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Article

Graphs whose l_p-optimal rankings are l_∞ optimal

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Abstract: A *ranking* on a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ with the following restriction: if $f(u) = f(v)$ for any $u, v \in V(G)$, then on every uv path in G, there exists a vertex w with $f(w) > f(u)$. The optimality of a ranking is conventionally measured in terms of the l_∞ norm of the sequence of labels produced by the ranking. In [1] we compared this conventional notion of optimality with the l_p norm of the sequence of labels in the ranking for any $p \in [0, \infty)$, showing that for any non-negative integer c and any non-negative real number p, we can find a graph such that the sets of l_p-optimal and l_∞-optimal rankings are disjoint. In this paper we identify some graphs whose set of l_p-optimal rankings and set of l_∞-optimal rankings overlap. In particular, we establish that for paths and cycles, if $p > 0$ then l_p optimality implies l_∞ optimality but not the other way around, while for any complete multipartite graph, l_p optimality and l_∞ optimality are equivalent.

Keywords: Ranking, l_p norm, l_∞ norm, l_p-optimal ranking, Max optimal ranking.

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1. Introduction

A *k-ranking* (or simply "ranking") on a graph G is defined to be a function $f : V(G) \to \{1, 2, \ldots, k\}$ with the following restriction: if $f(u) = f(v)$ for any $u, v \in V(G)$, then on every uv path in G, there exists a vertex w with $f(w) > f(u)$. This vertex labeling problem has generated great interest because of its applications to Cholesky factorization of matrices in parallel [2–4], VLSI layout [5, 6] and scheduling steps in manufacturing systems [7, 8].

The focus of much of the literature on rankings has been on minimizing k. That is, given a graph G, the goal is to determine the minimum number of labels required to produce a valid ranking on G. The minimum number of labels required to produce a ranking on a graph G is called the rank number of G, and is denoted $\chi_r(G)$. Note that on a graph G, a ranking with $\chi_r(G)$ labels is a ranking with smallest l_∞ or max norm.

However, in [9], Jamison and Narayan compare the rank number of G with the lowest possible sum, or l_1 norm, of all rankings on G. In particular, they show that for paths and cycles, l_1-optimal rankings are also max optimal.

The results in [9] give rise to the question of how changing the value of p affects l_p optimality of a ranking. While most commonly known applications of l_p norms with finite p use $p \in \{1, 2\}$, some applications have used finite p with $p \notin \{1, 2\}$. Some examples include machine learning [10, 11], compressed sensing [12], image deconvolution [13], and two-dimensional phase unwrapping [14]. Specific to k-rankings on graphs, a graph may have distinct l_p-optimal rankings for two different values of p, even if both values of p satisfy $1 < p < \infty$. For example, Figures 1–2 display a graph whose sets of l_2- and l_3-optimal rankings are disjoint.

In [1] we showed that for any $p \geq 0$ and any non-negative integer c, we can construct a graph with c cycles such that the sets of l_p-optimal and l_∞-optimal rankings are disjoint. In this paper, we show that there are families of graphs including paths, cycles, and complete multipartite graphs such any l_p-optimal ranking is also l_∞ optimal for any p with $0 \leq p < \infty$.

We will use the notion of minimality of a ranking throughout the paper.

Definition 1. *A ranking is* minimal *if decreasing any label to a smaller label violates the definition of a ranking.*

The concept of a reduction, defined by Ghoshal, Laskar and Pillone will also be used throughout.

Definition 2. [15] *For a graph G and a set $S \subseteq V = V(G)$, the* reduction of G with respect to S, *denoted $G^\flat_S$, is the subgraph of G induced by $V - S$, with an additional edge $uv \in E(G^\flat_S)$ for each $u, v \in V - S$ if and only if there exists a uv path in G with all internal vertices belonging to S.*

For a ranking f on a graph G, define $S_i(f)$ by $S_i(f) = \{v \in G \mid f(v) = i\}$. In general, a reduction can be applied with respect to any set $S \subseteq V(G)$. However, for the purposes of our paper, we primarily consider the case where $S = S_1(f)$ for some ranking f on G. When the set S is understood, we denote $G^\flat_S$ by $G^\flat$.

