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Article

A Note on Decomposition of Tensor Product of Complete Multipartite Graphs into Gregarious Kite

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Abstract: A *kite* K is a graph which can be obtained by joining an edge to any vertex of K_3. A kite with edge set $\{ab, bc, ca, cd\}$ can be denoted as $(a, b, c; cd)$. If every vertex of a kite in the decomposition lies in different partite sets, then we say that a kite decomposition of a multipartite graph is a *gregarious* kite decomposition. In this manuscript, it is shown that there exists a decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ into gregarious kite if and only if $n^2s^2m(m-1)r(r-1) \equiv 0 \pmod 8$, where $\otimes$, $\times$ denote the wreath product and tensor product of graphs respectively. We denote a gregarious kite decomposition as GK-decomposition.

Keywords: Decomposition, Kite, Tensor product, Wreath Product

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1. Introduction

Many authors have been studied on kite-design and kite decomposition/ kite factorization and their special properties such as gregarious kite decomposition/ gregarious kite factorization. Bermond and Schonheim [1] proved that a kite-design of order n exists if and only if $n \equiv 0, 1 \pmod 8$. Wang and Chang [2, 3] considered the problem of a resolvable $(K_3 + e)$ and $(K_3 + e, \lambda)$-group divisible designs of type $g^t u^1$. Wang [4] have proven that the necessary conditions to find a resolvable $(K_3 + e)$-group divisible design of type g^u are also sufficient. Fu et al. [5] have demonstrated that there exists a GK-decomposition of $K_m(n)$ if and only if $n \equiv 0, 1 \pmod 8$ for odd m or $n \geq 4$ for even m. Gionfriddo and Milici [6] considered the uniformly resolvable decompositions of K_v and $K_v - I$ into paths and kites. To know more about kite designs, refer [7–11]. Further, the authors A. Tamil Elakkiya and A. Muthusamy [12] have shown that there is a GK-decomposition of $K_m \times K_n$ if and only if $mn(m-1)(n-1) \equiv 0 \pmod 8$. Moreover, the authors A. Tamil Elakkiya and A. Muthusamy [13] considered the existence of a GK factorization of tensor product of complete graphs.

In this way, our main concern is to find a existence of a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. In this document, it is demonstrated that a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ holds, if and only if, $n^2s^2m(m-1)r(r-1) \equiv 0 \pmod 8$.

2. Preliminary Results

Definition 1. [12] *A partition of a graph G into a collection of subgraphs $G_1, G_2, \ldots, G_r$ such that each one of it is distinct mutually by their edges together with $E(G) = \cup_{i=1}^{r} E(G_i)$ is called a* decomposition *of G; We then write G as $G = G_1 \oplus G_2 \oplus \ldots \oplus G_r$, where $\oplus$ denotes edge-disjoint sum* *of subgraphs. For an integer s, sG denotes s copies of G.*

Definition 2. [12] *A* kite *K is a graph which can be obtained by joining an edge to any vertex of* K_3*, for instance see Figure 1. A kite with edge set* $\{ab, bc, ca, cd\}$ *can be denoted as* $(a, b, c; cd)$*. If every*

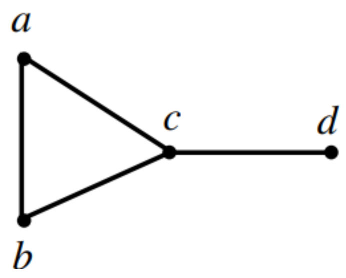

Figure 1

vertex of a kite in the decomposition lies in different partite sets, then we say that a kite decomposition of a multipartite graph is a gregarious *kite decomposition. Here we may consider a Gregarious Kite decomposition as a GK-decomposition for short.*

Definition 3. [12] *The* tensor product $G \times H$ *and the* wreath product $G \otimes H$ *of two graphs G and H are defined as follows:* $V(G \times H) = V(G \otimes H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$. $E(G \times H) = \{(u, v)(x, y) \mid ux \in E(G)$ *and* $vy \in E(H)\}$ *and* $E(G \otimes H) = \{(u, v)(x, y) \mid u = x$ *and* $vy \in E(H)$ *or* $ux \in E(G)\}$.

