Journal of Combinatorial Mathematics and Combinatorial Computing, 120: 31–42
DOI:10.61091/jcmcc120-003
http://www.combinatorialpress.com/jcmcc
Received 29 May 2024, Accepted 1 June 2024, Published 30 June 2024

Article

Carathéodory Number of P_3-convexity of Claw-Free Graphs

Athulyakrishna Kuruppankandy[1,*], **T.V. Ramakrishnan**[1], **and Manoj Changat**[2]

[1] Department of Mathematical Sciences, Kannur University, Mangattuparamba, Kannur, India-670567

[2] Department of Futures Studies, University of Kerala, Thiruvananthapuram-695581, India

* **Correspondence:** athulyakrishna@kannuruniv.ac.in

Abstract: Given a connected simple undirected graph $G = (V, E)$, a subset S of V is P_3-convex if each vertex of G not in S has at most one neighbor in S. The P_3-convex hull $\langle S \rangle$ of S is the smallest P_3-convex set containing S. A Carathéodory set of G is a set $S \subseteq V$ such that $\langle S \rangle \setminus \bigcup_{w \in S} \langle S \setminus \{w\} \rangle$ is non-empty. The Carathéodory number of G, denoted by $C(G)$, is the largest cardinality of a Carathéodory set of G. In this paper, we settle the conjecture posed by Barbosa et al. appeared in [SIAM J. Discrete Math. 26 (2012) 929–939] in the affirmative, which states that for a claw-free graph G of order $n(G)$, the Carathéodory number $C(G)$ of the P_3-convexity satisfies $C(G) \leq \frac{2n(G)+6}{5}$. Furthermore, we determine all graphs attaining the bound.

Keywords: Carathéodory number, P_3-convexity, Claw-free graph, Convex hull

---

1. Introduction

The theory of abstract convexity spaces was originally developed to generalize classical convexity invariants such as Helly, Carathéodory, and Radon numbers in $\mathbb{R}^d$. This idea was later extended to abstract convexity spaces by notable researchers such as Sierksma [1–3] and Duchet [4]. A comprehensive treatment of these concepts of abstract convexity can be found in the work of Van de Vel [5]. Graph convexity spaces have been a topic of much interest among the several structures of abstract convexity spaces. Many researchers have studied graph convexities from different perspectives. The most prominent types of graph convexities are defined in terms of paths in the graph, such as geodesic, induced path, and all-paths convexity [6–10]. An interesting type of path convexity in graphs is defined using the paths P_3 on three vertices, known as P_3-convexity. The P_3-convexity was initially studied for directed graphs by Parker et al. in [11]. For undirected graphs, the P_3-convexity has been discussed in [12–14], and [15].

The Carathéodory number is a classical convexity invariant whose relationship with two other classical convexity invariants, namely, the Helly and Radon numbers of abstract convexities, has been extensively studied in [5, 16]. Parker et al. studied the Carathéodory number of the P_3-convexity of multipartite tournaments in [11] and proved that the maximum possible Carathéodory number is 3. The Carathéodory number of the P_3-convexity in undirected graphs was investigated by Barbosa et al. in [13] and Coelho et al. in [15]. Some general results concerning the Carathéodory number of the P_3-convexity are presented in [13], where it is shown that determining the Carathéodory number is NP-hard even in bipartite graphs. However, efficient algorithms are presented to determine such parameter in trees and block graphs. Coelho et al. further demonstrated in [15] that the Carathéodory number of P_3-convexity in a chordal graph can be determined in polynomial time. In this paper, we aim to contribute to the study of the Carathéodory number of P_3-convexity in claw-free graphs. This paper builds upon a conjecture posed in [13], which states that the Carathéodory number of P_3-convexity on claw-free graphs is less than or equal to $\frac{2n(G)+6}{5}$. Our main objective is to prove this conjecture in the affirmative and to identify all extremal graphs that attain the bound. The paper is structured into two sections. In Section 2, we settle the conjecture for the Carathéodory number of the P_3-convexity in claw-free graphs. To aid in our discussion, we introduce the notations and terminology for the various concepts that we employ.

We restrict our attention to finite, simple, and undirected graphs. We denote a graph by $G = (V, E)$, where V and E are the sets of vertices and edges, respectively. Let C be a collection of subsets of V. We say that C is a convexity on V if the following conditions hold:

(1): $\emptyset, V \in C$, and
(2): C is closed under arbitrary intersections.

A convexity space is a pair (V, C) where V is a nonempty finite set and C is a convexity on V, with the members of C referred to as convex sets [11, 12]. For a connected graph $G = (V, E)$ and a convexity C on V such that (V, C) is a convexity space, we form the graph convexity space (G, C) [11, 12]. Given a subset $S \subseteq V$, the smallest convex set containing S is denoted by $\langle S \rangle$ and call it the convex hull of S. We say that a subset $A \subseteq V$ of a graph G is a Carathéodory set if $\langle A \rangle \neq \bigcup_{a\in A} \langle A \setminus \{a\} \rangle$. Equivalently, the set A is a Carathéodory set if $\langle A \rangle \setminus \bigcup_{a\in A} \langle A \setminus \{a\} \rangle \neq \emptyset$. We denote this set as $\partial H_G(S)$. The Carathéodory number $C(G)$ of a convexity space C is the maximum cardinality of a Carathéodory set [10, 11, 13, 15, 17]. Given a graph $G = (V, E)$, a set $S \subseteq V$ is said to be P_3-convex if for every path xyz in G where $x, z \in S$, it holds that the vertex y also belongs to S. In other words, a set $S \subseteq V$ is P_3-convex if every vertex of G outside of S has at most one neighbor in S. The collection of all P_3-convex sets in G form a convexity in G, called the P_3-convexity. We say that a graph G is claw-free if it does not contain the claw, which is the complete bipartite graph $K_{1,3}$, as an induced subgraph. Next Proposition states some elementary properties of Carathéodory sets.

Proposition 1. *Let G be a graph and let S be a Carathéodory set of G.*

a) If G has order at least 2 and is either complete, or a path, or a cycle, then $c(G) = 2$.
b) If S has order at least 2, then every vertex u in S lies on a path uvw of order 3 such that $v \in V(G) \setminus \langle S \setminus \{u\} \rangle$ and $w \in \langle S \setminus \{u\} \rangle$.
c) No proper subset S' of S satisfies $\langle S' \rangle = V(G)$.
d) The convex hull $\langle S \rangle$ of S induces a connected subgraph of G.

