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Article

Hamiltonian Laceability of Hypercubes with Prescribed Linear Forest and/or Faulty Edges

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Abstract: A node in the n-dimensional hypercube Q_n is called an odd node (resp. an even node) if the sum of all digits of the node is odd (resp. even). Let $F \subset E(Q_n)$ and let L be a linear forest in $Q_n - F$ such that $|E(L)| + |F| \leq n - 2$ for $n \geq 2$. Let x be an odd node and y an even node in Q_n such that none of the paths in L has x or y as internal node or both of them as end nodes. In this note, we prove that there is a Hamiltonian path between x and y passing through L in $Q_n - F$. The upper bound $n - 2$ on $|E(L)| + |F|$ is sharp.

Keywords: Hypercube, Bounds, Hamiltonian path

1. Introduction

We mostly follow [1] for graph-theoretical terminology and notation, and follow [2] for some similar graph-theoretical operations not defined here. Denote by N_n the set $\{0, 1, \cdots, n-1\}$ for a positive integer n.

Given a bipartite graph G, a *linear forest* in G is a subgraph each component of which is a path. Let F be an subset of $E(G)$ and let L be a linear forest in $G - F$ such that $|E(L)| + |F| \leq k$ for some non-negative integer k. For any two nodes x, y in different partite sets of G, $\{x, y\}$ and L are said to be *compatible* if none of the paths in L has x or y as internal node or both of them as end nodes. Denote by $\pi[x, y]$ a path between x and y. The graph G is said to be *k-fault-tolerant-prescribed hamiltonian laceable* if $G-F$ admits a hamiltonian path $\pi[x, y]$ passing through L under the condition that $\{x, y\}$ and L are compatible [3]. Particularly, G is said to be *k-fault-tolerant-hamiltonian laceable* if $E(L) = \emptyset$ [4], and *k-prescribed-hamiltonian laceable* if $F = \emptyset$, and *hamiltonian laceable* if $E(L) = F = \emptyset$ [5].

The n-dimensional hypercube is one of the most attractive interconnection networks for multiprocessor systems and it is bipartite, hamiltonian, n-regular, node-transitive, edge-transitive and recursive [6,7]. Many real multiprocessors have taken hypercubes as underlying interconnection networks. The problem of embedding hamiltonian paths and cycles with faulty edges and/or prescribed edges in hypercubes has drawn considerable attentions (see, for example, [4, 8–18] and references therein).

In [4], Tsai et al. investigated fault-tolerant hamiltonian laceability of hypercubes. One of their important results can be restated as follows:

Theorem 1 (See Lemma 3 in Tsai et al. [4]). *Given $n \geq 2$, Q_n is $(n-2)$-fault-tolerant hamiltonian laceable but not $(n-1)$-fault-tolerant hamiltonian laceable.*

In [18], Wang and Chen studied the problem of embedding fault-free hamiltonian cycles passing through a linear forest in hypercubes with or without faulty edges and they obtained the following generalized result.

Theorem 2 (See Theorem A in Wang and Chen [18])**.** *Given $n \geq 3$, let $F \subset E(Q_n)$ and let L be a linear forest in $Q_n - F$ such that $|E(L)| \geq 1$ and $|F| + |E(L)| \leq n-1$. Then $Q_n - F$ admits a hamiltonian cycle passing through L.*

In [14], Dvořák and Gregor investigated the problem of embedding hamiltonian paths passing through prescribed linear forests in hypercubes. One of their main results can be restated as follows:

Theorem 3 (See Theorem 10 in Dvořák and Gregor [14])**.** *The hypercube Q_n is $(2n-4)$-prescribed hamiltonian laceable if $n = 2$ or $n \geq 5$, and $(2n-5)$-prescribed hamiltonian laceable if $n \in \{3, 4\}$.*

In this note, we will generalize Theorems 1 and 2, and prove that Q_n with $n \geq 2$ is $(n-2)$-fault-tolerant-prescribed hamiltonian laceable but not $(n-1)$-fault-tolerant-prescribed hamiltonian laceable.