Because for our purposes the reduction often depends on the ranking we consider, we also define a new ranking associated with $G^\flat$ as follows. Let G be a graph, and f a ranking on G. For any subgraph H of G, we define a ranking f_H by setting $f_H(v) = f(v)$ for all $v \in V(H)$. If $H = G^\flat_{S_1(f)}$, set $f_H(v) = f(v) - 1$ for all $v \in V(H)$. Note that f_H is a ranking on H and is called a *reduction ranking*.

We can define a similar labeling if $S \neq S_1(f)$ by setting $f_H(v) = f(v)$ for all $v \in V(H)$ where H is the reduction. However, the resulting labeling is not guaranteed to be a ranking. In this case we call the labeling a *reduction labeling*. An *expansion* of G is a graph $G^\sharp$ such that $\left(G^\sharp\right)^\flat_S = G$ for some $S \subseteq V(G^\sharp)$.

Finally, we note the following result from [15].

Lemma 1. [15] *If f is a minimal ranking on G, then the reduction ranking $f_{G^\flat_{S_1(f)}}$ is a minimal ranking on $G^\flat_{S_1(f)}$.*

1.1. Optimality of Rankings

For a graph G with ranking f, and $1 \leq p < \infty$, the l_p *norm* of f is given by the l_p norm of the labels of f:

$$\|f\|_p = \left(\sum_{v \in V(G)} [f(v)]^p \right)^{1/p}. \tag{1}$$

If $p = 1$, this describes the norm considered in [9], and $p = 2$ corresponds to the standard Euclidean norm. The standard existing norm in the literature on graph rankings is the *supremum* or *max*, or l_∞ norm:

$$\|f\|_\infty = \max_{v \in V(G)} f(v).$$

For $0 < p < 1$, $\|\cdot\|_p$ is defined as in (1), but fails to be a norm. The case $p = 0$ is considered separately in Section 5.

We refer to a ranking f on G that achieves $\|f\|_\infty = \chi_r(G)$ as a *max optimal* ranking. Given a graph G, a ranking f and non-negative real number p, we say that f is l_p *optimal* if

$$\|f\|_p = \min\left\{ \|g\|_p \mid g \text{ is a ranking on } G \right\}.$$

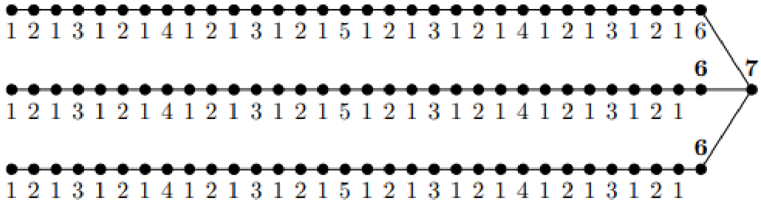

Figure 1. An l_3-Optimal Ranking, f

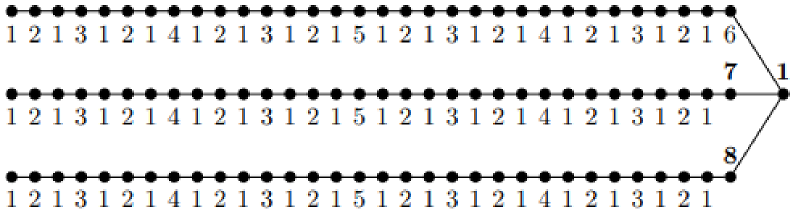

Figure 2. An l_2-Optimal Ranking, g

We define an analog of the rank number to l_p optimality.

Definition 3. *For a graph G and non-negative real number p, the l_p-*rank number *of G, $\chi_r^p(G)$ is $||f||_p$ where f is an l_p-optimal ranking on G.*

1.2. General Results about l_p Optimality

For any non-negative real number p, if f is not minimal then we can reduce one of the labels to produce a ranking $\tilde{f}$ with $||\tilde{f}||_p < ||f||_p$.

Observation 1. *For $p \in (0, \infty)$, if f is an l_p-optimal ranking on a graph G, then f is minimal.*

Note that observation 1 does not hold if $p = 0$ or $p = \infty$.

