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Article

A Sufficient Condition for Fractional $(2, b, k)$-Critical Covered Graphs

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Abstract:
Zhou, Xu and Sun [S. Zhou, Y. Xu, Z. Sun, Degree conditions for fractional (a, b, k)-critical covered graphs, Information Processing Letters 152(2019)105838] defined the concept of a fractional (a, b, k)-critical covered graph, namely, a graph G is a fractional (a, b, k)-critical covered graph if after removing any k vertices of G, the remaining graph of G is a fractional $[a, b]$-covered graph. In this paper, we prove that a graph G with $\delta(G) \geq 2 + k$ is fractional $(2, b, k)$-critical covered if $bind(G) > \frac{b+k}{b-1}$, where $k \geq 0$ and $b \geq 2 + k$ are two integers.

Keywords: Minimum degree, Binding number, Fractional $[a, b]$-factor, Fractional $[a, b]$-covered graph, Fractional (a, b, k)-critical Covered graph

1. Introduction

All graphs discussed here are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G, and let $N_G(x)$ denote the neighborhood of x in G. Let X be a subset of $V(G)$. We write $N_G(X) = \bigcup_{x \in X} N_G(x)$, and write $G[X]$ for the subgraph of G induced by X. Define $G - X = G[V(G) \setminus X]$. A subset $X \subseteq V(G)$ is called independent if $G[X]$ does not admit edges. The minimum degree of G is denoted by $\delta(G)$, the number of isolated vertices in G is denoted by $i(G)$, and the complete graph of order n is denoted by K_n. The binding number of G is defined by Woodall [1] as

$$bind(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let a, b and k be three nonnegative integers with $a \leq b$. Then a spanning subgraph F of G is called an $[a, b]$-factor of G if $a \leq d_F(x) \leq b$ holds for every $x \in V(G)$. Let h be a function from $E(G)$ to $[0, 1]$. Then we call $G[F_h]$ a fractional $[a, b]$-factor of G with an indicator function h if $a \leq \sum_{e \ni x} h(e) \leq b$ for all $x \in V(G)$, where $F_h = \{e : e \in E(G), h(e) > 0\}$. In particular, when $a = b = r$, a fractional $[a, b]$-factor is a fractional r-factor. A fractional $[a, b]$-factor is just an $[a, b]$-factor if $h(e) \in \{0, 1\}$ for each $e \in E(G)$. A graph G is a fractional $[a, b]$-covered graph if for any $e \in E(G)$, there exists a fractional $[a, b]$-factor $G[F_h]$ such that $h(e) = 1$. In particular, when $a = b = r$, a fractional $[a, b]$-covered graph is a fractional r-covered graph. A graph G is a fractional (a, b, k)-critical covered graph if after removing any k vertices of G, the remaining graph of G is a fractional $[a, b]$-covered graph.

In particular, when $a = b = r$, a fractional (a, b, k)-critical covered graph is a fractional (r, k)-critical covered graph.

Many scholars studied the problems of factors of graphs and fractional factors of graphs. Zhou, Xu and Sun [2] derived a degree condition for the existence of fractional (a, b, k)-critical covered graphs. Zhou [3] presented a neighborhood union condition for a graph being a fractional (a, b, k)-critical covered graph. In this paper, we study the fractional $(2, b, k)$-critical covered graph problem, and get a result on fractional $(2, b, k)$-critical covered graphs which is the following theorem.

Theorem 1. *Let $k \geq 0$ and $b \geq 2 + k$ be two integers, and let G be a graph with $\delta(G) \geq 2 + k$. If*

$$bind(G) > \frac{b+k}{b-1},$$

then G is fractional $(2, b, k)$-critical covered.

We immediately get the following corollary by setting $k = 0$ in Theorem 1.

Corollary 1. *Let $b \geq 2$ be an integer, and let G be a graph with $\delta(G) \geq 2$. If*

$$bind(G) > \frac{b}{b-1},$$

then G is fractional $[2, b]$-covered.

We promptly obtain the following corollary if $b = 2$ in Corollary 1.

