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Article

On Strong Regal and Strong Royal Colorings

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Abstract: Two colorings have been introduced recently where an unrestricted coloring c assigns nonempty subsets of $[k] = \{1, \ldots, k\}$ to the edges of a (connected) graph G and gives rise to a vertex-distinguishing vertex coloring by means of set operations. If each vertex color is obtained from the union of the incident edge colors, then c is referred to as a strong royal coloring. If each vertex color is obtained from the intersection of the incident edge colors, then c is referred to as a strong regal coloring. The minimum values of k for which a graph G has such colorings are referred to as the strong royal index of G and the strong regal index of G respectively. If the induced vertex coloring is neighbor distinguishing, then we refer to such edge colorings as royal and regal colorings. The royal chromatic number of a graph involves minimizing the number of vertex colors in an induced vertex coloring obtained from a royal coloring. In this paper, we provide new results related to these two coloring concepts and establish a connection between the corresponding chromatic parameters. In addition, we establish the royal chromatic number for paths and cycles.

Keywords: Edge coloring, Induced vertex coloring, Regal and royal coloring, Royal chromatic number

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1. Introduction

For a positive integer k, let $\mathcal{P}^*([k])$ denote the set of nonempty subsets of $[k] = \{1, 2, \ldots, k\}$. For a connected graph G, let $c : E(G) \to \mathcal{P}^*([k])$ be an edge coloring of G where adjacent edges may be colored the same. An induced vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ can defined by either

$$(1)\ c'(v) = \bigcup_{e \in E_v} c(e) \text{ or } (2)\ c'(v) = \bigcap_{e \in E_v} c(e),$$

where E_v is the set of edges incident with v. In the case of (1), if c' is a proper vertex coloring of G, that is, c' assigns adjacent vertices different colors, then c is called a *royal k-edge coloring* of G. The minimum positive integer k for which a graph G has a royal k-edge coloring is the *royal index* $\text{roy}(G)$ of G. If c' is vertex-distinguishing, that is, c' assigns a unique color to each vertex, then c is a *strong royal k-edge coloring* of G. The minimum positive integer k for which a graph G has a strong royal k-edge coloring is the *strong royal index* $\text{sroy}(G)$ of G. In the case of (2), if c' is a proper vertex coloring of G, then c is called a *regal k-edge coloring* of G. The minimum positive integer k for which a graph G has a regal k-edge coloring is the *regal index* $\text{reg}(G)$ of G. If c' is vertex-distinguishing, then c is a *strong regal k-edge coloring* of G. The minimum positive integer k for which a graph G

has a strong regal k-edge coloring is the *strong regal index* $\mathrm{sreg}(G)$ of G. In this paper, we focus on the strong royal and strong regal indexes of graphs.

The concept of royal colorings was introduced in [1] in 2017 by Bousquet et al and independently studied in [2]. Regal colorings were introduced and studied in [3] in 2018. It was shown that these parameters exist for any connected graph of order at least 3 and have been established for many well-known classes of graphs.

Theorem 1. [1–3] *For each integer $n \geq 4$,*

$$\mathrm{sreg}(K_n) = 1 + \lfloor \log_2 n \rfloor \text{ and } \mathrm{sroy}(K_n) = 1 + \lceil \log_2 n \rceil.$$

Here we can see that these indexes can take different values for an infinite class of graphs. Similarly, these indexes take on distinct values for a path of order 7, namely $\mathrm{sreg}(P_7) = 4$ while $\mathrm{sroy}(P_7) = 3$. However, these indexes agree for all of order n at least 4 with $n \neq 7$ and for all cycles of order n at least 4.

Theorem 2. [1–3] *If $n \geq 4$ is an integer with $n \neq 7$, then*

$$\mathrm{sreg}(P_n) = \mathrm{sroy}(P_n) = 1 + \lfloor \log_2 n \rfloor.$$

Theorem 3. [1,3,4] *For each integer $n \geq 4$,*

$$\mathrm{sreg}(C_n) = \mathrm{sroy}(C_n) = \begin{cases} 1 + \lceil \log_2 (n+1) \rceil & \text{if } n = 7 \\ 1 + \lfloor \log_2 n \rfloor & \text{if } n \neq 7. \end{cases}$$

For these classes of graphs, the strong regal index and the strong royal index differ by at most one. This phenomenon, in fact, is generally true for all connected graphs with order at least 3.