Proposition 1. *Let G be a graph, and let H be any subgraph of G. Then for any non-negative real number p, $\chi_r^p(H) \leq \chi_r^p(G)$.*

Proof. Suppose g is an l_p-optimal ranking on G. Then g_H is a ranking and

$$(\chi_r^p(H))^p \leq ||g_H||_p^p = \sum_{v \in H} g(v)^p \leq \sum_{v \in G} g(v)^p = ||g||_p^p = (\chi_r^p(G))^p \,.$$

□

Lemma 2. *Let f be any l_p-optimal ranking on a graph G. Also, let S be a non-empty subset of $S_1(f)$ and let $G^\flat = G_S^\flat$. Then $\chi_r^p(G^\flat) < \chi_r^p(G)$.*

Proof. Note that $f_{G^\flat}$ is a ranking on $G^\flat$ because the only vertices removed have label 1. By the definition of a reduction ranking, $||f_{G^\flat}||_p < ||f||_p$. Hence, $\chi_r^p(G^\flat) < \chi_r^p(G)$. □

We note here that a graph may have different l_p-optimal rankings depending on the value of p, even when p is restricted to $1 < p < \infty$, as illustrated by the example shown in Figures 1–2. Figure 1 shows a ranking f that is l_3 optimal but not l_2 optimal, while Figure 2 shows a ranking g of the same graph that is l_2 optimal but not l_3 optimal. Since $f(v) = g(v)$ for all $v \in V$ except for the vertex of degree 3 and vertices u and w with $g(u) = 7$ and $g(w) = 8$, we can compute that $||g||_3^3 - ||f||_3^3 = 81 > 0$, but $||f||_2^2 - ||g||_2^2 = 7 > 0$.

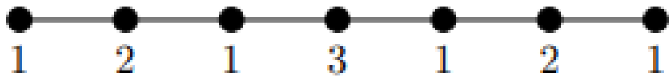

Figure 3. The Standard Ranking on P_7

1.3. Organization of This Paper

In Sections 2 and 3, we show any l_p-optimal ranking on a path or cycle where $p > 0$ is also max optimal, but that the converse does not hold. In Section 4 we show that on a complete multipartite graph a ranking is l_p optimal if and only if it is max optimal. Throughout Sections 2 – 4 we assume $0 < p < \infty$ unless otherwise stated. Finally, in Section 5, we show that if $p = 0$, then l_p optimality does not imply max optimality in paths and cycles, but that the result for complete multipartite graphs holds with the added assumption of minimality. We conclude with some open questions in Section 6.

2. Paths

Let P_n denote a path on n vertices, $v_1, v_2, \ldots, v_n$. We can construct a ranking f for P_n in a greedy fashion by starting with v_1, and defining $f(v_i)$ to be the lowest label possible without violating the definition of ranking, producing the *standard ranking* of a path as shown in Figure 3.

Bodlaender et al. [2] showed that $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$ and that the standard ranking f is a max optimal ranking on P_n. Note that the standard ranking f on P_n can also be defined by letting $f(v_i) = r + 1$ where r is the highest power of 2 that divides i.

In this section, we show that any l_p-optimal ranking on P_n is also a max optimal ranking. Throughout the section, given a ranking f on $G = P_n$, we use the notation $f^\flat$ to refer to the reduction ranking on $G_S^\flat$ where $S = S_1(f)$. Similarly, given a ranking f on a path $P_{\lfloor n/2 \rfloor}$, we use the notation $f^\sharp$ to refer to the expansion ranking on P_n formed by adding 1 to all existing labels, inserting a vertex labeled 1 between each consecutive pair of vertices in $P_{\lfloor n/2 \rfloor}$ and to the front of the path, and, if n is odd, adding one more vertex labeled 1 to the end of the path.

Observation 2. *Suppose f is an l_p-optimal ranking on P_n, and g an l_p-optimal ranking on P_{n+1}. Then $\|f\|_p < \|g\|_p$.*

Lemma 3. *If f is an l_p-optimal ranking on P_n, then $|S_1(f)| = \lceil n/2 \rceil$.*

Proof. Since f is a ranking, if $uv \in E(G)$, then $f(u) \neq f(v)$. Therefore, $|S_1(f)| \leq \lceil n/2 \rceil$. For the other direction, suppose g is a ranking on P_n with $|S_1(g)| < \lceil n/2 \rceil$. Let h be an l_p-optimal ranking on $P_{\lfloor n/2 \rfloor}$. Then, applying Observation 2 to the second step,

$$\begin{aligned}
\|g\|_p^p &= 1^p\,|S_1(g)| + \sum_{v \notin S_1(g)} \left(g^\flat(v) + 1\right)^p \\
&> |S_1(g)| + (\lceil n/2 \rceil - |S_1(g)|)\, 2^p + \sum_{v \in V(P_{\lfloor n/2 \rfloor})} (h(v) + 1)^p \\
&> \lceil n/2 \rceil + \sum_{v \in V(P_{\lfloor n/2 \rfloor})} (h(v) + 1)^p = \left\| h^\sharp \right\|_p^p
\end{aligned}$$