2. Upper bound of Carathéodory Number of Claw-free Graphs

In this section, we settle the Conjecture 4 on the upper bound of Carathéodory number of claw-free graphs from [13]. We state it as a theorem (Theorem1). To establish the theorem, we commence by introducing relevant notations and subsequently derive a lemma that specifically addresses a single case of the theorem. Let G be a graph with vertex set V and let S be a subset of V. We prove the lemma and the theorem using the discharging method, similar to the approach used to prove a weaker improvement of Conjecture 4, stated as Theorem 5 in [13]. This theorem asserts that if G is a claw-free graph of order $n(G)$, then $C(G) \leq \frac{8n(G)+12}{17}$. In [13], the proof technique for Theorem 5 involves constructing sets A, B and C in a specific manner and assigning initial charges $\frac{1}{3|A|}, \frac{1}{3|B|}$, and $\frac{1}{3|C|}$ to vertices in A, B, and C, respectively. In [13], only initial charges are assigned to the vertices. However, here we not only give initial charges to vertices but also distribute the charges of

some vertices to some of the neighbors of the vertices by decreasing the charges of these vertices and increasing the charges of their neighbors. For convenience, we use a function to manage the charges, ensuring that the total charge remains constant. We define the function f from V to the set of non-negative rational numbers $\mathbb{Q}^+$. We first choose a vertex sequence with certain properties.
By definition of P_3-convex hull, for any $v \in \langle S \rangle$, there exists a k such that $v \in P_3^k[S]$. Consequently, v has two neighbors in $P_3^{k-1}[S]$, denoted as v_i and v_j. As $v_i, v_j \in P_3^{k-1}[S]$, there exist two neighbors of v_i and two neighbors of v_j in $P_3^{k-2}[S]$. By iteration, this process yields a sequence $v_1, v_2, \ldots, v_r = v$ such that each v_i has at least two neighbors in $S \cup \{v_1, v_2, \ldots, v_{i-1}\}$. Now, assume that $S = \{v_1, v_2, \ldots, v_l\}$ is a Carathéodory set of G. Choose the sequence $v_{l+1}, v_{l+2}, \ldots, v_k$ consisting of minimum number of distinct vertices (having minimum index) in $V(G) \setminus S$ such that $v_k \in \partial H_G(S)$, and for every v_i with $l + 1 \leq i \leq k$, there are 2 neighbors of v_i in $v_1, v_2, \ldots, v_{i-1}$. Since S is a Carathéodory set of G, every v_i with $1 \leq i \leq l$ has at least one neighbor v_j with $j > i$ such that v_j has only one neighbor in $\{v_1, v_2, \ldots, v_{j-1}\} \setminus \{v_i\}$. For any v_i such that $l + 1 \leq i \leq k$, let S_{v_i} be the set of 2 neighbors of v_i in $\{v_1, v_2, \ldots, v_{i-1}\}$ with minimum indices, and let $s_{v_i v_j} \in S_{v_i}$ be the vertex with minimum index other than v_j. Since by the choice of the sequence $v_{l+1}, v_{l+2}, \ldots, v_k$, for every i with $l+1 \leq i \leq k$, there are 2 distinct vertices in $\{v_1, v_2, \ldots, v_k\}$ which are adjacent to v_i, S_{v_i} and $s_{v_i v_j}$ are well-defined. Now define

$$f(v) = \begin{cases} \dfrac{2}{5}, & \text{if } v \in \{v_1, v_2, \ldots, v_{k-1}\}, \\ \dfrac{2}{5} + \dfrac{6}{5} = \dfrac{8}{5}, & \text{if } v = v_k, \\ 0, & \text{otherwise.} \end{cases}$$

For every vertex v in $V(G)$, there is a $\frac{2}{5}$ contribution to the sum $\frac{2n(G)+6}{5}$, since $n(G) = |V|$. So

$$\sum_{i=1}^{k} f(v_i) = \frac{2|\{v_1, v_2, \ldots, v_k\}| + 6}{5} \leq \frac{2n(G) + 6}{5}.$$

We will adjust the f-values of certain vertices such that eventually, we achieve $f(v) \geq 1$ for all $v \in S$, while ensuring that $f(v_i) \geq 0$ for all v_i and maintaining the sum $\sum_{i=1}^{k} f(v_i)$ as a constant throughout. Note that after the transformations, $f(v_i) \geq 0$ for all v_i, and hence $\sum_{i=l+1}^{k} f(v_i) \geq 0$. Therefore, we have:

$$C(G) = |S| = l \leq \sum_{v \in S} f(v) \leq \sum_{i=1}^{k} f(v_i) \leq \frac{2n(G) + 6}{5},$$

which is the required inequality.

Lemma 1. *Let G be a claw-free graph of order $n(G)$ and S be a Carathéodory set of G. Let $v_{l+1}, v_{l+2}, \ldots, v_k$ be the sequence that possesses the properties described above . If $S \cap S_{v_k} \neq \emptyset$, then $C(G) \leq \dfrac{2n(G) + 6}{5}$.*

Proof. Let G be a claw-free graph, and let $S = \{v_1, v_2, \ldots, v_l\}$ be a Carathéodory set of cardinality $C(G)$. We now describe a procedure to adjust the values of $f(v)$ such that $f(v) \geq 1$ for all $v \in S$, $f(v) \geq 0$ for all v, and $\sum_{i=1}^{k} f(v_i)$ remains unchanged.

Now consider v_k. Let $u_1, u_2 \in S_{v_k}$. Since $S_{v_k} \cap S \neq \emptyset$, we have 2 cases.

Case 1: Both u_1, u_2 belongs to S.
Case 2: One of the u_1, u_2 belongs to S and other does not belong to S.

Case (1): If $\{u_1, u_2\} \subseteq S$, then $S = \{u_1, u_2\}$ because $v_k \in \langle \{u_1, u_2\} \rangle$ and $v_k \in \partial H_G(S)$. We can then adjust the values of $f(u_1)$ and $f(u_2)$ as follows: $f(u_1) \longrightarrow f(u_1) + \dfrac{3}{5}$, $f(u_2) \longrightarrow f(u_2) + \dfrac{3}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{6}{5}$. Note that this transformation does not change the value of $\Sigma_{i=1}^{k} f(v_i)$ and guarantees that $f(v) \geq 1$ for all v in S and $f(v) \geq 0$ for all v in $V(G)$.

Case (2): If one of the vertices u_1, u_2, say $u_1 \in S$ and the other $u_2 \notin S$. Then there are two cases: $u_1u_2 \in E(G)$ or $u_1u_2 \notin E(G)$.

Subcase (2.1): $u_1u_2 \in E(G)$.

Since $u_2 \notin S$ implies that $u_2 \in \{v_{l+1}, \ldots, v_k\}$. So $S_{u_2} \neq \emptyset$. Let $u_3 = s_{u_2u_1}$. Then change $f(u_1) \longrightarrow f(u_1) + \frac{3}{5}$, $f(u_3) \longrightarrow f(u_3) + \frac{4}{5} = \frac{6}{5}$, $f(u_2) \longrightarrow f(u_2) - \frac{2}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{5}{5}$. Note that $f(v_i) \geq 0$ and $\sum_{i=1}^{k} f(v_i)$ is not changed under the transformation.

If $u_3 \in S$ (as in Figure 1), then $S = \{u_1, u_3\}$ since $v_k \in \langle\{u_1, u_3\}\rangle$, $\{u_1, u_3\} \subset S$ and S is a Carathéodory set. Note that $f(v) \geq 1$ for all v in S and $f(v) \geq 0$ for all $v \in V(G)$. Also, $\sum_{i=1}^{k} f(v_i)$ is not changed.