Theorem 4. *If G is a connected graph of order at least 3, then*

$$|\,\mathrm{sroy}(G) - \mathrm{sreg}(G)| \leq 1.$$

Proof. First, we show that $\mathrm{sroy}(G) \leq \mathrm{sreg}(G) + 1$. Let $\mathrm{sreg}(G) = k$ and suppose that $c : E(G) \to \mathcal{P}^*([k])$ is a strong regal k-edge coloring of G where the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is a vertex-distinguishing vertex coloring induced by c. Define a complementary coloring $\overline{c} : E(G) \to \mathcal{P}^*([k+1])$ by $\overline{c}(e) = [k+1] - c(e)$. Then $\overline{c}' : V(G) \to \mathcal{P}^*([k+1])$ is a strong royal $(k+1)$-edge coloring of G. Indeed, observe that

$$\overline{c}'(v) = \bigcup_{e \in E_v} \overline{c}(e) = \bigcup_{e \in E_v} ([k+1] - c(e)) = [k+1] - c'(v)$$

where E_v is the set of edges incident to v. Note that $\overline{c}'(v)$ is nonempty since $c'(v) \subseteq [k]$. Since c' is vertex-distinguishing, it follows that $\overline{c}'$ is also vertex-distinguishing. Hence, $\mathrm{sroy}(G) \leq k + 1 = \mathrm{sreg}(G) + 1$.

Next, we show that $\mathrm{sreg}(G) \leq \mathrm{sroy}(G) + 1$. Let $\mathrm{sroy}(G) = k$ and suppose that $c : E(G) \to \mathcal{P}^*([k])$ is a strong royal k-edge coloring of G where the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is a vertex-distinguishing vertex coloring induced by c. Define a complementary coloring $\overline{c} : E(G) \to \mathcal{P}^*([k+1])$ by $\overline{c}(e) = [k+1] - c(e)$. Then $\overline{c}' : V(G) \to \mathcal{P}^*([k+1])$ is a strong regal $(k+1)$-edge coloring of G. Indeed, observe that

$$\overline{c}'(v) = \bigcap_{e \in E_v} \overline{c}(e) = \bigcap_{e \in E_v} ([k+1] - c(e)) = [k+1] - c'(v)$$

where E_v is the set of edges incident to v. Note that $\overline{c}'(v)$ is nonempty since $c'(v) \subseteq [k]$. Since c' is vertex-distinguishing, it follows that $\overline{c}'$ is also vertex-distinguishing. Hence, $\mathrm{sreg}(G) \leq k + 1 = \mathrm{sroy}(G) + 1$. Consequently, $\mathrm{sroy}(G) \geq \mathrm{sreg}(G) - 1$ and so

$$\mathrm{sreg}(G) - 1 \le \mathrm{sroy}(G) \le \mathrm{sreg}(G) + 1.$$

Therefore,

$$|\,\mathrm{sroy}(G) - \mathrm{sreg}(G)| \le 1.$$

□

It is possible for either $\mathrm{sreg}(G) < \mathrm{sroy}(G)$ or $\mathrm{sroy}(G) < \mathrm{sreg}(G)$ for a connected graph G. For example, we saw that $\mathrm{sreg}(K_5) = 3$ while $\mathrm{sroy}(K_5) = 4$. On the other hand, $\mathrm{sroy}(P_7) = 3$ while $\mathrm{sreg}(P_7) = 4$.

In addition, a common lower bound has been established for these parameters for a given a graph in terms of its order and upper bounds on these parameters relating to subgraph structure are known.

