\mathbf{Ax} + \mathbf{b} = \mathbf{0}$.

2. Main Results

To simplify expressions, we denote $\left\langle k^i \right\rangle = (k^i - 1)/(k-1)$.

Theorem 1. *The recurrence relation for $f_{k,b,q}(n)$ is given by*

$$f_{k,b,q}(n) = \left\lceil \frac{n(k-1)+b}{k} \right\rceil + f_{k,b,q}\left(\left\lfloor \frac{n - b\left\langle k^q \right\rangle}{k^q} \right\rfloor \right), \qquad \forall n > b$$

with $f_{k,b,q}(n) = 0, \forall n < 0$ and $f_{k,b,q}(n) = n, \forall n \in [0,b]$. Similarly,

$$g_{k,b,q}(n) = \left\lceil \frac{n(k-1)+b}{k} \right\rceil + g_{k,b,q}\left(\left\lfloor \frac{n - b\left\langle k^{2q-1} \right\rangle}{k^{2q-1}} \right\rfloor \right), \quad \forall n > b$$

with $g_{k,b,q}(n) = 0, \forall n < 0$ and $g_{k,b,q}(n) = n, \forall n \in [0,b]$.

Proof. Assume that F is a maximal (k,b,q)-linear-free subset of $[n]$ with $f_{k,b,q}(n)$ elements. Fix k and q, and partition $[n]$ into

$$X = \left\{ 1, 2, 3, ..., \left\lfloor \frac{n - b\left\langle k^q \right\rangle}{k^q} \right\rfloor \right\} \quad \text{and}$$

$$Y = \left\{ \left\lfloor \frac{n - b\langle k^q \rangle}{k^q} \right\rfloor + 1, \left\lfloor \frac{n - b\langle k^q \rangle}{k^q} \right\rfloor + 2, ..., n \right\}.$$

We will prove that, from the set Y, F cannot have more than $\left\lceil \frac{n(k-1)+b}{k} \right\rceil$ elements. Indeed, for each $i = \left\lfloor \frac{n-b}{k} \right\rfloor + 1, \left\lfloor \frac{n-b}{k} \right\rfloor + 2, ..., n$, let $h(i)$ be the greatest non-negative integer such that $p^{-h(i)}(i) \in [n]$, where $p^{-i}(x)$ is the inverse function of $p^i(x)$. Define

$$Y_i = \{p^{-t}(i), 0 \le t \le \min\{h(i), q-1\}\}.$$

The union of Y_i, $i = \left\lfloor \frac{n-b}{k} \right\rfloor + 1, \left\lfloor \frac{n-b}{k} \right\rfloor + 2, \dots, n$ $\left(\text{there are } n - \left\lfloor \frac{n-b}{k} \right\rfloor = \left\lceil \frac{n(k-1)+b}{k} \right\rceil \text{ such sets}\right)$ covers Y. This is because, for every $c \in Y$, there exists $j \in [1, q]$ satisfying $\left\lfloor \frac{n-b\langle k^j \rangle}{k^j} \right\rfloor + 1 \le c \le \left\lfloor \frac{n-b\langle k^{j-1} \rangle}{k^{j-1}} \right\rfloor$. As $\left\lfloor \frac{n-b}{k} \right\rfloor + 1 \le p^{j-1}(c) \le n$, this guarantees $c \in Y_{p^{j-1}(c)}$.

Because F cannot include more than one element in each Y_i, and can have at most $f_{k,b,q}\left(\left\lfloor \frac{n-b\langle k^q \rangle}{k^q} \right\rfloor\right)$ elements in X, so

$$f_{k,b,q}(n) \le \left\lceil \frac{n(k-1)+b}{k} \right\rceil + f_{k,b,q}\left(\left\lfloor \frac{n-b\langle k^q \rangle}{k^q} \right\rfloor\right).$$

On the other hand, let A_1 be a maximal (k, b, q)-linear-free subset of X and $A_2 = \left\{\left\lfloor \frac{n-b}{k} \right\rfloor + 1, \left\lfloor \frac{n-b}{k} \right\rfloor + 2, ..., n\right\}$. Then $A = A_1 \cup A_2$ is a maximal (k, b, q)-linear-free subset of $[n]$, hence

$$f_{k,b,q}(n) \ge \left\lceil \frac{n(k-1)+b}{k} \right\rceil + f_{k,b,q}\left(\left\lfloor \frac{n-b\langle k^q \rangle}{k^q} \right\rfloor\right).$$

The upper and lower bounds complete the proof for the first part of Theorem 1.