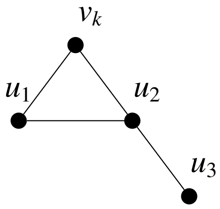

Figure 1

Suppose $u_3 \notin S$. Then we know that either u_1 is in S_{u_3} or it is not. In the former case, as shown in Figure 2(i), we can select the vertex u_4 (distinct from u_1) and update f as follows:$f(u_4) \longrightarrow f(u_4) + \frac{4}{5}$, $f(u_3) \longrightarrow f(u_3) - \frac{4}{5}$. Here $f(u_3) = \frac{6}{5} - \frac{4}{5} \geq 0$. We can then continue the process with u_4 in place of u_3. If $u_1 \notin S_{u_3}$, we let $S_{u_3} = \{u_4, u_5\}$. Since the induced subgraph $G[\{u_2, u_3, u_4, u_5\}]$ cannot be a claw, we know that either $u_2u_4 \in E(G)$ or $u_2u_5 \in E(G)$, or $u_4u_5 \in E(G)$. However, by the choice of u_2, u_3, u_4, u_5, we have u_2u_4 and u_2u_5 not in $E(G)$. Therefore, we conclude that u_4u_5 must be an edge in G, as depicted in Figure 2(ii).

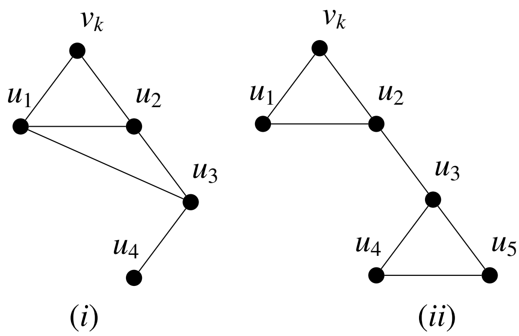

Figure 2

If $u_4, u_5 \in S$, then $S = \{u_1, u_4, u_5\}$ similarly as above. We can now change $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(u_5) \longrightarrow f(u_5) + \frac{3}{5}$, $f(u_3) \longrightarrow f(u_3) - \frac{6}{5} = 0$ to ensure that $f(v) \geq 1$ for all v in S, without changing $\Sigma_{i=1}^{k} f(v_i)$ and $f(v) \geq 0$ for all $v \in V(G)$.

If $|S_{u_3} \cap S| = 1$, we let $u_4 \in S$ and $u_5 \notin S$. Then we let $u_6 = s_{u_5u_4}$. We can now change $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(u_6) \longrightarrow f(u_6) + \frac{4}{5}$, $f(u_5) \longrightarrow f(u_5) - \frac{2}{5}$, $f(u_3) \longrightarrow f(u_3) - \frac{5}{5}$, and continue with u_6. If both u_4 and u_5 are not in S, then there are three possibilities for $|S_{u_4} \cap S_{u_5}|$: it can be 0, 1, or 2.

Subsubcase 2.1.1: $|S_{u_4} \cap S_{u_5}| = 0$.

If either $u_5 \in S_{u_4}$ or $u_4 \in S_{u_5}$. Then rename the vertex which has 2 neighbors in $(S_{u_4} \cup S_{u_5}) \setminus \{u_4, u_5\}$ as u_4, and the other vertex as u_5 if necessary. Let $u_6 = s_{u_5u_4}$. If $u_5 \notin S_{u_4}$ and $u_4 \notin S_{u_5}$, we rename

u_4 and u_5 if necessary such that S_{u_5} contains the vertex w in $S_{u_4} \cup S_{u_5}$ with the maximum index. Let $u_6 = s_{u_5 w}$. In both cases, as shown in Figure 3, we change $f(u_4) \longrightarrow f(u_4) + \frac{4}{5}$, $f(u_6) \longrightarrow f(u_6) + \frac{4}{5}$, $f(u_5) \longrightarrow f(u_5) - \frac{2}{5}$, and $f(u_3) \longrightarrow f(u_3) - \frac{6}{5}$, and continue with u_4 and u_6.

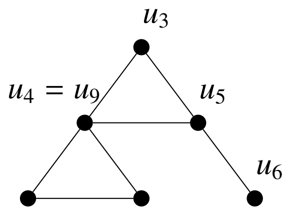

Figure 3

Subsubcase 2.1.2: $|S_{u_4} \cap S_{u_5}| = 1$.

Let $u_6 \in S_{u_4} \cap S_{u_5}$. If $u_4 \in S_{u_5}$ or $u_5 \in S_{u_4}$, rename the vertex in $\{u_4, u_5\}$ that has two neighbors in $S_{u_4} \cup S_{u_5} \setminus \{u_5, u_4\}$ as u_4, and the other vertex as u_5 if necessary. Then change $f(u_4) \longrightarrow f(u_4) + \dfrac{4}{5}$ and $f(u_3) \longrightarrow f(u_3) - \dfrac{4}{5}$, and continue with u_4. If $u_5 \notin S_{u_4}$ and $u_4 \notin S_{u_5}$, rename the vertices in u_4, u_5 if necessary such that S_{u_5} contains the vertex v with the largest index in $S_{u_4} \cup S_{u_5}$ other than u_6. Let $S_{u_4} = \{u_6, u_7\}$. If $u_6 \in S$ or $v \notin S_{u_6}$, then change $f(u_4) \longrightarrow f(u_4) + \dfrac{4}{5}$ and $f(u_3) \longrightarrow f(u_3) - \dfrac{4}{5}$, and continue with the vertices u_6 and u_7 as similar to u_4 and u_5. (Note that if $u_6 \notin S$ and $v \notin S_{u_6}$, then the vertices in S_{u_6} have lower indices than v, since $vu_6 \in E(G)$). Otherwise, $u_6 \notin S$ and $v \in S_{u_6}$. Then $S_{u_6} = \{u_7, v\}$ (since the vertex u_7 is adjacent to u_6 and has a lower index than v). Change $f(u_3) \longrightarrow f(u_3) - \dfrac{4}{5}$, $f(v) \longrightarrow f(v) + \dfrac{4}{5}$, $f(u_7) \longrightarrow f(u_7) + \dfrac{4}{5}$, $f(u_4) \longrightarrow f(u_4) - \dfrac{2}{5}$, and $f(u_6) \longrightarrow f(u_6) - \dfrac{2}{5}$, and continue with u_7 and v.

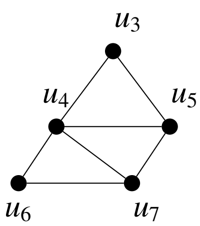

Figure 4

Sub-subcase 2.1.3: If both neighbours are same (Figure 5).

Then change $f(u_4) \longrightarrow f(u_4) + \dfrac{4}{5}$, $f(u_3) \longrightarrow f(u_3) - \dfrac{4}{5}$ and continue.