Therefore g is not l_p optimal since $h^\sharp$ is a ranking on P_n. Hence, if f is an l_p-optimal ranking, $|S_1(f)| = \lceil n/2 \rceil$. □

Lemma 4. *A ranking f on P_n is l_p optimal if and only if $f^\flat$ is an l_p-optimal ranking on $P_{\lfloor n/2 \rfloor}$.*

Proof. Let f be an l_p-optimal ranking on P_n. We know that $\|f\|_p^p = \lceil n/2 \rceil + \sum_{v \notin S_1(f)} \left(f^\flat(v) + 1\right)^p$ by Lemma 3. If $f^\flat$ is not an l_p-optimal ranking on $P_{\lfloor n/2 \rfloor}$, then there exists an l_p-optimal ranking g on

$P_{\lfloor n/2 \rfloor}$, and an expansion $g^\sharp$ such that

$$\begin{aligned}\|g^\sharp\|_p^p &= \lceil n/2 \rceil + \sum_{v \in V(P_{\lfloor n/2 \rfloor})} (g(v)+1)^p \\ &< \lceil n/2 \rceil + \sum_{v \notin S_1(f)} \left(f^\flat(v)+1\right)^p = \|f\|_p^p,\end{aligned}$$

contradicting the optimality of f. Hence, $f^\flat$ is an l_p-optimal ranking on $P_{\lfloor n/2 \rfloor}$.

Conversely, suppose that $f^\flat$ is an l_p-optimal ranking on $P_{\lfloor n/2 \rfloor}$. Then by construction, f is a ranking on P_n with $|S_1(f)| = \lceil n/2 \rceil$. If f is not l_p optimal, then there exists an l_p-optimal ranking g on P_n. By Lemma 3, $|S_1(g)| = \lceil n/2 \rceil$. But then the reduction ranking $g^\flat$ on $P_{\lfloor n/2 \rfloor}$ has $\|g^\flat\|_p < \|f^\flat\|_p$, contradicting the optimality of $f^\flat$. □

From Lemma 4, we get the following corollary.

Corollary 1. *Let f be an l_p-optimal ranking on P_n. Then for any j with $2 \le j \le \chi_r^p(P_n)$,*

$$\left|S_j(f)\right| = \left\lceil \frac{n - \sum_{i=1}^{j-1} |S_i(f)|}{2} \right\rceil.$$

Lemma 5. *A ranking g is a standard ranking on $P_{\lfloor n/2 \rfloor}$ if and only if $g^\sharp$ is a standard ranking on P_n.*

Proof. The ranking g is a standard ranking on $P_{\lfloor n/2 \rfloor}$ if and only if for each i, $1 \le i \le \lfloor n/2 \rfloor$, $g(v_i) = r+1$ where 2^r is the largest power of 2 that divides i if and only if (1) $g^\sharp(v_{2i}) = r+2$ where 2^{r+1} is the highest power of 2 that divides $2i$, and (2) $g^\sharp(v_{2i-1}) = 1$ for any i. This is true if and only if $g^\sharp(v_j) = \tilde{r}+1$ where $2^{\tilde{r}}$ is the highest power of 2 that divides j for each j, $1 \le j \le n$. Specifically, for even j, $\tilde{r} = r+1$ and for odd j, $\tilde{r} = 0$. □

Observation 3. *Suppose f and g are rankings on a graph G, such that for any positive integer i, $|S_i(f)| = |S_i(g)|$. Then for any positive integer i, $|S_i(f_{G^\flat_{S_1(f)}})| = |S_i(g_{G^\flat_{S_1(g)}})|$. The expansions of f and g that label all new vertices with label 1 also have the same number of vertices of each label.*

Given a path P_n, let $\mathcal{S}_{j;n}$ denote the set of vertices with label j under the standard ranking on P_n.