2. Preliminaries

The *n-dimensional hypercube Q_n* is a simple graph consists of 2^n nodes, each of which is an n-bit binary string with the form $\delta_0\delta_1 \ldots \delta_{n-1}$. Two nodes are adjacent if and only if they differ in exactly one bit. An edge $(u, v) \in E(Q_n)$ is called *an edge in dimension i* if u and v differ in the i^{th} bit, where $i \in N_n$. A node in Q_n is called an *odd node* (resp. *even node*) if the sum of all digits of the node is odd (resp. even). Let X and Y be the set of all odd nodes and all even nodes in Q_n, respectively. Then (X, Y) is a bipartition of Q_n. Fix $n \geq 2$. Let $F \subset E(Q_n)$ and let L be a linear forest in $Q_n - F$. There are n different ways to decompose Q_n into two disjoint copies, $Q[0]$ and $Q[1]$, of Q_{n-1} by deleting all the edges in dimension i of Q_n for an arbitrary $i \in N_n$, where each node of $Q[i]$ has j in the i^{th} bit for $j \in N_2$. For any $j \in N_2$, abbreviated $V(Q[j])$ as V_j, and denote by L_j and F_j the restrictions of L and F in $Q[j]$, respectively. For an arbitrary node $u \in V_j$, there is exactly one neighbour of u in $Q[1-j]$, and we denote it by u^{1-j}.

The following two lemmas will be used in the proof of our main result in Section 3.

Lemma 1. *Given $j \in N_2$, let P be a hamiltonian path passing through L_j in $Q[j]$. If $E(L_j) \cup E(L_{1-j}) = E(L)$, $E(L_{j-1}) \neq \emptyset$ and $2^{n-1} - |E(L)| - |E(L_{j-1})| > 0$, then there exists an edge $(u, v) \in E(P) \setminus E(L_j)$ such that $\{u^{1-j}, v^{1-j}\}$ and L_{1-j} are compatible.*

Proof of Lemma 1. There are $|E(P) \setminus E(L_j)| = 2^{n-1} - 1 - |E(L_j)|$ edge candidates in total. Note that each component of a linear forest is a path. Let $(u, v) \in E(P) \setminus E(L_j)$. Then it fails the lemma only if
(i). u^{1-j} or v^{1-j} is an internal node of L_{1-j}, or
(ii). u^{1-j} and v^{1-j} are the two end nodes of some component of L_{1-j}.

Let y be an internal node of L_{1-j}, if any. Then it makes at most one candidate in $E(P) \setminus E(L_j)$ fail if y^j is an end node of P, and at most two candidates fail otherwise (see, for example, in Figure 1, an internal node y of the path $\langle x, y, z \rangle$, a component of L_{1-j}, makes (s, y^j) and (t, y^j) fail if $(x^j, z^j) \in E(P) \setminus E(L_j)$). Note that each component of L_{1-j} is a path. Let $\pi[x, z]$ be a component of L_{1-j} of length at least 1. Then such a pair $\{x, z\}$ makes no candidate in $E(P) \setminus E(L_j)$ fail if x^j is not adjacent to z^j in P, and makes at most one candidate (x^j, z^j) fail otherwise (see, for example, in Figure 1, the pair $\{x, z\}$ of the end nodes of a component of L_{1-j} makes (x^j, z^j) fail if $(x^j, z^j) \in E(P) \setminus E(L_j)$).

Note that $E(L_{j-1}) \neq \emptyset$. Let m be the number of components of length at least 1 in L_{j-1}, and let $P_1, P_2, \ldots, P_m$ be the m components. Then $m \geq 1$ and the number of internal nodes of L_{1-j} is $\sum_{i=1}^{m}(|E(P_i)| - 1) = |E(L_{1-j})| - m$. Thus, the total number of edge candidates that fail the lemma does not exceed $m + 2(|E(L_{1-j})| - m) = 2|E(L_{1-j})| - m$. Since $|E(P) \setminus E(L_j)| - (2|E(L_{1-j})| - m) = 2^{n-1} - 1 - |E(L_j)| - 2|E(L_{1-j})| + m = 2^{n-1} - 1 - |E(L)| - |E(L_{1-j})| + m \geq 2^{n-1} - |E(L)| - |E(L_{j-1})| > 0$, the lemma follows. □