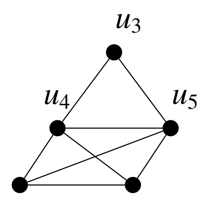

Figure 5

In general, if u_i is a vertex not in S such that $f(u_i) \geq \frac{6}{5}$, then $S_{u_i} \neq \phi$, say $u_{i+1}, u_{i+2} \in S_{u_i}$. If one of them has been considered, we can simply rename the other vertex as u_{i+1}, if necessary. We then update the values of $f(u_{i+1})$ and $f(u_i)$ as follows: $f(u_{i+1}) \longrightarrow f(u_{i+1}) + \frac{4}{5}$ and $f(u_i) \longrightarrow f(u_i) - \frac{4}{5}$, and continue with u_{i+1}. If both neighbors are already considered, we stop and continue with another vertex u_j with $f(u_j) \geq \frac{6}{5}$, if there is any such vertex.

Now continue the process with u_i, u_{i+1}, and u_{i+2} in place of u_3, u_4, and u_5. It is important to note that in every step, the sum $\Sigma_{i=1}^{k} f(v_i)$ remains unchanged and $f(v_i) \geq 0$ for all v_i. Continuing like this, we obtain a sequence of vertices $u_1, u_2, \ldots$. Since G is finite, this sequence must terminate. Let the terminating vertex be u_j. By our construction of the u_i's, we have $v_k \in [S \cap u_1, u_2, \ldots, u_j]$. Furthermore, since $v_k \in \partial H_G(S)$, it follows that $S \subseteq \{u_1, u_2, \ldots, u_j\}$. Moreover, we have modified the values of $f(u_i)$ such that $f(u_i) \geq 1$ whenever $u_i \in S$.

Subcase 2.2: $u_1u_2 \notin E(G)$ and $u_1 \in S$, $u_2 \notin S$.

We change $f(u_1) \longrightarrow f(u_1) + \frac{3}{5}$, $f(u_2) \longrightarrow f(u_2) + \frac{4}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{7}{5}$. Since $u_2 \notin S$, we have $S_{u_2} \neq \phi$. Let $u_3, u_4 \in S_{u_2}$. Then $u_3u_4 \in E(G)$ since G is claw-free (as shown in Figure 6). Next, we continue with u_2 as we did with u_3 in subcase 2.1.

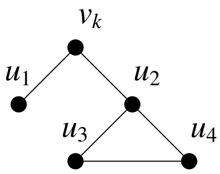

Figure 6

We obtain a sequence of vertices $u_1, u_2, \ldots, u_j$ that includes all vertices in S and satisfies $f(v) \geq 1$ for all $v \in S$, as per our construction. Thus we get $C(G) \leq \frac{2n(G)+6}{5}$. □

Theorem 1. *Let G be a claw-free graph of order n(G). Then,*

$$C(G) \leq \frac{2n(G)+6}{5}.$$

Proof. Let G be a claw-free graph of order $n(G)$ and $S, v_{l+1}, v_{l+2}, \ldots v_k, f$ be the same as described earlier. If $S_{v_k} \cap S \neq \emptyset$ then $C(G) \leq \frac{2n(G)+6}{5}$ by Lemma 1.

If $S_{v_k} \cap S = \emptyset$, let $S_{v_k} = \{u_1, u_2\}$. If $u_1u_2 \in E(G)$, we can continue as in above lemma with v_k inplace of u_i since $f(v_k) \geq \frac{6}{5}$. If $u_1u_2 \notin E(G)$, let $S_{u_1} = \{u_3, u_4\}$ and $S_{u_2} = \{u_5, u_6\}$. Then there are three cases:

Case 1: $S_{u_1} = S_{u_2}$.

If $S_{u_1} = S_{u_2}$, then $S = \{u_3, u_4\}$ (as in Figure 7(i)). Otherwise, we must have $u_1u_2 \in E(G)$ since G is claw-free, which leads to a contradiction.

Case 2: If $|S_{u_1} \cap S_{u_2}| = 1$, let's say $u_4 = u_6$ (as depicted in Figure 7(ii)). Then, either $u_4 \in S$ or $S_{u_4} = \{u_3, u_5\}$; otherwise, there exists a neighbor u_7 with an index less than u_4, and $u_7 \neq u_3, u_5$. Since u_2, u_1, u_4, u_7 do not form a claw, there are three possibilities: either $u_2u_7 \in E(G)$, or $u_1u_7 \in E(G)$, or $u_1u_2 \in E(G)$. However, since u_4 and u_5 are two neighbors of u_2 with minimum indices, and the index of u_7 is less than u_4, we have $u_2u_7 \notin E(G)$. Similarly, $u_1u_7 \notin E(G)$.It follows that $u_1u_2 \in E(G)$, which contradicts our assumption that $u_1u_2 \notin E(G)$. Therefore, either $u_4 \in S$ or $S_{u_4} = \{u_3, u_5\}$ and since G is claw-free, we have $u_4u_3, u_4u_5 \in E(G)$.

Subcase 2.1: If $u_4 \in S$.

If $u_3u_5 \in E(G)$ and u_3 or u_5 is in S, say u_3, then $S = \{u_3, u_4\}$ since $v_k \in \partial H_G(S)$. We then change $f(u_3) \longrightarrow f(u_3) + \frac{3}{5}$ and $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{6}{5}$.

If $u_3u_5 \notin E(G)$ and u_3 or u_5 is in S, say u_3 (since $u_4 \in S$, not both u_3 and u_5 belong to S), we let $u_6 = S_{u_5u_4}$ and change $f(u_3) \longrightarrow f(u_3) + \frac{3}{5}$, $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(v_k) \longrightarrow f(v_k) - \frac{6}{5}$, $f(u_6) \longrightarrow f(u_6) + \frac{4}{5}$, $f(u_5) \longrightarrow f(u_5) - \frac{2}{5}$, and $f(u_2) \longrightarrow f(u_2) - \frac{2}{5}$, and continue with u_6 as above.

If $u_3, u_5 \notin S$ and $u_3u_5 \notin E(G)$, we consider $u_6 = S_{u_3u_4}$ and $u_7 = S_{u_5u_4}$. If $u_6 = u_7$, then $S = \{u_4, u_6\}$ since $v_k \in \langle\{u_4, u_6\}\rangle, u_4, u_6 \in S$ and S is a Carathéodory set. Change $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(u_6) \longrightarrow f(u_6) + \frac{3}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{6}{5}$. If $u_6 \neq u_7$, we change $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(u_6) \longrightarrow f(u_6) + \frac{4}{5}$, $f(v_k) \longrightarrow f(v_k) - \frac{7}{5}$, $f(u_7) \longrightarrow f(u_7) + \frac{4}{5}$, $f(u_5) \longrightarrow f(u_5) - \frac{2}{5}$, and $f(u_2) \longrightarrow f(u_2) - \frac{2}{5}$, and continue with u_6 and u_7 similar to the above case.

If $u_3, u_5 \notin S$ and $u_3u_5 \in E(G)$, we let u_3 have a lower index than u_5 and consider $u_6 = S_{u_3u_4}$. Then change $f(u_6) \longrightarrow f(u_6) + \frac{4}{5}$, $f(u_4) \longrightarrow f(u_4) + \frac{3}{5}$, $f(v_k) \longrightarrow f(v_k) - \frac{7}{5}$ and continue with u_6

Subcase 2.2: If $u_4 \notin S$, then $u_3, u_5 \in S_{u_4}$ and change $\longrightarrow f(u_4) + \frac{6}{5}$, $f(v_k) \longrightarrow f(v_k) - \frac{6}{5}$ and continue with u_4.