The second recurrence relation regarding $g_{k,b,q}(n)$ can be proven in the same fashion. □

Based on the proof of Theorem 1, a straightforward algorithm to construct a maximal (k, b, q)-linear-free subset F of $[n]$ with $f_{k,b,q}(n)$ elements is proposed: first, check the largest number in $[n]$, which is n. As $p^i(n) \notin F, \forall i = 1, 2, \dots, q-1$, we add n to F (initially, F is empty). Next, assume that we have checked until $x > 1$. We then proceed to check $x-1$. If $p^i(x-1) \notin F, \forall i = 1, 2, \dots, q-1$, then $x-1$ is added to F. Otherwise, we proceed to check $x-2$. This algorithm terminates after number 1 is checked.

The next theorem provides formulae to determine $f_{k,b,q}(n)$ and $g_{k,b,q}(n)$.

Theorem 2. *For every positive integer $n, n \ge b \ge 1$:*

$$f_{k,b,q}(n) = f_{k,b,q}\left(\left\lfloor \frac{n - b\left\langle k^{q(m+1)} \right\rangle}{k^{q(m+1)}} \right\rfloor\right) + \sum_{i=0}^{m} \left\lceil \frac{k-1}{k} \left\lfloor \frac{n - b\left\langle k^{qi} \right\rangle}{k^{qi}} \right\rfloor + \frac{b}{k} \right\rceil,$$

where $m = \left\lfloor \frac{\log_k((n(k-1)+b)/bk)}{q} \right\rfloor$. Similarly,

$$g_{k,b,q}(n) = g_{k,b,q}\left(\left\lfloor \frac{n - b\left\langle k^{(2q-1)(m+1)} \right\rangle}{k^{(2q-1)(m+1)}} \right\rfloor\right) + \sum_{i=0}^{m} \left\lceil \frac{k-1}{k} \left\lfloor \frac{n - b\left\langle k^{(2q-1)i} \right\rangle}{k^{(2q-1)i}} \right\rfloor + \frac{b}{k} \right\rceil,$$

where $m = \left\lfloor \frac{\log_k((n(k-1)+b)/bk)}{2q-1} \right\rfloor$.

It should be remarked that, by Theorem 1, $f_{k,b,q}\left(\left\lfloor \frac{n-b\langle k^{q(m+1)} \rangle}{k^{q(m+1)}} \right\rfloor\right)$ is either 0 or $\left\lfloor \frac{n-b\langle k^{q(m+1)} \rangle}{k^{q(m+1)}} \right\rfloor$, depending on whether $\left\lfloor \frac{n-b\langle k^{q(m+1)} \rangle}{k^{q(m+1)}} \right\rfloor$ is negative or not, and $g_{k,b,q}\left(\left\lfloor \frac{n-b\langle k^{(2q-1)(m+1)} \rangle}{k^{(2q-1)(m+1)}} \right\rfloor\right)$ is determined correspondingly.

Proof. For $f_{k,b,q}(n)$, this is obtained by repeatedly applying Theorem 1 until $\left\lfloor \frac{n-b\left\langle k^{qi}\right\rangle}{k^{qi}} \right\rfloor \leq b$, i.e., $i \geq \left\lfloor \frac{\log_k((n(k-1)+b)/bk)}{q} \right\rfloor$. Also, notice that

$$\left\lfloor \frac{\left\lfloor \frac{n-b\left\langle k^{qi}\right\rangle}{k^{qi}} \right\rfloor - b\left\langle k^q \right\rangle}{k^q} \right\rfloor = \left\lfloor \frac{n - b\left\langle k^{q(i+1)}\right\rangle}{k^{q(i+1)}} \right\rfloor.$$