Lemma 6. *A ranking f on P_n is l_p optimal if and only if $|S_j(f)| = |\mathcal{S}_{j;n}|$ for each $j \ge 1$.*

Proof. For $n = 1, 2$ or 3, we can see that the l_p-optimal rankings are precisely the standard rankings on P_n. Suppose that for any n up to $l-1$, a ranking f on P_n is l_p optimal if and only if $|S_j(f)| = |\mathcal{S}_{j;n}|$ for each $j \ge 1$.

Using Lemmas 3 and 4, a ranking f on P_l is l_p optimal if and only if $S_1(f) = \lceil l/2 \rceil$ and $f^\flat$ is l_p optimal. Using the inductive hypothesis, $f^\flat$ is l_p optimal if and only if $|S_i(f^\flat)| = |\mathcal{S}_{i;\lfloor l/2 \rfloor}|$ for each $i \ge 1$. For $i \ge 1$, note that by definition $|S_i(f^\flat)| = |S_{i+1}(f)|$, and $|\mathcal{S}_{i;\lfloor l/2 \rfloor}| = |\mathcal{S}_{i+1;l}|$ by Lemma 5. Therefore $f^\flat$ is l_p optimal if and only if $|S_j(f)| = |\mathcal{S}_{j;l}|$ for each $j \ge 2$.

Thus, f is l_p optimal if and only if $|S_j(f)| = |\mathcal{S}_{j;l}|$ for each $j \ge 1$. □

Theorem 4. *Any l_p-optimal ranking on P_n is a max optimal ranking.*

Proof. This result is immediate from Lemma 6 and the fact that the standard ranking on P_n is max optimal. □

Figure 4 illustrates that the converse of Theorem 4 does not hold, which we now state formally.

Proposition 2. *A minimal max optimal ranking on P_n is not necessarily l_p optimal.*

Figure 4. Two Max Optimal Rankings on P_5, but only the Ranking on the Left is l_p Optimal

3. Cycles

In [16], Bruoth and Horňák showed the following.

Theorem 5. [16] $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$.

The *standard ranking* f on the cycle C_n is as follows. Starting with any vertex, label the first $n-1$ vertices $v_1, v_2, \ldots, v_{n-1}$ greedily in a clockwise direction. Thus, the first $n-1$ vertices are labeled as the standard ranking on P_{n-1}. Then the final vertex v_n is labeled $f(v_n) = \chi_r(P_{n-1}) + 1$. It is well known that the standard ranking f on a cycle C_n is a max optimal ranking.

Throughout this section, given a ranking f on $G = C_n$, we use $f^\flat$ to denote the reduction ranking on $G^\flat_S$ where $S = S_1(f)$. Given a ranking g on $C_{\lceil n/2 \rceil}$, we use $g^\sharp$ to denote the expansion ranking on C_n obtained as follows: add 1 to all labels, then insert $\lfloor n/2 \rfloor$ vertices labeled 1, one between each consecutive pair of vertices in $C_{\lceil n/2 \rceil}$. If n is odd, there will be one pair of consecutive vertices in $C_{\lceil n/2 \rceil}$ that do not receive a new vertex between them. For consistency, we omit the new vertex just before the vertex with highest label in $C_{\lceil n/2 \rceil}$.

Using similar arguments to the proof of Lemma 3, and applying Lemma 2, we get the following lemma.

Lemma 7. *If f is an l_p-optimal ranking on C_n, then $|S_1(f)| = \lfloor \frac{n}{2} \rfloor$.*

Lemma 8. *A ranking f on C_n is l_p optimal if and only if $f^\flat$ is an l_p-optimal ranking on $C_{\lceil n/2 \rceil}$.*

Proof. The proof follows from applying Lemma 7 to arguments similar to those in the proof of Lemma 4. □

Corollary 2. *If f is an l_p-optimal ranking on C_n, then for any j with $1 < j \leq \chi_r^p(C_n)$,*

$$\left|S_j(f)\right| = \left\lfloor \frac{n - \sum_{i=1}^{j-1} |S_i(f)|}{2} \right\rfloor.$$

Proof. This follows from Lemmas 7 and 8. □

Lemma 9. *A ranking g on $C_{\lceil n/2 \rceil}$ is a standard ranking if and only if $g^\sharp$ is a standard ranking on C_n.*

Proof. For v_1 through v_{n-1}, the lemma holds by noting that the standard ranking on a cycle is defined to be the same as the standard ranking on a path for the first $n-1$ vertices, and Lemma 5.