Proposition 1. [2, 3] *If G is a connected graph of order $n \ge 4$, then*

$$\mathrm{sroy}(G) \ge 1 + \lfloor \log_2 n \rfloor \ \textit{and}\ \ \mathrm{sroy}(G) \ge 1 + \lfloor \log_2 n \rfloor.$$

Proposition 2. [2, 3] *For a connected graph G of order 4 or more, let $\mathcal{H}$ be the set of connected spanning subgraphs of G. Then*

$$\mathrm{sreg}(G) \le \min\{\mathrm{sreg}(H) : \ H \in \mathcal{H}\}$$

and

$$\mathrm{sroy}(G) \le 1 + \min\{\mathrm{sroy}(H) : \ H \in \mathcal{H}\}.$$

For a graph G of order $n \ge 3$, let k be the unique positive integer such that $2^{k-1} \le n \le 2^k - 1$. In [2, 3], corresponding versions of the following conjecture were stated.

Conjecture 1. *If G is a connected graph of order n at least 3 where $2^{k-1} \le n \le 2^k - 1$, then*

$$\mathrm{sroy}(G) \in \{k, k+1\} \ \textit{and}\ \mathrm{sreg}(G) \in \{k, k+1\}.$$

A connected graph G of order $n \ge 3$ where $2^{k-1} \le n \le 2^k - 1$ is a *royal-zero graph* if $\mathrm{sroy}(G) = k$ and is a *royal-one graph* if $\mathrm{sroy}(G) = k + 1$. Similarly, G is a *regal-zero graph* if $\mathrm{sreg}(G) = k$ and is a *regal-one graph* if $\mathrm{sreg}(G) = k + 1$. As a consequence of Proposition 4, we have the following result.

Corollary 1. *If G is royal-zero or regal-zero, then*

$$\mathrm{sroy}(G) \in \{k, k+1\} \ \textit{and}\ \mathrm{sreg}(G) \in \{k, k+1\}.$$

2. Graph Operations

The *corona* $\mathrm{cor}(G)$ of a graph G is the graph obtained from G by adding a pendant edge at each vertex of G. Thus, if the order of G is n, then the order of $\mathrm{cor}(G)$ is $2n$. It was shown in [4] that the strong royal index of $\mathrm{cor}(G)$ never exceeds $\mathrm{sroy}(G)$ by more than 1. We prove a similar result for the strong regal index of $\mathrm{cor}(G)$.

Proposition 3. *If G is a connected graph of order $n \ge 4$, then*

$$\mathrm{sreg}(\mathrm{cor}(G)) \le \mathrm{sreg}(G) + 1.$$

Furthermore, if G is regal-zero, then $\mathrm{cor}(G)$ *is regal-zero.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $H = \mathrm{cor}(G)$ be obtained from G by adding the pendant edge $u_i v_i$ at v_i for $1 \le i \le n$. Let $c_G : E(G) \to \mathcal{P}^*([k])$ be a strong regal k-edge coloring of G. Define an edge coloring $c_H : E(H) \to \mathcal{P}^*([k+1])$ by

$$c_H(e) = \begin{cases} c_G(e) & \text{if } e \in E(G) \\ c'_G(v_i) \cup \{k+1\} & \text{if } e = u_i v_i \text{ for } 1 \le i \le n. \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'_H(u_i) = c'_G(v_i) \cup \{k+1\} \text{ and } c'_H(v_i) = c'_G(v_i) \text{ for } 1 \le i \le n.$$

Since c'_H is vertex-distinguishing, it follows that c_H is a strong regal $(k+1)$-edge coloring of cor(G) and so sreg$(H) \le$ sreg$(G) + 1$.