The same argument can be applied to deduce the formula for $g_{k,b,q}(n)$. □

In Theorem 2, when $q = 2$, formulae for $f_{k,b}(n)$ and $g_{k,b}(n)$ are obtained, which are, however, different from those of You, Yu, and Liu [5]. Particularly, You, Yu, and Liu suggested that

$$f_{k,b}(n) = \sum_{i=1}^{n(1)} |N_i| \left\lceil \frac{i+1}{2} \right\rceil \quad \text{and} \quad g_{k,b}(n) = \sum_{i=1}^{n(1)} |N_i| \left\lceil \frac{i+1}{3} \right\rceil,$$

where $n(1) = \left\lfloor \log_k \frac{n+b/(k-1)}{1+b/(k-1)} \right\rfloor$ and

$$|N_i| = \begin{cases} \left\lfloor \frac{n-b\left\langle k^i\right\rangle}{k^i} \right\rfloor - 2\left\lfloor \frac{n-b\left\langle k^{i+1}\right\rangle}{k^{i+1}} \right\rfloor + \left\lfloor \frac{n-b\left\langle k^{i+2}\right\rangle}{k^{i+2}} \right\rfloor & \text{for } i < n(1), \\ \left\lfloor \frac{n-b\left\langle k^i\right\rangle}{k^i} \right\rfloor & \text{for } i = n(1). \end{cases}$$

These are essentially different from Theorem 2 as the two have different strategies: our approach utilises recurrence relations while You, Yu, and Liu partition $[n]$ from the beginning. Furthermore, it seems that their approach can be suitably modified to accommodate for the more general cases when $q \geq 3$.

These somewhat complicated formulae are illustrated in the following example:

Example 1. *To determine $f_{2,1,3}(2021)$ and $g_{2,1,3}(2021)$, Theorem 2 gives:*

$$f_{2,1,3}(2021) = f_{2,1,3}\left(\left\lfloor \frac{2021}{8^4} - \frac{8^4-1}{8^4} \right\rfloor\right) + \sum_{i=0}^{3} \left\lceil \frac{1}{2} \left\lfloor \frac{2021}{8^i} - \frac{8^i-1}{8^i} \right\rfloor + \frac{1}{2} \right\rceil = 1155$$

and

$$g_{2,1,3}(2021) = g_{2,1,3}\left(\left\lfloor \frac{2021}{32^2} - \frac{32^2-1}{32^2} \right\rfloor\right) + \sum_{i=0}^{1} \left\lceil \frac{1}{2} \left\lfloor \frac{2021}{32^i} - \frac{32^i-1}{32^i} \right\rfloor + \frac{1}{2} \right\rceil = 1043.$$

Corollary 1.

$$f_{k,0,q}(n) = \frac{k^{q-1}(k-1)}{k^q-1} n + O(\log n).$$

This is based on

$$f_{k,0,q}(n) = \sum_{i \geq 0} \left\lceil \frac{k-1}{k} \left\lfloor \frac{n}{k^{qi}} \right\rfloor \right\rceil,$$

which is a direct corollary of Theorem 2. In fact, Corollary 1 can be extended by generalising the method of Wakeham and Wood [7]. Particularly, for positive integers a, b whose greatest common divisor is g and $a < b$, one has

$$f_{b/a,0,q}(n) = \frac{b^{q-1}(b-g)}{b^q - g^q} n + O(\log n).$$

Next, still in the case $b = 0$, if n is a perfect power of k then a much neater formula for $f_{k,0,q}(n)$ is as follows:

Theorem 3.

$$f_{k,0,q}(k^n) = \left\lfloor \frac{k^{q-1}(k^n-1)(k-1)}{k^q-1} \right\rfloor + 1.$$

Proof. We will induct on n. First, let $0 \le n < q$. The theorem is true for $n = 0$ as $f(1) = 1$. If $1 \le n < q$, Theorem 1 suggests

$$f_{k,0,q}(k^n) = \left\lceil \frac{k-1}{k} k^n \right\rceil + f_{k,0,q}(0) = k^{n-1}(k-1).$$

We can then prove that, for $0 \le n < q$:

$$f_{k,0,q}(k^n) = \left\lfloor \frac{k^{q-1}(k^n-1)(k-1)}{k^q-1} \right\rfloor + 1.$$

Lastly, the inductive step from n to $n+q$ can be done as follows

$$\left\lfloor \frac{k^{q-1}(k^n-1)(k-1)}{k^q-1} \right\rfloor + 1 = \left\lfloor \frac{k^{q-1}(k^{n-q}-1)(k-1)}{k^q-1} \right\rfloor + 1 + (k-1)k^{n-1}.$$

This completes the proof. □

There is an alternative to determine $f_{k,0,q}(n)$. Partition $[n]$ into geometric progressions of the form $\{t, kt, k^2t, \ldots, k^jt\}$, where t is not divisible by k and $1 \le t \le n$. Each such subset's cardinality is $\lfloor \log_k(n/t) \rfloor + 1$. Within each subset, if its cardinality is M, then at most $\lceil M/q \rceil$ elements can belong to the same maximal $(k,0,q)$-linear-free subset of $[n]$. Therefore,

$$f_{k,0,q}(n) = \sum_{\substack{1 \le t \le n \\ k \nmid t}} \left\lceil \frac{\lfloor \log_k(n/t) \rfloor + 1}{q} \right\rceil.$$