Corollary 2. *Let G be a graph with $\delta(G) \geq 2$. If*

$$bind(G) > 2,$$

then G is fractional 2-covered.

2. The Proof of Theorem 1

We begin this section with one definition. For any $X \subseteq V(G)$ and $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \leq a\}$, we define $\varepsilon(X, Y)$ as follows:

$$\varepsilon(X,Y) = \begin{cases} 2, & \text{if } X \text{ is not independent,} \\ 1, & \text{if } X \text{ is independent and there is an edge joining} \\ & X \text{ and } V(G) \setminus (X \cup Y), \text{ or there is an edge } e = xy \\ & \text{joining } X \text{ and } Y \text{ such that } d_{G-X}(y) = a \text{ for } y \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

Li, Yan and Zhang [4] posed a necessary and sufficient condition for a graph to be a fractional $[a, b]$-covered graph, which is a special case of Li, Yan and Zhang's fractional (g, f)-covered graph theorem.

Theorem 2. *([4]) Let $b \geq a \geq 0$ be two integers. Then a graph G is a fractional $[a, b]$-covered graph if and only if for any $X \subseteq V(G)$,*

$$a|Y| - d_{G-X}(Y) \leq b|X| - \varepsilon(X, Y),$$

where $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \leq a\}$.

The proof of Theorem 1 depends heavily on the following theorem, which is equivalent to Theorem 2.

Theorem 3. *([4]) Let $b \geq a \geq 0$ be two integers. Then a graph G is a fractional $[a, b]$-covered graph if and only if for any $X \subseteq V(G)$,*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-X) \leq b|X| - \varepsilon(X, Y),$$

where $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \leq a\}$ and $p_j(G-X)$ denotes the number of vertices in $G - X$ with degree j.

Proof of Theorem 1. Let $Q \subseteq V(G)$ with $|Q| = k$, and let $H = G - Q$. It suffices to verify that H is a fractional $[2, b]$-covered graph. Suppose, to the contrary, that H is not a fractional $[2, b]$-covered graph. Then it follows from Theorem 3 that

$$2p_0(H-X) + p_1(H-X) \geq b|X| - \varepsilon(X, Y) + 1 \tag{1}$$

for some $X \subseteq V(H)$, where $Y = \{x : x \in V(H) \setminus X, d_{H-X}(x) \leq 2\}$.

We write $Y_0 = \{x : x \in V(H)\setminus X, d_{H-X}(x) = 0\}$ and $Y_1 = \{x : x \in V(H)\setminus X, d_{H-X}(x) = 1\}$. Obviously, we obtain $|Y_0| = p_0(H-X) = i(H-X)$ and $|Y_1| = p_1(H-X)$. The following proof will be divided into three cases.

Case 1. $Y_1 = \emptyset$.

Clearly, $p_1(H-X) = 0$. Combining this with (1) and $\varepsilon(X, Y) \leq |X|$, we admit

$$2p_0(H-X) = 2p_0(H-X) + p_1(H-X) \geq b|X| - \varepsilon(X, Y) + 1 \geq b|X| - |X| + 1 \geq 1,$$

namely,

$$p_0(H-X) \geq \frac{1}{2}. \tag{2}$$

According to (2) and the integrity of $p_0(H-X)$, we get

$$p_0(H-X) \geq 1,$$

which implies that there exists $x_0 \in V(H) \setminus X$ with $d_{H-X}(x_0) = 0$. Combining this with $\delta(G) \geq 2 + k$ and $H = G - Q$, we have

$$\begin{aligned} 2 + k &\leq \delta(G) \leq d_G(x_0) \leq d_{G-Q}(x_0) + |Q| \\ &= d_H(x_0) + k \leq d_{H-X}(x_0) + |X| + k = |X| + k, \end{aligned}$$

that is,

$$|X| \geq 2. \tag{3}$$

Note that $Y_0 \neq \emptyset$ by $p_0(H-X) \geq 1$, and so $N_G(Y_0) \neq V(G)$. In terms of (1), (3), $\varepsilon(X, Y) \leq 2$, $p_1(H-X) = |Y_1| = 0$ and $b \geq 2 + k \geq 2$, we get