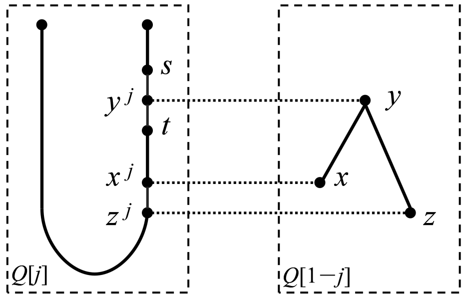


Figure 1. Illustration of an Example in the Proof of Lemma 1

Lemma 2. *Given $j \in N_2$, let $x \in V_j \cap X$, $y \in V_{1-j} \cap Y$ be two nodes such that they are incident with at most one edge of L_j and L_{1-j}, respectively. If $2^{n-2} - (|E(L_j)| + |E(L_{1-j})|) > 0$, then there exists a node $u \in V_j \cap Y$ such that $\{x, u\}$ and L_j are compatible, and $\{y, u^{1-j}\}$ and L_{1-j} are compatible.*

Proof of Lemma 2. There are $|V_0 \cap Y|$ node candidates in total. Let $u \in V_0 \cap Y$. Then it fails the lemma only if
(i). u is an internal node of L_j, or
(ii). u^{1-j} is an internal node of L_{1-j}, or
(iii). x and u are the two end nodes of some component of L_j, or
(iv). y and u^{1-j} are the two end nodes of some component of L_{1-j}.

If $E(L_j) = \emptyset$, then there is no $u \in V_0 \cap Y$ such that (i) or (iii) hold. If $E(L_j) \neq \emptyset$, then the number of internal nodes of L_j does not exceed $|E(L_j)| - 1$ and the number of such u that (iii) holds does not exceed 1. Therefore, the total number of such a node u that (i) or (iii) hold does not exceed $|E(L_j)|$. Similarly, the total number of such a node u that (ii) or (iv) hold does not exceed $|E(L_{1-j})|$. Therefore, the number of such a node u that fails the lemma does not exceed $|E(L_j)| + |E(L_{1-j})|$. Since $|V_0 \cap Y| - (|E(L_j)| + |E(L_{1-j})|) = 2^{n-2} - (|E(L_j)| + |E(L_{1-j})|) > 0$, the lemma follows. □

3. Main Result

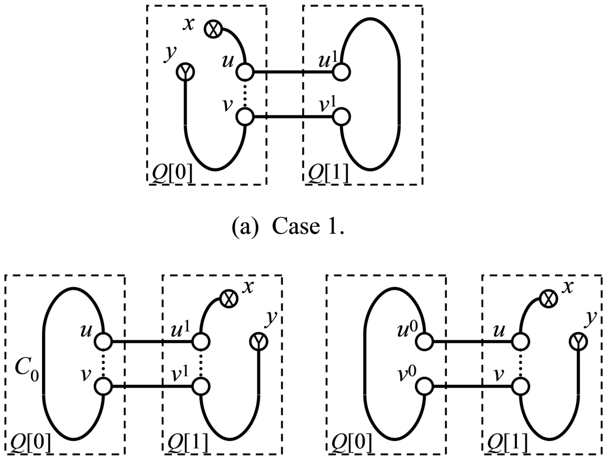
Theorem 4. *Let $n \geq 2$. The hypercube Q_n is $(n-2)$-fault-tolerant-prescribed hamiltonian laceable but not $(n-1)$-fault-tolerant-prescribed hamiltonian laceable.*

Proof of Theorem 4. By Theorem 1, Q_n with $n \geq 2$ is not $(n-1)$-fault-tolerant hamiltonian laceable, and so it is not $(n-1)$-fault-tolerant-prescribed hamiltonian laceable. It remains to prove that Q_n is $(n-2)$-fault-tolerant hamiltonian laceable for $n \geq 2$. Let $F \subset E(Q_n)$ and let L be a linear forest in $Q_n - F$ such that $|E(L)| + |F| \leq n-2$. Let $x \in X$ and $y \in Y$ such that $\{x, y\}$ and L are compatible. In the following, it is enough to prove that $Q_n - F$ admits a hamiltonian path $\pi[x, y]$ passing through L, and it suffices to consider the case that $|E(L)| + |F| = n-2$.