Case 3: If all neighbours are distinct(as given in Figure 7(iii)), then change $f(u_1) \longrightarrow f(u_1) + \frac{4}{5}$, $f(u_2) \longrightarrow f(u_2) + \frac{4}{5}$, $f(v_k) \longrightarrow f(v_k) - \frac{8}{5}$ and continue with u_1 and u_2.

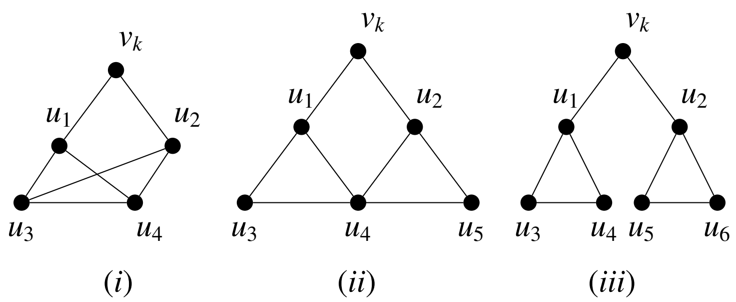

Figure 7

By following the procedure mentioned above, we eventually obtain a finite sequence of vertices $u_1, u_2, \ldots, u_j$ that includes all vertices in the set S, while ensuring that $f(v) \geq 1$ for all $v \in S$ and without changing the sum $\sum_{i=1}^{k} f(v_i)$, and $f(v_i) \geq 0$ for all v_i. Thus, we can conclude that $C(G) = |S| = l \leq \Sigma_{v \in S} f(v) \leq \Sigma_{i=1}^{k} f(v_i) \leq \frac{2n(G)+6}{5}$. □

Next, our goal is to identify all claw-free graphs that attain the upper bound of the Carathéodory number for P_3-convexity as given in the above inequality. Let G be a graph that satisfies equality in the above inequality. Then, $\frac{2n(G) + 6}{5}$ is an integer, implying that $n(G) = 5k + 2$ for some integer k. Since the Carathéodory number of a graph of order 2 is 1, and the Carathéodory number of G_0 in Figure 8 (which has order 7) is 4, the smallest order of a graph that attains the bound in the above inequality is 7.

Now, we will construct a collection of possible graphs from G_0 in Figure 8 that attain the bound in the above inequality. We define the extension of G_0 through u_6 and u_7 as the graph H_0, which is obtained by attaching a triangle at vertex u_6 and a paw (a graph obtained by joining a cycle graph C_3 to a singleton graph K_1 with a bridge) at vertex u_7, as shown in Figure 8. If u_i and u_{i+1} are the endpoints of a branch in H_0 (i.e., they lie in a triangle, and no vertex lies below them), we can extend the graph H_0 through u_i and u_{i+1} by attaching a triangle at u_i and a paw at u_{i+1}.

More generally, if G is a graph obtained by a finite extension of G_0, and u and v are endpoints of a particular branch, we can define the extension of G through u and v as the graph obtained by attaching a triangle at u and a paw at v. Let $\mathcal{G}$ be the family of graphs that consists of G_0 and every graph obtained from G_0 by a finite extension.

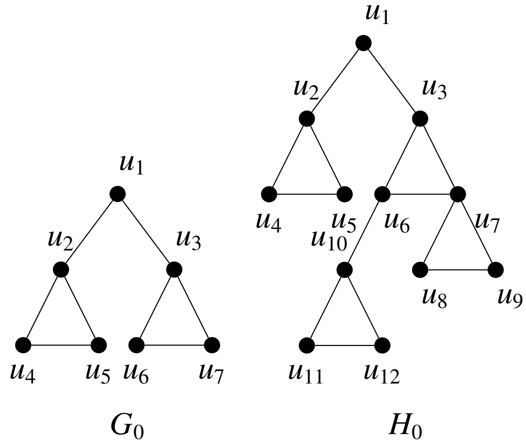

Figure 8. Extension of G_0 Through u_6, u_7

Theorem 2. *Let G be a graph. if $G \in \mathcal{G}$, then* $C(G) = \dfrac{2n(G)+6}{5}$.

Proof. We can verify that $S = \{u_6, u_7, u_8, u_9\}$, which is the set of all end vertices of branches in G_0, is a Carathéodory set of G_0 with cardinality $\dfrac{2n(G_0)+6}{5}$.

Now, we will prove that if G is a graph in $\mathcal{G}$ and let S be the set of all end vertices of branches in G, which is a Carathéodory set of G, with $|S| = \dfrac{2n(G)+6}{5}$, then $S_1 = S \setminus \{u, v\} \cup \{v_1, v_2, v_4, v_5\}$ is a Carathéodory set of G_1. Here, G_1 arises from G by replacing vertex u with a triangle u, v_1, v_2, and vertex v with a paw v, v_3, v_4, v_5.

It should be noted that u and v are both elements of set S. We have $\partial H_G(S) \subseteq \partial H_{G_1}(S_1)$ because

$$\begin{aligned}
\partial H_{G_1}(S_1) &= \langle S_1\rangle \setminus \Big(\bigcup_{w\in S\setminus\{u,v\}} \langle S_1 \setminus \{w\}\rangle \bigcup \bigcup_{w\in\{v_1,v_2,v_4,v_5\}} \langle S_1 \setminus \{w\}\rangle\Big) \\
&= \langle S\rangle \bigcup \{v_1, v_2, v_3, v_4, v_5\} \ \setminus \Big(\bigcup_{w\in S\setminus\{u,v\}} \langle S \setminus \{w\}\rangle \\
&\qquad \bigcup \{v_1, v_2, v_3, v_4, v_5\} \cup \langle S \setminus \{u, v\}\rangle \bigcup \langle S \setminus \{v\}\rangle \bigcup \{v_1, v_2, v_3, v_4, v_5\}\Big) \\
&= \langle S\rangle \setminus \Big(\bigcup_{w\in S\setminus\{u,v\}} \langle S \setminus \{w\}\rangle \bigcup \langle S \setminus \{v\}\rangle \bigcup \langle S \setminus \{u, v\}\rangle\Big) \\
&\supseteq \langle S\rangle \setminus \bigcup_{w\in S} \langle S \setminus \{w\}\rangle \\
&= \partial H_G(S).
\end{aligned}$$

Thus, S_1 is a Carathéodory set of G_1. Therefore,

$$\begin{aligned}
C(G_1) \geq |S_1| &= |S| - 2 + 4 = |S| + 2 \\
&= \frac{2n(G)+6}{5} + 2 = \frac{2n(G)+10+6}{5} \\
&= \frac{2(n(G)+5)+6}{5} = \frac{2n(G_1)+6}{5}.
\end{aligned}$$