Replacing n with k^{n-1}, we obtain a result in The American Mathematical Monthly [8]:

> Let k, q and n be positive integers with $k \ge 2$, and let P be the set of all positive integers less than k^n that are not divisible by k. Prove
>
> $$\sum_{p \in P} \left\lceil \frac{\lfloor n - \log_k p \rfloor}{q} \right\rceil = \left\lfloor \frac{k^{q-1}(k^{n-1}-1)(k-1)}{k^q-1} \right\rfloor + 1.$$

Next, inspired by Theorem 2 in [5], we investigate the possible cardinalities of maximal (k,b,q)-linear-free subsets. The following lemma is crucial to the proof of Theorem 4.

Lemma 1. *A set S is said to be q-*distanced *if for all $x, y \in S, x \ne y$, then $|x-y| \ge q$. Let the set $H_q(n)$ include every subset S of $[n]$ such that: S is q-distanced but $S \cup \{t\}, \forall t \in [n] \backslash S$ is not. Then, there exists a set $A \in H_q(n)$ whose cardinality is x if and only if $\left\lceil \frac{n}{q} \right\rceil \ge x \ge \left\lceil \frac{n}{2q-1} \right\rceil$.*

Proof. Let a be an integer in $[0,n]$. From a numbers from 1 to a, pick all the numbers which are congruent to 1 modulo q to be elements of S (initially, S is empty). For the other $n-a$ numbers, pick all the numbers which are congruent to $a+q$ modulo $2q-1$ to be in S. Then $S \in H_q(n)$ and its cardinality is $\left\lceil \frac{a}{q} \right\rceil + \left\lceil \frac{n-a}{2q-1} \right\rceil$. It suffices to prove that there exists an integer $a \in [0,n]$ such that $\left\lceil \frac{a}{q} \right\rceil + \left\lceil \frac{n-a}{2q-1} \right\rceil = x$ if and only if $\left\lceil \frac{n}{q} \right\rceil \ge x \ge \left\lceil \frac{n}{2q-1} \right\rceil$.

Indeed, consider a to be of the form $(2q-1)k$, where k is a non-negative integer. Let the function $s : \left[0, \left\lfloor \frac{n}{2q-1} \right\rfloor\right] \longrightarrow \mathbb{N}$ be

$$s(k) = \left\lceil \frac{(2q-1)k}{q} \right\rceil + \left\lceil \frac{n-(2q-1)k}{2q-1} \right\rceil = k + \left\lceil \frac{-k}{q} \right\rceil + \left\lceil \frac{n}{2q-1} \right\rceil.$$

Then

$$s(k+1) - s(k) = \left(1 + \left\lceil \frac{-k-1}{q} \right\rceil - \left\lceil \frac{-k}{q} \right\rceil\right) \in \{0, 1\}.$$

Hence, s is a non-decreasing function and each of its jump is exactly 1 unit. Furthermore, $s(0) = \left\lceil \frac{n}{2q-1} \right\rceil, \left\lceil \frac{n}{q} \right\rceil + \left\lceil \frac{n-n}{2q-1} \right\rceil = \left\lceil \frac{n}{q} \right\rceil$ and

$$s\left(\left\lfloor \frac{n}{2q-1} \right\rfloor\right) = \left\lfloor \frac{n}{2q-1} \right\rfloor + \left\lceil \frac{-\left\lfloor \frac{n}{2q-1} \right\rfloor}{q} \right\rceil + \left\lceil \frac{n}{2q-1} \right\rceil.$$

Hence,

$$s\left(\left\lfloor \frac{n}{2q-1} \right\rfloor\right) \geq \frac{n}{2q-1} - 1 - \frac{n}{(2q-1)q} + \frac{n}{2q-1} = \frac{n}{q} - 1$$

$$\implies s\left(\left\lfloor \frac{n}{2q-1} \right\rfloor\right) \geq \left\lceil \frac{n}{q} \right\rceil - 1.$$