As tensor product has commutative and distributive property over an edge-disjoint sum of subgraphs, if decomposition of $G = G_1 \oplus G_2 \oplus \ldots \oplus G_r$ *holds, we then write as* $G \times H = (G_1 \times H) \oplus (G_2 \times H) \oplus \ldots \oplus (G_r \times H)$.

Definition 4. [12] *A graph G having partite sets* $V_1, V_2, \ldots, V_m$ *with* $|V_i| = n$, $1 \leq i \leq n$ *and* $E(G) = \{uv \mid u \in V_i$ *and* $v \in V_j, \forall i \neq j\}$ *is called* complete m-partite graph *and is denoted by* $K_m(n)$*. Note that* $K_m(n)$ *is same as the* $K_m \otimes \overline{K}_n$*. where* $\overline{K}_n$ *is the complement of a complete graph on n vertices.*

In order to prove our key result, we require the following:

Theorem 1. [5] *There exists a GK-decomposition of* $K_m(n)$ *if and only if* $m \equiv 0, 1 \pmod 8$, *for all odd n (or)* $m \geq 4$, *for even n.*

Remark 1. [5] *Let K is a kite. Then there exists a GK-decomposition of* $K \otimes \overline{K}_s$, *for all s.*

Lemma 1. [12] *For all n,* $n \neq 2$*, there exists a GK-decomposition of* $K \times K_n$*, where K is a kite.*

Theorem 2. [12] *There exists a GK-decomposition of* $K_m \times K_n$ *if and only if* $mn(m-1)(n-1) \equiv 0 \pmod 8$.

3. Existence of GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$

Lemma 2. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $m \equiv 0, 1 \pmod 8$, $n, s \equiv 1 \pmod 2$ *and for all r,* $r \neq 2$.

Proof. By Theorem 1, we have a GK-decomposition of $(K_m \otimes \overline{K}_n)$. Then $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) = (K \oplus K \oplus \ldots \oplus K) \times (K_r \otimes \overline{K}_s) = K \times (K_r \otimes \overline{K}_s) \oplus K \times (K_r \otimes \overline{K}_s) \oplus \ldots \oplus K \times (K_r \otimes \overline{K}_s)$. Now $K \times (K_r \otimes \overline{K}_s) \cong (K \times K_r) \otimes \overline{K}_s$. As we have a kite decomposition of $K \times K_r$ from Lemma 1, we then write as $(K \times K_r) \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. A GK-decomposition of $K \otimes \overline{K}_s$ can be obtained from Remark 1. Thus all together provides a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. □

Lemma 3. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $n \equiv 0 \pmod 2$ *and for all m, r, s,* $\{m, r\} \neq 2$.

Proof. **Case 1:** Let $m = 3$, $n = 2b$, $b \geq 0$ and for all r, s, $(r \neq 2)$. We write $(K_3 \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) \cong [(K_3 \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s = [(K_3 \oplus K_3 \oplus \ldots \oplus K_3) \times K_r] \otimes \overline{K}_s = [(K_3 \times K_r) \oplus (K_3 \times K_r) \oplus \ldots \oplus (K_3 \times K_r)] \otimes \overline{K}_s = [(K_3 \times K_r) \otimes \overline{K}_s] \oplus [(K_3 \times K_r) \otimes \overline{K}_s] \oplus \ldots \oplus [(K_3 \times K_r) \otimes \overline{K}_s]$. Theorem 2 gives a GK-decomposition for $K_3 \times K_r$. Now, we write as $(K_3 \times K_r) \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. A GK-decomposition exists for $K \otimes \overline{K}_s$ according to Remark 1. As a consequence of it, a GK-decomposition of $(K_3 \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ has been derived.