Now, $C(G_1) \leq \dfrac{2n(G_1)+6}{5}$ since the graph G_1 is claw-free. Thus, $C(G_1) = \dfrac{2n(G_1)+6}{5}$. Therefore, if $G \in \mathcal{G}$ then $C(G) = \dfrac{2n(G)+6}{5}$. □

Theorem 3. *Let G be a claw-free graph of order n(G). Then,* $C(G) = \dfrac{2n(G)+6}{5}$ *if and only if $G \in \mathcal{G}$.*

Proof. The sufficiency of the statement is proven by the theorem mentioned above. To prove the converse part of the theorem, suppose G is a graph of order $n(G)$ such that $C(G) = \frac{2n(G)+6}{5}$, and let S be a Carathéodory set of cardinality $C(G)$. Then G is minimal in the sense that G has no proper subgraph with the same Carathéodory number. Let $v_1, v_2, \ldots, v_k$ be the subset of V as described in the main theorem's proof. Since $V(G) \neq \{v_1, v_2, \ldots, v_k\}$ would imply that $G[v_1, v_2, \ldots, v_k]$ is a proper subgraph of G with fewer than $n(G)$ vertices and a Carathéodory set of cardinality $C(G)$, this contradicts the minimality of G. Therefore, $V(G) = \{v_1, v_2, \ldots, v_k\}$.

For G to attain equality in the upper bound of Theorem 1,we need $|S| = \sum_{i=1}^{k} f(v_i) = \sum_{i=1}^{k} f(v_k) = \frac{2n(G)+6}{5}$. After the transformation described in the proof, this implies that $f(v) = 1$ for all $v \in S$ and $f(v) = 0$ for all $v \notin S$. Therefore, cases (1) and (2), where $f(v_k) > 0$, cannot happen since $v_k \notin S$.

For subcases (a) and (b) of case 3, we have $f(v_k) > 0$, and since $v_k \notin S$, neither case (a) nor (b) is possible. Therefore, subcase (c) of case 3 applies to G.

Now, as in the proof, let u_3 and u_4 be vertices in S_{u_1}. Either u_3 and u_4 are both in S or both are not in S; otherwise, as described in the proof, $f(u_1) > 0$, which is not possible .

If they are not in S, let $u_5, u_6 \in S_{u_3}$, and let $u_7, u_8 \in S_{u_4}$. Then u_5, u_6, u_7, and u_8 must all be distinct; otherwise, we would have $f(u_1) > 0$, which is not possible since $u_1 \notin S$. Suppose u_5 and u_6 are neighbors of u_3, and u_7 is a neighbor of u_4. Then, $u_7 \notin S$, or otherwise $f(u_7) > 1$, which is not possible. Consider two neighbors u_8 and u_9 of u_7. In general, let u_i be a vertex not in S, and let u_{i+1} and u_{i+2} be two neighbors of u_i in S. If either u_{i+1} or u_{i+2} has already been considered, then we would have $f(u_i) > 0$, which is not possible. Thus, both $u_{i+1} \neq u_j$ and $u_{i+2} \neq u_j$ for all $j < i$.

We continue by considering the vertices u_i, u_{i+1}, and u_{i+2}, and observe that either both u_{i+1} and u_{i+2} are in S, or neither of them is. Otherwise, we would have $f(u_i) > 0$, which is not possible since $u_i \notin S$. We must also have $S_{u_{i+1}} \cap S_{u_{i+2}} = \emptyset$; otherwise, $f(u_i) > 0$.

Now, as in the proof, let $u_{i+3}, u_{i+4} \in S_{u_{i+1}}$ and $u_{i+5} \in S_{u_{i+2}}$. Then $u_{i+5} \notin S$, otherwise $f(u_{i+5}) > 1$, which is not possible. Thus, $u_{i+5} \notin S$. Now consider $u_{i+6}, u_{i+7} \in S_{u_{i+5}}$.

Note that if any of these five vertices has already been considered, then we would have $f(u_{i+1}) > 0$, $f(u_{i+2}) > 0$, or $f(u_{i+5}) > 0$, which is not possible. Therefore, either both u_{i+1} and u_{i+2} are in S, or we obtain five other vertices u_{i+3}, u_{i+4}, u_{i+5}, u_{i+6}, and u_{i+7} with edges $u_{i+3}u_{i+4}$, $u_{i+3}u_{i+1}$, $u_{i+4}u_{i+1}$, $u_{i+2}u_{i+5}$, $u_{i+6}u_{i+5}$, $u_{i+5}u_{i+7}$, $u_{i+6}u_{i+7}$.

Assume that there exists an edge between a vertex u in one branch to another vertex v in another branch. Let U_i be the set of vertices in the same branch below the vertex u_i including u_i. Then $S \cap U_i \neq \emptyset$.

Then $u \neq v_k$(if $u = v_k$, then $v_k \in \langle S \cap U_1 \rangle$) or $v_k \in \langle S \cap U_2 \rangle$, which is not possible since $v_k \in \partial H_G(S)$. Thus, $u = u_i$ for some i.

Now there are two cases: (1) $u_i \in S$ and (2) $u_i \notin S$.

Case (1): If $u_i \in S$, then as in the previous argument, u_i is adjacent to some vertex in S in the same branch (if u_j is the vertex such that $u_i, u_{i+1} \in S_{u_j}$, then either both belong to S or both do not belong to S, since $u_i \in S$ implies $u_{i+1} \in S$). Then, $u_i \in \langle (S \cap U_{i'}) \cup \{u_{i+1}\} \rangle \subseteq \langle S \setminus \{u_i\} \rangle$ where $v = u_{i'}$, which is not possible since S is a Carathéodory set.

Case (2): $u_i \notin S$. Then, we have three cases, which are shown in Figure 9:

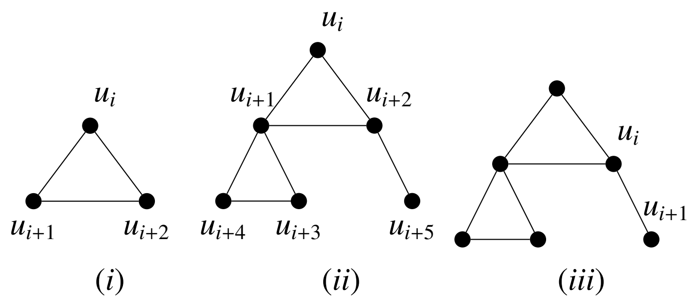

Figure 9

(i) $u_i \notin S$ and $u_{i+1}, u_{i+2} \in S$ where $u_{i+1}, u_{i+2} \in S_{u_i}$ as in Figure 9(i). Then, $v_k \in \langle S \setminus \{u_{i+1}\}\rangle$, which is not possible since $v_k \in \partial H_G(S)$.

(ii) $u_i \notin S$, both u_{i+1} and u_{i+2} are not in S as in Figure 9(ii). Then, $v_k \in \langle S \setminus (S \cap U_{i+5})\rangle$, which is not possible since $v_k \in \partial H_G(S)$.