We will prove the above statement by induction on n. It is trivial for the base case $n = 2$. For $n = 3$, $|E(L)| + |F| \leq 1$, Theorems 1 and 3 imply that the statement holds. In the remainder, assume that the statement holds for Q_{n-1} and prove that it also holds for Q_n, where $n \geq 4$. By Theorems 1 and 3, the statement holds for $E(L) = \emptyset$ or $F = \emptyset$, respectively. It remains to show that it also holds for $E(L) \neq \emptyset$ and $F \neq \emptyset$. Since $|E(L)| + |F| = n-2 < n$, there is a dimension, say dimension 0, such that there is no edge of $E(L) \cup F$ in this dimension. Let $Q[0]$, $Q[1]$ be a decomposition of Q_n by deleting all the edges in dimension 0. Without loss of generality, assume that $|E(L_0)| + |F_0| \geq |E(L_1)| + |F_1|$. There are three cases to be considered.

Case 1. $x \in V_0$ and $y \in V_0$.

Suppose first that $|E(L_0)| + |F_0| = n-2$. Then $L_0 = L$, $F_0 = F$ and $E(L_1) = F_1 = \emptyset$. Let $f \in F_0$. Then $|E(L_0) \cup F_0 \setminus \{f\}| = n-3$. By the induction hypothesis, $Q[0] - F_0 \setminus \{f\}$ has a hamiltonian path $\pi[x, y]$ passing through L_0. Let $(u, v) = f$ if f lies on $\pi[x, y]$, and let (u, v) be an arbitrary edge in



(a) Case 1.

(b) Case 2.1. (c) Case 2.2.

Figure 2. Illustrations of Cases 1-2 of the Proof of Theorem 4

$E(\pi[x,y]) \setminus E(L_0)$ otherwise. Clearly, u^1 and v^1 lie in different partite sets of $Q[1]$, and so $Q[1]$ has a hamiltonian path $\pi[u^1, v^1]$.

Suppose now that $|E(L_0)| + |F_0| \leq n-3$. By the induction hypothesis, $Q[0] - F_0$ has a hamiltonian path $\pi[x,y]$ passing through L_0. Note that $|F| \geq 1$ and $|E(L_1)| \leq |E(L)| < |E(L)| + |F| = n-2$. Then $2^{n-1} - |E(L)| - |E(L_1)| > 2^{n-1} - 2(n-2) > 0$, and so Lemma 1 implies that there exists an edge $(u,v) \in E(\pi[x,y]) \setminus E(L_0)$ such that $\{u^1, v^1\}$ and L_1 are compatible. Since $|E(L_1)| + |F_1| \leq |E(L_0)| + |F_0| \leq n-3$, by the induction hypothesis, $Q[1] - F_1$ has a hamiltonian path $\pi[u^1, v^1]$ passing through L_1.

For all the possible scenarios above, $\pi[x,y] \cup \pi[u^1, v^1] + \{(u,u^1),(v,v^1)\} - (u,v)$ is a hamiltonian path of $Q_n - F$ between x and y passing through L as illustrated in Figure 2 (a).

Case 2. $x \in V_1$ and $y \in V_1$.

Case 2.1. $|E(L_0)| + |F_0| = n-2$.

In this case, $|F_0| = |F| \geq 1$, $1 \leq |E(L_0)| = |E(L)| \leq n-3$, and $E(L_1) = F_1 = \emptyset$. Theorem 2 implies that $Q[0] - F_0$ has a hamiltonian cycle C_0 passing through L_0. Since $|E(C_0) \setminus E(L_0)| = |E(C_0)| - |E(L_0)| \geq 2^{n-1} - (n-3) > 1$, there exists an edge $(u,v) \in E(C) \setminus E(L_0)$ such that $\{u^1, v^1\} \neq \{x,y\}$. Thus, $\{x,y\}$ and the path $\langle u^1, v^1 \rangle$ are compatible. Combing this with $|\{(u^1, v^1)\}| = 1 \leq n-3$ ($n \geq 4$), by Theorem 3, $Q[1] - F_1$ has a hamiltonian path $\pi[x,y]$ passing through $\langle u^1, v^1 \rangle$. Thus, $C_0 \cup \pi[x,y] + \{(u,u^1),(v,v^1)\} - \{(u,v),(u^1,v^1)\}$ is a hamiltonian path of $Q_n - F$ between x and y passing through L as illustrated in Figure 2 (b).

Case 2.2. $|E(L_0)| + |F_0| \leq n-3$.