Case 2: A GK-decomposition for $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ have been described as follows:

Let $m \geq 4$, $n = 2b$, $b \geq 0$ and for all r, s, $(r \neq 2)$. One can obtain a GK-decomposition of $(K_m \otimes \overline{K}_n)$ from Theorem 1. Then $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) = (K \oplus K \oplus \ldots \oplus K) \times (K_r \otimes \overline{K}_s) = K \times (K_r \otimes \overline{K}_s) \oplus K \times (K_r \otimes \overline{K}_s) \oplus \ldots \oplus K \times (K_r \otimes \overline{K}_s)$. Now $K \times (K_r \otimes \overline{K}_s) \cong (K \times K_r) \otimes \overline{K}_s$. A kite decomposition exits for $K \times K_r$ by using Lemma 1. Moreover, we then write, $(K \times K_r) \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. According to Remark 1, a GK-decomposition of $K \otimes \overline{K}_s$ has been proved.

Consequently, Cases 1 and 2 lead a GK-decomposition for $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ when $n \equiv 0 \pmod 2$ and for all m, r, s, $\{m, r\} \neq 2$. □

Lemma 4. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $n \equiv 1 \pmod 2$, $s \equiv 0 \pmod 2$, *and for all* m, r, $\{m, r\} \neq 2$.

Proof. **Case 1:** Let $m = 3$, $n = 2b + 1$, $b \geq 0$, $s = 2c$, $c \geq 0$ and for all r, $(r \neq 2)$. Let us consider $(K_3 \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) \cong [(K_3 \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s = [(K_3 \oplus K_3 \oplus \ldots \oplus K_3) \times K_r] \otimes \overline{K}_s = [(K_3 \times K_r) \oplus (K_3 \times K_r) \oplus \ldots \oplus (K_3 \times K_r)] \otimes \overline{K}_s = [(K_3 \times K_r) \otimes \overline{K}_s] \oplus [(K_3 \times K_r) \otimes \overline{K}_s] \oplus \ldots \oplus [(K_3 \times K_r) \otimes \overline{K}_s]$. By Theorem 2, a GK-decomposition exists for $K_3 \times K_r$. Now $(K_3 \times K_r) \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. Now, a GK-decomposition of $K \otimes \overline{K}_s$ follows from Remark 1. Thus, the above construction provides a GK-decomposition of $(K_3 \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$.

Case 2: Let $m \geq 4$, $n = 2b + 1$, $b \geq 0$, $s = 2c$, $c \geq 0$ and for all r, $(r \neq 2)$. By Theorem 1, there is a GK-decomposition for $(K_r \otimes \overline{K}_s)$. Then $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) = (K_m \otimes \overline{K}_n) \times (K \oplus K \oplus \ldots \oplus K) = [(K_m \otimes \overline{K}_n) \times K] \oplus [(K_m \otimes \overline{K}_n) \times K] \oplus \ldots \oplus [(K_m \otimes \overline{K}_n) \times K]$.

Now $[(K_m \otimes \overline{K}_n) \times K] \cong K \times (K_m \otimes \overline{K}_n) \cong (K \times K_m) \otimes \overline{K}_n$. By Lemma 1, we have a kite decomposition of $K \times K_m$. Then $(K \times K_m) \otimes \overline{K}_n = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_n = (K \otimes \overline{K}_n) \oplus (K \otimes \overline{K}_n) \oplus \ldots \oplus (K \otimes \overline{K}_n)$. A GK-decomposition of $K \otimes \overline{K}_n$ follows from Remark 1. Thus, all together gives a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. □

Lemma 5. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $m \equiv 4, 5 \pmod 8$, n, $s \equiv 1 \pmod 2$ *and for all* r, $r \neq 2$.