If G is a connected graph of order $n \ge 4$ where $2^{k-1} \le n \le 2^k - 1$, then cor$(G)$ is a connected graph of order $2n \ge 8$ where $2^k \le 2n \le 2^{k+1} - 2$. Since sreg(cor$(G)$) $\ge k+1$ by Proposition 1 and there is a strong regal $(k+1)$-edge coloring of cor(G), it follows that sreg(cor(G)) $= k+1$ and so cor(G) is regal-zero. □

Though thus far we consider only connected graphs of order at least 3, any graph whose connected components have order at least 3 also have strong royal and strong regal colorings and, consequently, the associated chromatic parameters. Let F and H be two graphs with disjoint vertex sets. The *union* $G = F + H$ of F and H has vertex set $V(G) = V(F) \cup V(H)$ and edge set $E(G) = E(F) \cup E(H)$. The union $H + H$ of two disjoint copies of H is denoted by $2H$. If a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$. □

Proposition 4. *If F and H are connected graphs of order at least* 3*, then*

$$\max\{\text{sroy}(F),\ \text{sroy}(H)\} \le \text{sroy}(F+H) \le \max\{\text{sroy}(F),\ \text{sroy}(H)\} + 1$$

and

$$\max\{\text{sreg}(F),\ \text{sreg}(H)\} \le \text{sreg}(F+H) \le \max\{\text{sreg}(F),\ \text{sreg}(H)\} + 1.$$

Proof. Since the argument for both the royal and regal cases are similar, we prove the result for only the regal case. Suppose that sreg$(F) = k_F$ and sreg$(H) = k_H$, where say $k_F \le k_H$. Since the restriction of a strong regal coloring of $F + H$ to H is also a strong regal coloring of H, it follows that

$$\text{sreg}(H) = \max\{\text{sreg}(F),\ \text{sreg}(H)\} \le \text{sreg}(F+H).$$

Thus, the lower bound holds. For the upper bound, we show that there is a strong regal k_H-edge coloring of $F+H$. Let c_F be a strong regal k_F-edge coloring of F and let c_H be a strong regal k_H-edge coloring of H. Define an edge coloring $c : E(F+H) \to \mathcal{P}^*([k_H+1])$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_H(e) \cup \{k_H+1\} & \text{if } e \in E(H). \end{cases}$$

Then the induced coloring $c' : V(F+H) \to \mathcal{P}^*([k_H+1])$ is given by

$$c'(v) = \begin{cases} c'_F(v) & \text{if } v \in V(F) \\ c'_H(v) \cup \{k_H+1\} & \text{if } v \in V(H). \end{cases}$$

Since c'_F and c'_H are vertex-distinguishing and $c'(x) \ne c'(y)$ if $x \in V(F)$ and $y \in V(H)$, it follows that c' is a vertex-distinguishing vertex coloring of $F + H$. Therefore, c is a strong regal k_H-edge coloring of $F+H$. □Note that Proposition 4 cannot be improved. For example, we saw that sreg$(P_3) =$ sreg$(K_3) =$ sreg$(P_3 + K_3) = 3$, but sreg$(P_3) =$ sreg$(P_7) = 3$ and sreg$(P_3 + P_7) = 4$.

Let F and H be two graphs with disjoint vertex sets. The *join* $G = F \vee H$ of F and H has vertex set $V(G) = V(F) \cup V(H)$ and edge set

$$E(G) = E(F) \cup E(H) \cup \{uv : u \in V(F), v \in V(H)\}.$$

□

Proposition 5. *If F and H are connected graphs of order at least* 3*, then*

$$\text{sreg}(F \vee H) \le \max\{\text{sreg}(F),\ \text{sreg}(H)\} + 1$$

and

$$\text{sroy}(F \vee H) \leq \max\{\text{sroy}(F),\ \text{sroy}(H)\} + 2.$$

Proof. Let $G = F \vee H$. It suffices to show that a strong regal s-edge coloring and a strong royal t-edge coloring of G exist for values of s and t corresponding to each desired upper bound.