(iii) $u_i \notin S$, both u_{i+1} and u_{i+2} are not in S as in Figure 9(iii). Then, $v_k \in \langle S \setminus (S \cap U_{i+1})\rangle$, which is not possible since $v_k \in \partial H_G(S)$. We have discussed all the cases since $u \neq v_k$ and all other vertices in G must belong to any of these four cases (by the above argument, since G satisfies $C(G) = \frac{2n(G)+6}{5}$). Thus, there is no edge between branches of G.

Thus, If G be a graph such that $C(G) = \dfrac{2n(G)+6}{5}$, then $G \in \mathcal{G}$. □

Example 1. *Consider a graph G with* $C(G) < \dfrac{2n(G)+6}{5}$.

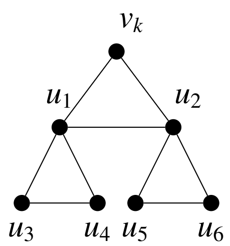

Figure 10

Here $C(G) = 3$ *but* $n(G) = 7$ *thus* $\dfrac{2n(G)+6}{5} = 4$.

Definition 1. *Let A be the set of vertices in a graph G that are the centers of an induced claw. A graph G is almost claw-free if A is independent, and for every x in A, the subgraph induced by N(x) has a dominating set of atmost 2.*

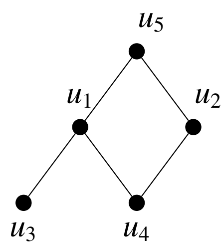

Figure 11. An Almost Claw-Free Graph Which Is Not Claw-Free

Theorem 4. *If G is an almost claw-free graph of order n(G) then,*

$$C(G) \leq \frac{2n(G)+6}{5}.$$

Proof. Let G be an almost claw-free graph of order $n(G)$, and let $S = \{v_1, v_2, \ldots, v_l\}$ be a Carathéodory set of cardinality $C(G)$. Let the sequence $v_{l+1}, v_{l+2}, \ldots, v_k$, the set S_{v_i}, the vertex $s_{v_i v_j}$, and the f-values of v_i be as described in Theorem 1.

We now describe a procedure that adjusts the values $f(v)$ of some vertices in $\{v_1, v_2, \ldots, v_k\}$ such that $f(v) \geq 1$ for all $v \in S$ and $f(v) \geq 0$ without changing the value of $\sum_{i=1}^{k} f(v_i)$ as in Theorem 1. Let $u_1, u_2 \in S_{v_k}$.

Case 1: If $u_1, u_2 \in S$, then $S = \{u_1, u_2\}$, and we change $f(u_1) = f(u_1) + \frac{3}{5}$, $f(u_2) = f(u_2) + \frac{3}{5}$, and $f(v_k) = f(v_k) - \frac{6}{5}$ as in Theorem 1

Case 2: If only one of them belongs to S, say u_1, and $u_2 \notin S$.

Subcase 2.1: If $u_1u_2 \in E(G)$, let $u_3 = s_{u_2u_1}$. Then change $f(u_3) = f(u_3) + \frac{4}{5}$, $f(u_1) = f(u_1) + \frac{3}{5}$, $f(u_2) = f(u_2) - \frac{2}{5}$, and $f(v_k) = f(v_k) - \frac{5}{5}$ as in Theorem 1.

If $u_3 \notin S$, then let $u_4, u_5 \in S_{u_3}$. If $u_1 \in S_{u_3}$, then continue with $s_{u_3u_1}$, say u_4, by changing $f(u_3) = f(u_3) - \frac{4}{5}$ and $f(u_4) = f(u_4) + \frac{4}{5}$.

If $u_1 \notin S_{u_3}$ and $u_4u_5 \in E(G)$, then change the f values of u_4, u_5, u_3 similarly to the proof of Theorem 1.

If $u_4u_5 \notin E(G)$, then u_2, u_3, u_4, u_5 form a claw, implying that the subgraph induced by $N(u_3)$ has a dominating set of order at most 2. Then there exists a vertex w_3 which is adjacent to at least 2 of u_4, u_5, u_2 and belongs to $N(u_3)$. Continue with u_4, u_5 by changing $f(u_4) = f(u_4) + \frac{4}{5}$, $f(u_5) = f(u_5) + \frac{4}{5}$, $f(u_3) = f(u_3) - \frac{6}{5}$, and $f(w_3) = f(w_3) - \dfrac{2}{5}$.

Now, in general, if u_i is a vertex not in S such that $f(u_i) \geq \frac{6}{5}$ and $u_{i+1}, u_{i+2} \in S_{u_i}$, if u_{i+1} or u_{i+2} is equal to w_j for some w_j which has already been considered, say $u_{i+2} = w_j$, then there exists a vertex u_j adjacent to w_j, and the subgraph induced by $N(u_j)$ has a claw which has vertices in $\{u_1, u_2, \ldots, u_{j+2}\}$ and has a dominating set containing w_j with cardinality at most 2. Then, w_j is adjacent to 2 vertices in $N(u_j)$ which are non-adjacent to each other, say u and v. If u_i, w_j, u, v form a claw with center w_j, then $w_j \in A$.

Since $u_j \in A$, we get that A is dependent, a contradiction. Thus, $u_{i+1} = w_j, u, v, u_i$ is not a claw. Therefore, one of u_iv, u_iu, or uv belongs to $E(G)$. Since $uv \notin E(G)$, we have u_iu or u_iv is in $E(G)$.

Since $u, v \in \{u_1, u_2, \ldots, u_{i-1}\}$, we then continue with u_{i+1} by changing $f(u_i) \longrightarrow f(u_i) - \frac{4}{5}$ and $f(u_{i+1}) \longrightarrow f(u_{i+1}) + \frac{4}{5}$.

If both of them belong to $\{u_1, u_2, \ldots, u_{i-1}\}$, then stop and continue with another vertex u_j with $f(u_j) \geq \frac{6}{5}$. If only one of them belongs to $\{u_1, u_2, \ldots, u_{i-1}\}$, then it is enough to consider the vertex which is not equal to any u_j where $j < i$, say u_{i+1}, and continue by changing $f(u_{i+1}) = f(u_{i+1}) + \frac{4}{5}$ and $f(u_i) = f(u_i) - \frac{4}{5}$.

If $u_{i+1}u_{i+2} \in E(G)$, then we continue by changing the f-values similarly as in the proof of Theorem 1.

If $u_{i+1}u_{i+2} \notin E(G)$, then $u_{i-1}, u_i, u_{i+1}, u_{i+2}$ form a claw, and since G is almost claw-free, there exists $w_i \in N(u_i)$ such that w_i is adjacent to 2 vertices in $u_{i+1}, u_{i+2}, u_{i-1}$, say u and v.

If $w_i = w_j$ for some $j < i$, then there exists a vertex u_j such that the subgraph induced by $N(u_j)$ contains a claw of vertices in $u_1, u_2, \ldots, u_{i-1}$, and w_j dominates at least 2 vertices in the claw, say u', v'. Then, u', v', w_i, u_i is not a claw; otherwise, A is dependent, which is not possible. Since $u'v' \notin E(G)$, this implies that either u_iu' or u_iv' belongs to $E(G)$; let it be u'. Then we continue with $s_{u_iu'}$ by changing $f(u_i) \longrightarrow f(u_i) - \dfrac{4}{5}$, and $f(s_{u_iu'}) \longrightarrow f(s_{u_iu'}) + \dfrac{4}{5}$.