If $g^\sharp$ is a standard ranking on C_n, then $g^\sharp(v_n) = \chi_r(P_{n-1}) + 1 = \lfloor \log_2(n-1) \rfloor + 2$. Then by definition of a reduction ranking, $g(v_{\lceil n/2 \rceil}) = \chi_r(P_{n-1}) = \lfloor \log_2(n-1) \rfloor + 1 = \lfloor \log_2(\lceil n/2 \rceil - 1) \rfloor + 2 = \chi_r(P_{\lceil n/2 \rceil - 1}) + 1$, the standard label. Conversely, if g is a standard ranking, then $g(v_{\lceil n/2 \rceil}) = \chi_r(P_{\lceil n/2 \rceil - 1}) + 1 = \lfloor \log_2(\lceil n/2 \rceil - 1) \rfloor + 2 = \lfloor \log_2(n-1) \rfloor + 1 = \chi_r(P_{n-1})$. Then by definition of an expansion, $g^\sharp(v_n) = \chi_r(P_{n-1}) + 1$, the standard label, completing the proof. □

Lemma 10. *A ranking f on C_n is l_p optimal if and only if the number of $v \in V$ with $f(v) = j$ is the same as the number of vertices in the standard ranking on C_n with label j.*

Proof. The proof follows by applying Lemma 9 and Observation 3 to similar arguments to those in the proof of Lemma 6. □

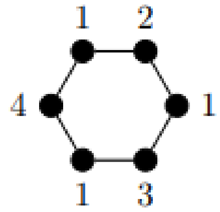

Figure 5. f is l_p optimal and max optimal.

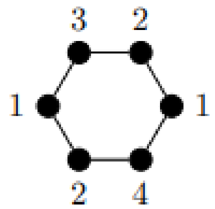

Figure 6. g is max optimal but not l_p optimal.

Theorem 6. *Any l_p-optimal ranking on C_n is also max optimal.*

Proof. This is a result of Lemma 10 and the observation that the standard ranking on the cycle is a max optimal ranking. □

The converse of Theorem 6 does not hold, as we demonstrate with the following proposition.

Proposition 3. *A max optimal ranking on C_n is not necessarily l_p optimal.*

Proof. Figures 5 and 6 show max optimal rankings f and g on C_6, but only f is l_p optimal. □

4. Graphs for which Max Optimality Implies l_p Optimality

We have shown that for paths and cycles, l_p optimality implies max optimality, but that max optimality does not necessarily imply l_p optimality. We now show that for any complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$ with partite sets of order $m_1 \geq m_2 \geq \cdots \geq m_r$, the set of l_p-optimal rankings and the set of max optimal rankings are the same.

Lemma 11. *If f is an l_p-optimal ranking on $K_{m_1,m_2,\ldots,m_r}$, then under f every vertex in one of the largest partite sets is labeled* 1*, and all other vertices are labeled using distinct labels.*

Proof. Suppose f is l_p optimal. Then by observation 1, f is minimal and thus by [15] f must label each vertex in one of the partite sets W the label 1, and every other vertex using a distinct label. If W is not the largest partite set, then a ranking g can be obtained by labeling every vertex in one of the largest partite sets 1, and every other vertex using distinct labels. Now, $||g||_p < ||f||_p$, which implies that f is not l_p optimal, a contradiction. □

Lemma 12. *Let f be a minimal ranking on $K_{m_1,m_2,\ldots,m_r}$ such that under f every vertex in some largest partite set is labeled* 1*, and other vertices are labeled using distinct labels. Then f is l_p optimal.*

Proof. Suppose f is a minimal ranking such that every vertex in one of the largest partite sets is labeled 1 under f, and other vertices are labeled using distinct labels. Suppose g is an l_p optimal ranking. Then, by Lemma 11, g labels each vertex in one of the largest partite sets 1 and other vertices using distinct labels. This means $||f||_p = ||g||_p$, and thus f is l_p optimal. □

Theorem 7. *A ranking on a complete multipartite graph is max optimal if and only if it is l_p optimal.*