Proof. Let us consider $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) \cong [(K_m \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s$. We write as $(K_m \otimes \overline{K}_n) \times K_r \cong K_r \times (K_m \otimes \overline{K}_n) \cong (K_r \times K_m) \otimes \overline{K}_n$. By Theorem 2, we have a GK-decomposition of $K_r \times K_m$. Then $(K_r \times K_m) \otimes \overline{K}_n = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_n = (K \otimes \overline{K}_n) \oplus (K \otimes \overline{K}_n) \oplus \ldots \oplus (K \otimes \overline{K}_n)$. By Remark 1, we can obtain a GK-decomposition of $K \otimes \overline{K}_n$. Further, $[(K_m \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. A GK-decomposition of $K \otimes \overline{K}_s$ follows from Remark 1. Thus, all together gives a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. □

Lemma 6. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $m \equiv 2, 3 \pmod 4$, $m \neq 2$, n, $s \equiv 1 \pmod 2$ *and* $r \equiv 0, 1 \pmod 8$.

Proof. A GK-decomposition of $K_r \otimes \overline{K}_s$ can be obtained from Theorem 1. We then write, $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) = (K_m \otimes \overline{K}_n) \times (K \oplus K \oplus \ldots \oplus K) = [(K_m \otimes \overline{K}_n) \times K] \oplus [(K_m \otimes \overline{K}_n) \times K] \oplus \ldots \oplus [(K_m \otimes \overline{K}_n) \times K]$. Now, we write as $[(K_m \otimes \overline{K}_n) \times K] \cong K \times (K_m \otimes \overline{K}_n) \cong (K \times K_m) \otimes \overline{K}_n$. A kite decomposition of $K \times K_m$ has been derived from Lemma 1. Then $(K \times K_m) \otimes \overline{K}_n = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_n = (K \otimes \overline{K}_n) \oplus (K \otimes \overline{K}_n) \oplus \ldots \oplus (K \otimes \overline{K}_n)$. A GK-decomposition of $K \otimes \overline{K}_n$ exists according to Remark 1. Consequently, a GK-decomposition exists for $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. □

Lemma 7. *A GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *exists, if* $m \equiv 2, 3 \pmod 4$, $m \neq 2, n, s \equiv 1 \pmod 2$ *and* $r \equiv 4, 5 \pmod 8$.

Proof. We write $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s) \cong [(K_m \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s$. Now $(K_m \otimes \overline{K}_n) \times K_r \cong K_r \times (K_m \otimes \overline{K}_n) \cong (K_r \times K_m) \otimes \overline{K}_n$. By Theorem 2, we have a GK-decomposition of $K_r \times K_m$. Then $(K_r \times K_m) \otimes \overline{K}_n = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_n = (K \otimes \overline{K}_n) \oplus (K \otimes \overline{K}_n) \oplus \ldots \oplus (K \otimes \overline{K}_n)$. A GK-decomposition of $K \otimes \overline{K}_n$ can be obtained from Remark 1. Further, $[(K_m \otimes \overline{K}_n) \times K_r] \otimes \overline{K}_s = (K \oplus K \oplus \ldots \oplus K) \otimes \overline{K}_s = (K \otimes \overline{K}_s) \oplus (K \otimes \overline{K}_s) \oplus \ldots \oplus (K \otimes \overline{K}_s)$. A GK-decomposition of $K \otimes \overline{K}_s$ follows from Remark 1. Thus the above describes a GK-decomposition for $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. □

4. Main Result

Theorem 3. *There exists a GK-decomposition of* $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ *if and only if* $n^2 s^2 m(m-1)r(r-1) \equiv 0 \pmod 8$.

Proof. Necessity: The number of edges of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ is $\frac{n^2 s^2 m(m-1)r(r-1)}{2}$ and number of edges of a kite K is 4. Therefore, the edge divisibility condition for a graph $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$ is $\frac{n^2 s^2 m(m-1)r(r-1)}{8}$.

Sufficiency: It has been derived from Lemmas 2 - 7. □

5. Conclusion

In Section 3, we give a complete solution for the existence of a GK-decomposition of $(K_m \otimes \overline{K}_n) \times (K_r \otimes \overline{K}_s)$. In future, one can find the existence of a GK-factorization of Tensor product complete multipartite graph.

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Conflict of Interest

The author have no conflict of interest.

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