This shows that $\left\lceil \frac{a}{q} \right\rceil + \left\lceil \frac{n-a}{2q-1} \right\rceil$ can take any integral values between $\left\lceil \frac{n}{q} \right\rceil$ and $\left\lceil \frac{n}{2q-1} \right\rceil$, inclusively, completing the proof. □

This leads to

Theorem 4. *For any integer x in $\left[g_{k,b,q}(n), f_{k,b,q}(n)\right]$, there is at least one maximal (k, b, q)-linear-free subset of $[n]$ containing exactly x elements.*

Proof. Equipped with Lemma 1, one can proceed by making obvious modifications to the proof in [5]. □

Lastly, we determine values of n for which the lower bound and upper bound of the cardinalities of (k, b, q)-linear-free subsets of $[n]$ are, in fact, identical.

Theorem 5. *$g_{k,b,q}(n) = f_{k,b,q}(n)$ if and only if $n < p^q(1)$.*

Proof. One can show that

$$f_{k,b,q}(n+1) = \begin{cases} f_{k,b,q}(n) & \text{if } q \nmid h(n+1), \\ f_{k,b,q}(n) + 1 & \text{if } q \mid h(n+1). \end{cases}$$

Similarly,

$$g_{k,b,q}(n+1) = \begin{cases} g_{k,b,q}(n) & \text{if } 2q-1 \nmid h(n+1), \\ g_{k,b,q}(n) + 1 & \text{if } 2q-1 \mid h(n+1). \end{cases}$$

Hence, when $n < p^q(1)$, for any $x \in [n], q \mid h(x)$ if and only if $h(x) = 0$, and so $(2q-1) \mid h(x)$. This implies that $f_{k,b,q}(n) = g_{k,b,q}(n)$ due to the recurrence relations. On the other hand, when $n \geq p^q(1)$, the rate of growth of $f_{k,b,q}(n)$ is evidently faster than that of $g_{k,b,q}(n)$, and thus, they cannot be equal. □

3. Further Research

For future research, it will be interesting to investigate the problem in a multi-dimensional space. For instance, one can start from the following generalisation for the double-free subsets:

Let m and $n_1, n_2, \dots, n_m$ be positive integers. Define $T = \{(x_1, x_2, \dots, x_m) : 1 \leq x_i \leq n_i, \forall i = \overline{1, m}\}$. Let $g(n_1, \dots, n_m)$ be the maximum number of lattice points one can choose from T such that if $(x_1, \dots, x_m)$ is chosen, then $(2x_1, \dots, 2x_m)$ is not. Determine $g(n_1, \dots, n_m)$.

It can be proven that

$$g(n_1, \dots, n_m) = \prod_{i=1}^{m} n_i - \prod_{i=1}^{m} \left\lfloor \frac{n_i}{2} \right\rfloor + g\left(\left\lfloor \frac{n_1}{4} \right\rfloor, \dots, \left\lfloor \frac{n_m}{4} \right\rfloor\right).$$

Hence,

$$g(n_1, \dots, n_m) = \sum_{i \geq 0} (-1)^i \prod_{j=1}^{m} \left\lfloor \frac{n_j}{2^i} \right\rfloor.$$

It is unclear whether Lemma 1 can be extended in this case, even to the Cartesian plane, which is critical to generalise Theorem 4.

References

1. Wang, E.T., 1989. On double-free sets of integers. *Ars Combinatoria, 28*, pp.97-100.
2. Leung, J.Y. and Wei, W.D., 1994. Maximal k-multiple-free sets of integers. *Ars Combinatoria, 38,* pp.113-117.
3. Chung, F., Erdos, P. and Graham, R., 2002. On sparse sets hitting linear forms. *J. Number Theory*. to appear, 2.
4. Spencer, J., Graham, R.L. and Witsenhausen, H.S., 1977. On extremal density theorems for linear forms. In *Number Theory and Algebra* (pp. 103-107). Academic Press.
5. You, L., Yu, G. and Liu, B., 2001. On maximal (k, b)-linear-free sets of integers and its spectrum. *Australasian Journal of Combinatorics, 23*, pp.211-216.
6. Liu, B.L. and Bo, Z., 1999. A generalization of maximal k-multiple-free sets of integers. *Ars Combinatoria, 52*, pp.285-288.
7. Wakeham, D. and Wood, D.R., 2013. On multiplicative Sidon sets. *Annual Volume 2013, 13*, p.392-401.
8. Minh, N. Q., 2021. Problems 12252. *The American Mathematical Monthly, 128*(5), p.467.