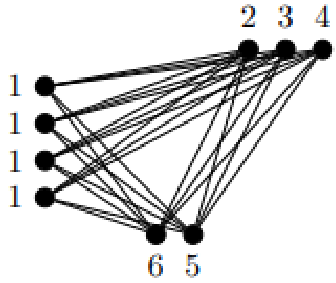

Figure 7. A Max Optimal and l_p-Optimal Ranking on $K_{4,3,2}$

Proof. By [15, Theorem 8], f is a minimal ranking if and only if some partite set W has $f(w) = 1$ for all $w \in W$, and all $v \in V\backslash W$ have unique labels from 2 up to $|V| - |W| + 1$. The minimal ranking f is a max optimal ranking if and only if the partite set W has $|W| = m_1$. Then by Lemmas 11 and 12, f is max optimal if and only if f is l_p optimal. □

By noting the a complete graph is a complete multipartite graph $K_{m_1,m_2,\ldots m_r}$ with $m_1 = m_2 = \cdots m_r = 1$, we get the following immediate corollary.

Corollary 3. *A ranking f on the complete graph K_n is max optimal if and only if it is l_p optimal.*

5. The Special Case of $p = 0$

Given a sequence of numbers, the l_0 *"norm"* is defined as the number of non-zero terms in the sequence. For a graph G with ranking f,

$$||f||_0 = |\{v \in V(G) \mid f(v) > 0\}|.$$

The quotation marks indicate that the map fails to satisfy the requirement of homogeneity. Depending on the application [17–20] it is referred to as the *zero*, *counting* or *Hamming* "norm."

In this section, we use labels 0 through $k-1$ instead of 1 through k, because starting with 1 leads to $||f||_0 = |V(G)|$ for every ranking f on G. We do this on the standard rankings on paths and cycles as well, subtracting 1 from each label in the standard ranking. For reductions and expansions, we remove or add vertices with label 0 instead of 1.

Lemma 13. *If f is an l_0-optimal ranking on P_n, then $|S_0(f)| = \lceil n/2 \rceil$.*

Proof. For any ranking f on a path P_n, $||f||_0 = n - |S_0(f)|$. Thus, maximizing the cardinality of $S_0(f)$ results in an l_0-optimal ranking. Since $|S_0(f)| \leq \lceil n/2 \rceil$ for any ranking f, and the standard ranking g starting with label 0 has $|S_0(g)| = \lceil n/2 \rceil$, the result follows. □

Corollary 4. $\chi_r^0(P_n) = \lfloor n/2 \rfloor$.

The *arank number* of a graph G, denoted $\psi_r(G)$, is given by $\psi_r(G) = \max\{||f||_\infty\}$ over any minimal ranking f on G.

Lemma 14. [21] *Let $n = 3$ or $n \geq 5$. Then $\psi_r(P_n) > \chi_r(P_n)$.*

The following theorem shows that neither Theorem 4 nor its converse applies if $p = 0$.

Theorem 8. *There exist minimal max optimal rankings on paths that are not l_0 optimal. If $n = 6, 7$ or $n \geq 10$, then there exists a minimal l_0-optimal ranking on P_n that is not max optimal.*

Proof. For the first statement, Figure 9 demonstrates a minimal max optimal ranking that is not l_0 optimal. For the second statement, if $n = 6$ or 7, then we have $\lfloor n/2 \rfloor = 3$; otherwise we have $\lfloor n/2 \rfloor \geq 5$. In either case, by Lemma 14 we can find a minimal ranking on $P_{\lfloor n/2 \rfloor}$ that is not max

Figure 8. A Minimal l_0-Optimal Ranking that is not Max Optimal

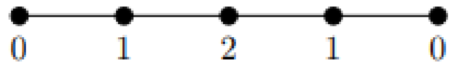

Figure 9. A Minimal Max Optimal Ranking that is not l_0 Optimal

optimal. Choose one such ranking f. Then the ranking $f^\sharp$ is a minimal ranking of P_n: if $f^\sharp(v) > 0$, then we know it cannot be lowered to a label greater than 0 by minimality of f; it cannot be lowered to 0 because every vertex not labeled 0 is adjacent to a vertex labeled 0. Also, $f^\sharp$ is not a max optimal ranking, but is l_0 optimal by Corollary 4 since $\|f\|_0 = n - |S_0| = \lfloor n/2 \rfloor$. □

Lemma 15. *If f is an l_0-optimal ranking on C_n, then $|S_0(f)| = \lfloor \frac{n}{2} \rfloor$.*

Corollary 5. $\chi_r^0(C_n) = \lceil n/2 \rceil$.