$$\begin{aligned} bind(G) &\leq \frac{|N_G(Y_0)|}{|Y_0|} \leq \frac{|Q| + |X|}{|Y_0|} = \frac{k + |X|}{p_0(H-X)} \\ &= \frac{2k + 2|X|}{2p_0(H-X) + p_1(H-X)} \\ &\leq \frac{2k + 2|X|}{b|X| - \varepsilon(X, Y) + 1} \leq \frac{2k + 2|X|}{b|X| - 1} \\ &= \frac{2}{b} \cdot \Big(1 + \frac{bk + 1}{b|X| - 1}\Big) \leq \frac{2}{b} \cdot \Big(1 + \frac{bk + 1}{2b - 1}\Big) \\ &= \frac{4 + 2k}{2b - 1} \leq \frac{2b + 2k}{2b - 1} < \frac{2b + 2k}{2b - 2} = \frac{b + k}{b - 1}, \end{aligned}$$

which contradicts $bind(G) > \frac{b+k}{b-1}$.

Case 2. $Y_1 \neq \emptyset$ and $Y_1 = N_{H-X}(Y_1)$.

We easily see that $|Y_1| = 2t \geq 2$, where t is a positive integer. For $x_1 \in Y_1$, we admit $d_{H-X}(x_1) = 1$. Then using $H = G - Q$ and $\delta(G) \geq 2 + k$, we derive

$$\begin{aligned} 2 + k &\leq \delta(G) \leq d_G(x_1) \leq d_{G-Q}(x_1) + |Q| \\ &= d_H(x_1) + k \leq d_{H-X}(x_1) + |X| + k \\ &= 1 + |X| + k, \end{aligned}$$

namely,

$$|X| \geq 1.$$

If $|X| = 1$, then $p_0(H - X) = 0$ (since $\delta(G) \geq 2 + k$) and $\varepsilon(X, Y) \leq 1$. It follows from (1) and the hypothesis of Theorem 1 that

$$\begin{aligned} \frac{b+k}{b-1} &< bind(G) \leq \frac{|N_G(Y_1 \setminus \{x_1\})|}{|Y_1 \setminus \{x_1\}|} \leq \frac{|Q| + |X| + |Y_1| - 1}{|Y_1| - 1} \\ &= \frac{k + |Y_1|}{|Y_1| - 1} = \frac{k + p_1(H-X)}{p_1(H-X) - 1} = 1 + \frac{1+k}{p_1(H-X) - 1} \\ &= 1 + \frac{1+k}{2p_0(H-X) + p_1(H-X) - 1} \\ &\leq 1 + \frac{1+k}{b|X| - \varepsilon(X,Y)} \leq 1 + \frac{1+k}{b|X| - 1} = 1 + \frac{1+k}{b-1} \\ &= \frac{b+k}{b-1}, \end{aligned}$$

which is a contradiction. Next, we consider $|X| \geq 2$.

In light of (1), $|X| \geq 2$, $\varepsilon(X, Y) \leq 2$ and $b \geq 2 + k$, we deduce

$$\begin{aligned} bind(G) &\leq \frac{|N_G(Y_0 \cup (Y_1 \setminus \{x_1\}))|}{|Y_0 \cup (Y_1 \setminus \{x_1\})|} \leq \frac{|Q| + |X| + |Y_1| - 1}{|Y_0| + |Y_1| - 1} \\ &= \frac{k + |X| + p_1(H-X) - 1}{p_0(H-X) + p_1(H-X) - 1} \\ &= 1 + \frac{k + |X| - p_0(H-X)}{p_0(H-X) + p_1(H-X) - 1} \\ &\leq 1 + \frac{k + |X| - p_0(H-X)}{b|X| - \varepsilon(X,Y) - p_0(H-X)} \\ &\leq 1 + \frac{k + |X| - p_0(H-X)}{b|X| - 2 - p_0(H-X)} \leq 1 + \frac{k + |X|}{b|X| - 2} \\ &= 1 + \frac{1}{b} \cdot \left(1 + \frac{2 + bk}{b|X| - 2}\right) \leq 1 + \frac{1}{b} \cdot \left(1 + \frac{2 + bk}{2b - 2}\right) \\ &= \frac{2b + k}{2b - 2} \leq \frac{2b + 2k}{2b - 2} = \frac{b+k}{b-1}, \end{aligned}$$

which contradicts $bind(G) > \frac{b+k}{b-1}$.