If $w_i = u_r$ where $u_r, u_i \in S_{u_{i-1}}$ and $f(u_r) = \dfrac{2}{5}$, then change $f(u_i) \longrightarrow f(u_i) - \dfrac{6}{5}$, $f(u_r) \longrightarrow f(u_r) - \dfrac{2}{5}$, $f(u_{i+1}) \longrightarrow f(u_{i+1}) + \dfrac{4}{5}$, $f(u_{i+2}) \longrightarrow f(u_{i+2}) + \dfrac{4}{5}$ and continue.

If $w_i = u_r$ where $u_r, u_i \in S_{u_{i-1}}$ and $f(u_r) = 0$, then $S_{u_r} \cap S_{u_i} = \emptyset$. Since $f(u_i) \geq \dfrac{6}{5}$, this implies S_{u_r} contains vertices with larger indices in $v_1, v_2, \ldots, v_k$ than u_{i+1}, u_{i+2}. Since u_ru_{i+1} or u_ru_{i+2} is in $E(G)$, say $u_{i+1}u_r \in E(G)$, this implies that vertices in S_{u_r} have smaller indices than u_{i+1} in $\{v_1, v_2, \ldots, v_k\}$, a contradiction.

If $w_i = u_j$ for some $j < i$ and $u_j \notin S_{u_{i-1}}$, then u', v', w_i, u_{j-1} is not a claw since $u_iw_i \in E(G)$ and A is independent. Thus, $u'u_{j-1}$ or $v'u_{j-1}$ belongs to $E(G)$. Let $u'u_{j-1} \in E(G)$, then $u' \in \langle u_j, u_{j-1} \rangle$. Then continue by changing $f(u_i) \longrightarrow f(u_i) - \dfrac{4}{5}$ and $f(v') \longrightarrow f(v') + \dfrac{4}{5}$.

Similarly, if $v'u_{j-1} \in E(G)$, continue by changing $f(u_i) \longrightarrow f(u_i) - \dfrac{4}{5}$ and $f(u') \longrightarrow f(u') + \dfrac{4}{5}$.

If $w_i \neq w_j$ and $w_i \neq u_j$ for $j < i$, then continue by changing $f(u_{i+1}) = f(u_{i+1}) + \dfrac{4}{5}$, $f(u_{i+2}) =$

$f(u_{i+2}) + \frac{4}{5}$, $f(u_i) = f(u_i) - \frac{6}{5}$, and $f(w_i) = f(w_i) - \frac{2}{5}$.

Subcase 2.2: $u_1u_2 \notin E(G)$. Continue the process as in case a by changing $f(u_1) \longrightarrow f(u_1) + \frac{3}{5}$, $f(u_2) \longrightarrow f(u_2) + \frac{4}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{7}{5}$.

Case 3: Both u_1 and u_2 are not in S. Continue the process similar to the above cases by changing $f(u_1) \longrightarrow f(u_1) + \frac{4}{5}$, $f(u_2) \longrightarrow f(u_2) + \frac{4}{5}$, and $f(v_k) \longrightarrow f(v_k) - \frac{8}{5}$.

Continuing like this, we get a sequence that contains S and $v_k \in \langle\{u_1, u_2, \ldots, u_s\}\rangle$ such that $f(v) \geq 1$ for all v in S and $f(v) \geq 0$ for all v in $V(G)$ without changing the sum $\sum_{i=1}^{n} f(v_i)$. Thus, we get $C(G) \leq \frac{2n(G)+6}{5}$. □

References

1. Sierksma, G., 1975. Carathéodory and Helly-numbers of convex-product-structures. *Pacific Journal of Mathematics, 61*(1), pp.275-282.
2. Sierksma, G., 1976. *Axiomatic theory and convex product space*, Ph. D. thesis, University of Groningen.
3. Sierksma, G., 1976. Relationships between Carathéodory, Helly, Radon and exchange numbers of convexity spaces. Rijksuniversiteit, *Nieuw Archief voor Wiskunde,3*(1977)no.25, pp.115-132
4. Duchet, P., 1987. Convexity in combinatorial structures. In *Proceedings of the 14th Winter School on Abstract Analysis* (pp. 261-293). Circolo Matematico di Palermo.
5. van De Vel, M. L., 1993. *Theory of convex structures*. Elsevier.
6. Changat, M., Mulder, H. M. and Sierksma, G., 2005. Convexities related to path properties on graphs. *Discrete Mathematics, 290*(2-3), pp.117-131.
7. Changat, M., Klavzar, S. and Mulder, H. M., 2001. The all-paths transit function of a graph. *Czechoslovak Mathematical Journal, 51*(2), pp.439-448.
8. Kannan, B. and Changat, M., 2008. Hull numbers of path convexities on graphs. In *Proceedings of the International Instructional Workshop on Convexity in Discrete Structures* (Vol. 5, pp. 11-23).
9. Duchet, P., 1988. Convex sets in graphs, II. Minimal path convexity. *Journal of Combinatorial Theory, Series B, 44*(3), pp.307-316.
10. Changat, M. and Mathew, J., 1999. On triangle path convexity in graphs. *Discrete Mathematics, 206*(1-3), pp.91-95.
11. Parker, D.B., Westhoff, R.F. and Wolf, M.J., 2008. On two-path convexity in multipartite tournaments. *European Journal of Combinatorics, 29*(3), pp.641-651.
12. Centeno, C.C., Dantas, S., Dourado, M.C., Rautenbach, D. and Szwarcfiter, J.L., 2010. Convex partitions of graphs induced by paths of order three. *Discrete Mathematics & Theoretical Computer Science, 12*(Graph and Algorithms), pp.175–184.
13. Barbosa, R.M., Coelho, E.M., Dourado, M.C., Rautenbach, D. and Szwarcfiter, J.L., 2012. On the Carathéodory number for the convexity of paths of order three. *SIAM Journal on Discrete Mathematics, 26*(3), pp.929-939.
14. Dourado, M.C., Rautenbach, D., dos Santos, V.F., Schäfer, P.M., Szwarcfiter, J.L. and Toman, A., 2012. On the Radon number for P 3-convexity. In LATIN 2012: Theoretical Informatics: 10th Latin American Symposium, Arequipa, Peru, April 16-20, 2012. *Proceedings 10* (pp. 267-278). Springer Berlin Heidelberg.

15. . Coelho, E.M., Dourado, M.C., Rautenbach, D. and Szwarcfiter, J.L., 2014. The Carathéodory number of the P3 convexity of chordal graphs. *Discrete Applied Mathematics, 172,* pp.104-108.

16. Kay, D. and Womble, E.W., 1971. Axiomatic convexity theory and relationships between the Carathéodory, Helly, and Radon numbers. *Pacific Journal of Mathematics, 38*(2), pp.471-485.

17. Duchet, P., 1988. Convex sets in graphs, II. Minimal path convexity. *Journal of Combinatorial Theory, Series B, 44*(3), pp.307-316.