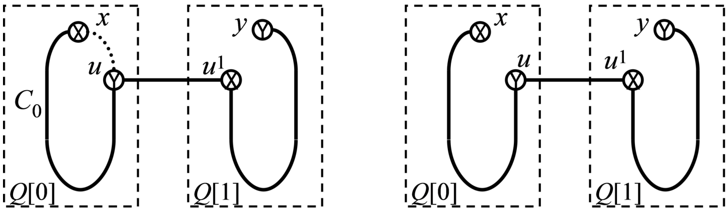
Note that $|E(L_1)| + |F_1| \leq |E(L_0)| + |F_0| \leq n-3$. By the induction hypothesis, $Q[1] - F_1$ has a hamiltonian path $\pi[x,y]$ passing through L_1. Since $2^{n-1} - |E(L)| - |E(L_0)| > 2^{n-1} - 2(n-2) > 0$, Lemma 1 implies that there exists an edge $(u,v) \in E(\pi[x,y]) \setminus E(L_1)$ such that $\{u^0, v^0\}$ and L_0 are compatible. Since $|E(L_0)| + |F_0| \leq n-3$, by the induction hypothesis, $Q[0] - F_0$ has a hamiltonian path $\pi[u^0, v^0]$ passing through L_0. Thus, $\pi[u^0, v^0] \cup \pi[x,y] + \{(u^0,u),(v^0,v)\} - (u,v)$ is a hamiltonian path of $Q_n - F$ between x and y passing through L as illustrated in Figure 2 (c).

Case 3. $x \in V_0$ and $y \in V_1$, or $y \in V_0$ and $x \in V_1$.

Without loss of generality, assume that $x \in V_0$ and $y \in V_1$.

Case 3.1. $|E(L_0)| + |F_0| = n-2$.

In this case, $E(L_1) = F_1 = \emptyset$. Theorem 2 implies that $Q[0] - F_0$ has a hamiltonian cycle C_0 passing through L_0. Since $\{x,y\}$ and L are compatible, x is not an internal node in L, and so there is a neighbour u of x on C_0 such that $(x,u) \notin E(L_0)$. Recall that x is an odd node and y is an even node. Then u is an even node and u^1 is an odd node. Thus, u^1 and y are in different partite sets of $Q[1]$, and so there is a hamiltonian path $\pi[u^1, y]$ in $Q[1]$. Therefore, $C_0 \cup \pi[u^1, y] + (u,u^1) - (x,u)$ is a hamiltonian



(a) Case 3.1. (b) Case 3.2.

Figure 3. Illustrations of Case 3 of the Proof of Theorem 4

path of $Q_n - F$ between x and y passing through L as illustrated in Figure 3 (a).

Case 3.2. $|E(L_0)| + |F_0| \leq n - 3$.

Since $2^{n-2} - (|E(L_0)| + |E(L_1)|) = 2^{n-2} - |E(L)| \geq 2^{n-2} - (n-3) > 0$, Lemma 2 implies that there exists a node $u \in V_0 \cap Y$ such that $\{x, u\}$ and L_0 are compatible and $\{y, u^1\}$ and L_1 are compatible. Since $|E(L_1)| + |F_1| \leq |E(L_0)| + |F_0| \leq n - 3$, by the induction hypothesis, $Q[0] - F_0$ has a hamiltonian path $\pi[x, u]$ passing through L_0, and $Q[1] - F_1$ has a hamiltonian path $\pi[y, u^1]$ passing through L_1. Thus, $\pi[x, u] \cup \pi[y, u^1] + (u, u^1)$ is a hamiltonian path of $Q_n - F$ between x and y passing through L as illustrated in Figure 3 (b). □

4. Conclusion

In this note, we investigated the fault-tolerant prescribed hamiltonian laceability of hypercubes and proved that the hypercube Q_n with $n \geq 2$ is $(n-2)$-fault-tolerant-prescribed hamiltonian laceable but not $(n-1)$-fault-tolerant-prescribed hamiltonian laceable. That is to say, given an arbitrary set $F \subset E(Q_n)$ and a linear forest L in Q_n with $n \geq 2$ such that $|F| + |E(L)| \leq n - 2$, for two nodes x and y of different partite sets in Q_n, $Q_n - F$ has a hamiltonian path between x and y passing through L if $\{u, v\}$ and L are compatible. Since Q_n is n-regular, the upper bound $n - 2$ on $|E(L)| + |F|$ cannot be improved. The main result in this note generalized some known results.

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