Neither Theorem 6 nor its converse applies to the case $p = 0$, as shown by the following theorem.

Theorem 9. *There exist cycles with minimal max optimal rankings that are not l_0 optimal. If $\psi_r(C_{\lceil n/2 \rceil}) > \chi_r(C_{\lceil n/2 \rceil})$, then C_n has a minimal ranking f that is l_0 optimal but not max optimal.*

Proof. For the first statement, note that subtracting 1 from each of the labels in Figure 6 results in a max optimal ranking that is not l_0 optimal. For the second statement, suppose $\psi_r(C_{\lceil n/2 \rceil}) > \chi_r(C_{\lceil n/2 \rceil})$. Let f be a minimal ranking on $C_{\lceil n/2 \rceil}$ that is not max optimal. Then $f^\sharp$ is a minimal l_0-optimal ranking of C_n, but is not max optimal. □

Figure 10 shows a minimal l_0-optimal ranking on C_{14} that is not max optimal. Figure 11 shows the standard ranking on C_{14}, which is max optimal.

Finally, we show that unlike paths and cycles, complete multipartite graphs have the same property with regard to l_0 optimality as with l_p optimality, as long as we require minimality of the l_0-optimal ranking.

Proposition 4. *A ranking f on a complete multipartite graph is max optimal if and only if it is minimal and l_0 optimal.*

Proof. Let $K_{m_1,m_2,\ldots m_n}$ be a complete multipartite graph with partite sets of order $m_1 \geq m_2 \geq \cdots \geq m_n$. By [15], and subtracting 1 from all labels, f is a max optimal ranking if and only if a partite set W with $|W| = m_1$ has all vertices labeled 0, and all $v \in V \backslash W$ have unique labels from 1 up to $|V| - |W|$.

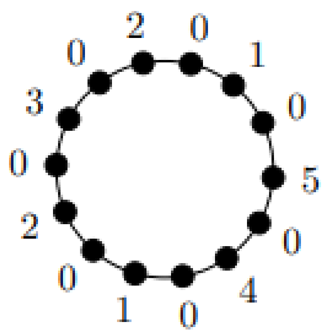

Figure 10. An l_0-Optimal Ranking

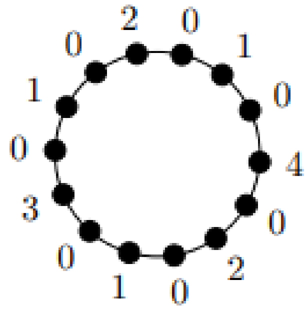

Figure 11. A Max Optimal Ranking

Let g be a minimal l_0-optimal ranking on $K_{m_1,m_2,\ldots m_n}$. Since g is minimal, some partite set W has all vertices labeled 0, and all $v \in V\backslash W$ have unique labels from 1 up to $|V| - |W|$. Then $\|g\|_0 = |V| - |W|$. To minimize $|V| - |W|$, we must have $|W| = m_1$, and then $\|g\|_0 = |V| - m_1$. □

6. Conclusion and Open Problems

For paths and cycles, we have shown that l_p optimality of a ranking implies max optimality for all $p > 0$. For $p = 0$, we have shown that this does not hold. We have also shown that max optimality does not imply l_p optimality for these graphs. For complete multipartite graphs, however, we showed that the set of l_p-optimal rankings is the same as the set of max optimal rankings, including in the case $p = 0$ if we restrict l_0-optimal rankings to be minimal.

In [1], we constructed graphs — including trees — whose l_p-optimal rankings and max optimal rankings are disjoint. The construction included a cut vertex whose removal resulted in a graph with three components. One question to investigate is whether any graph that has l_p-optimal rankings and max optimal rankings disjoint must have a cut vertex. Another potential problem is to consider the analog of an arank number $\psi_r(G)$ of a graph G under a norm different from the max norm. For example, on a graph G, among all minimal rankings f, what is the largest possible l_1 norm, or in other words, what is $\max_f \sum_{i=1}^{|V(G)|} f(v)$? Finally, can we characterize graphs that have l_p optimality and max optimality equivalent, as with the complete multipartite graph?

Conflict of Interest

The authors declare no conflict of interest.

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