First, we consider $\text{sreg}(F \vee H)$. We may assume that

$$\text{sreg}(F) \leq \text{sreg}(H) = k$$

and so $\max\{\text{sreg}(F),\ \text{sreg}(H)\} + 1 = k + 1$. Then there are strong regal k-edge colorings $c_F : E(F) \to \mathcal{P}^*([k])$ and $c_H : E(H) \to \mathcal{P}^*([k])$ of F and H with vertex-distinguishing induced vertex colorings c'_F and c'_H respectively. Define an edge coloring $c : E(G) \to \mathcal{P}^*([k+1])$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_H(e) \cup \{k+1\} & \text{if } e \in E(H) \\ [k+1] & \text{otherwise.} \end{cases}$$

Then the induced coloring $c' : V(G) \to \mathcal{P}^*([k+1])$ is given by

$$c'(v) = \begin{cases} c'_F(v) & \text{if } v \in V(F) \\ c'_H(v) \cup \{k+1\} & \text{if } v \in V(H). \end{cases}$$

Since c'_F and c'_H are vertex-distinguishing and $c'(x) \neq c'(y)$ if $x \in V(F)$ and $y \in V(H)$, it follows that c' is a vertex-distinguishing vertex coloring of G. Therefore, c is a strong regal $(k+1)$-edge coloring of $G = F \vee H$.

Next, we consider $\text{sroy}(F \vee H)$. We may assume that

$$\text{sroy}(F) \leq \text{sroy}(H) = k$$

and so $\max\{\text{sroy}(F),\ \text{sroy}(H)\} + 2 = k + 2$. Then there are strong royal k-edge colorings $c_F : E(F) \to \mathcal{P}^*([k])$ and $c_H : E(H) \to \mathcal{P}^*([k])$ of F and H with vertex-distinguishing induced vertex colorings c'_F and c'_H respectively. Define an edge coloring $c : E(G) \to \mathcal{P}^*([k+2])$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_H(e) \cup \{k+1\} & \text{if } e \in E(H) \\ \{k+2\} & \text{otherwise.} \end{cases}$$

Then the induced coloring $c' : V(G) \to \mathcal{P}^*([k+2])$ is given by

$$c'(v) = \begin{cases} c'_F(v) \cup \{k+2\} & \text{if } v \in V(F) \\ c'_H(v) \cup \{k+1, k+2\} & \text{if } v \in V(H). \end{cases}$$

Since c'_F and c'_H are vertex-distinguishing and $c'(x) \neq c'(y)$ if $x \in V(F)$ and $y \in V(H)$, it follows that c' is a vertex-distinguishing vertex coloring of G. Therefore, c is a strong royal $(k+2)$-edge coloring of $G = F \vee H$. □

Equality can be obtained for both bounds. Note that $\text{sreg}(P_3) = 3$ and $\text{sroy}(P_3 \vee P_3) = 4$ while $\text{sroy}(P_3) = 2$ and $\text{sroy}(P_3 \vee P_3) = 4$.

Let F and H be two graphs with disjoint vertex sets. The *Cartesian product* $G = F \square H$ of F and H has vertex set $V(G) = V(F) \times V(H)$, where two distinct vertices (u, v) and (x, y) of $F \square H$ are adjacent if either $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(F)$. In particular, the graph $G \square K_2$ consists of two copies G_1 and G_2 of G. If $V(G_1) = \{u_1, u_2, \ldots u_n\}$ where u_i is labeled v_i in G_2, then $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and

$$E(H) = E(G_1) \cup E(G_2) \cup \{u_i v_i : 1 \leq i \leq n\}.$$

If G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ for some integer k, then $G \square K_2$ is a connected graph of order $2n \geq 8$ where $2^k \leq 2n \leq 2^{k+1} - 2$. Therefore, $\text{sreg}(G \square K_2) \geq k + 1$ by Proposition 1. A bound was given for $\text{sroy}(G \square K_2)$ in terms of $\text{sroy}(G)$ in [4]. □

Proposition 6. [4] *If G is a connected graph of order at least* 3*, then*

$$\text{sroy}(G \square K_2) \leq \text{sroy}(G) + 1.$$

Here we extend this result to the strong regal indexes of graphs.