Case 3. $Y_1 \neq \emptyset$ and $Y_1 \neq N_{H-X}(Y_1)$.

In this case, there exists $x_2 \in N_{H-X}(Y_1) \setminus Y_1$ with $d_{H-X}(x_2) \geq 2$, and so there exists $x_3 \in Y_1$ such that $x_3x_2 \in E(H - X)$ and $x_3x \notin E(H - X)$ for any $x \in V(H) \setminus (X \cup \{x_2\})$. Thus, we deduce $|N_G(Y_0 \cup Y_1)| \leq |Q| + |X| + |Y_1|$. Combining this with (1), $|X| \geq 1$ (since $Y_1 \neq \emptyset$ and $\delta(G) \geq 2 + k$), $\varepsilon(X, Y) \leq 2$, $b \geq 2 + k$ and the hypothesis of Theorem 1, we infer

$$\frac{b+k}{b-1} < bind(G) \leq \frac{|N_G(Y_0 \cup Y_1)|}{|Y_0 \cup Y_1|} \leq \frac{|Q| + |X| + |Y_1|}{|Y_0| + |Y_1|}$$

$$
\begin{aligned}
&= \frac{k + |X| + p_1(H - X)}{p_0(H - X) + p_1(H - X)} \\
&= 1 + \frac{k + |X| - p_0(H - X)}{p_0(H - X) + p_1(H - X)} \\
&= 1 + \frac{k + |X| - p_0(H - X)}{2p_0(H - X) + p_1(H - X) - p_0(H - X)} \\
&\leq 1 + \frac{k + |X| - p_0(H - X)}{b|X| - \varepsilon(X, Y) + 1 - p_0(H - X)} \\
&\leq 1 + \frac{k + |X| - p_0(H - X)}{b|X| - 1 - p_0(H - X)} \\
&\leq 1 + \frac{k + |X|}{b|X| - 1} = 1 + \frac{1}{b} \cdot \Big(1 + \frac{1 + bk}{b|X| - 1}\Big) \\
&\leq 1 + \frac{1}{b} \cdot \Big(1 + \frac{1 + bk}{b - 1}\Big) = \frac{b + k}{b - 1},
\end{aligned}
$$

which is a contradiction. This finishes the proof of Theorem 1. □

3. Sharpness of Theorem 1

We show that the binding number condition $bind(G) > \frac{b+k}{b-1}$ in Theorem 1 cannot be replaced by $bind(G) \geq \frac{b+k}{b-1}$.

We construct a graph $G = K_{1+k} \vee (\frac{b}{2}K_2)$, where $\vee$ means "join", and b and k are two integers such that $b \geq 2 + k$ and b is even. We easily deduce that $bind(G) = \frac{b+k}{b-1}$. Let $Q = V(K_k) \subseteq V(K_{1+k})$ and $H = G - Q = K_1 \vee (\frac{b}{2}K_2)$. We select $X = V(K_1)$ and $Y = \{x : x \in V(H) \setminus X, d_{H-X}(x) \leq 2\}$ in H. It is obvious that $\varepsilon(X, Y) = 1$ by the definition of $\varepsilon(X, Y)$. Hence, we admit

$$2p_0(H - X) + p_1(H - X) = b = b|X| - \varepsilon(X, Y) + 1 > b|X| - \varepsilon(X, Y).$$

In light of Theorem 3, H is not a fractional $[2, b]$-covered graph, and so G is not a fractional $(2, b, k)$-critical covered graph.

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