Proposition 7. *If G is a connected graph of order at least* 3*, then*

$$\text{sreg}(G \square K_2) \leq \text{sreg}(G) + 1.$$

Proof. Suppose that $\text{sreg}(G) = k$. Then it suffices to show that there exists a strong regal $(k+1)$-edge coloring of $G \square K_2$. By definition, $G \square K_2$ consists of two copies of G, say G_1 and G_2 as describe previously. There exist strong regal k-edge colorings $c_1 : E(G_1) \to \mathcal{P}^*([k])$ and $c_2 : E(G_2) \to \mathcal{P}^*([k])$ with vertex-distinguishing induced vertex colorings c'_1 and c'_2 respectively.

Define an edge coloring $c : E(G \square K_2) \to \mathcal{P}^*([k+1])$ by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E(G_1) \\ c_2(e) \cup \{k+1\} & \text{if } e \in E(G_2) \\ [k+1] & \text{otherwise.} \end{cases}$$

Then the induced coloring $c' : V(G \square K_2) \to \mathcal{P}^*([k+1])$ is given by

$$c'(v) = \begin{cases} c'_1(v) & \text{if } v \in V(G_1) \\ c'_2(v) \cup \{k+1\} & \text{if } v \in V(G_2). \end{cases}$$

Since c'_1 and c'_2 are vertex-distinguishing and $c'(x) \neq c'(y)$ if $x \in V(G_1)$ and $y \in V(G_2)$, it follows that c' is a vertex-distinguishing vertex coloring of $G \square K_2$. Therefore, c is a strong regal $(k+1)$-edge coloring of $G \square K_2$. □ □

3. Cubic Caterpillars

Since every connected graph contains a spanning tree, Proposition 2 suggests that studying these indexes for common classes of trees can aid in verifying Conjecture 1 for many graphs. For this reason, we consider a well-known class of trees. A *caterpillar* is a tree T of order 3 or more, the removal of whose leaves produces a path and a tree T is called *cubic* if every vertex of T that is not an end-vertex has degree 3. The strong royal index of cubic caterpillars has been determined.

Proposition 8. [4] *If T is a cubic caterpillar of order at least* 4*, then T is royal-zero.*

Here we determine the strong regal index for cubic caterpillars. We begin with a lemma. For integers a and b with $a < b$, let

$$[a, b] = \{a, a+1, \ldots, b\}.$$

Lemma 1. *For every integer $n \geq 4$ with $n \neq 7$ and $k = 1 + \lfloor \log_2 n \rfloor$, there exist a strong regal k-coloring c of P_n such that if $c'(u) = [k]$ for some vertex $u \in V(P_n)$, then u is a leaf of P_n.*

Proof. Let $k = 1 + \lfloor \log_2 n \rfloor$, where $n \geq 4$ and $n \neq 7$. Figure 1 shows that P_n has such a coloring for $n \in [4, 6] \cup [12, 15]$. Notice in each case that if $[k]$ is an induced color, it is induced on an end vertex. Observe that the induced vertex coloring of each path P_n in Figure 1 for $n \in [4, 5] \cup [12, 15]$ contains two adjacent vertices whose colors are disjoint. For an integer $n \in [4, 5] \cup [12, 15]$, let

$$H = (v_1, v_2, \dots, v_n)$$

be the path P_n of order n and let

$$H^\star = (v_n, v_{n-1}, \dots, v_1)$$

be the path P_n in reverse order. Let c_H be the edge coloring of H shown in Figure 1. Notice that for the colorings of P_4 and P_{13} of Figure 1, since $[k]$ (where $k = 3$ for P_4 and $k = 4$ for P_{13}) is not the induced vertex color of any vertex in each of these two colorings, it follows that both colorings satisfy the conditions in the statement vacuously.

$$
\begin{aligned}
\mathcal{S}_c(P_3) &= (12, 23)\\
\mathcal{S}_{c'}(P_3) &= (\mathbf{12}, \mathbf{2}, \mathbf{23},)\\[4pt]
\mathcal{S}_c(P_4) &= (12, 23, 13)\\
\mathcal{S}_{c'}(P_4) &= (\mathbf{12}, \underline{\mathbf{2}, \mathbf{3}}, \mathbf{13})\\[4pt]
\mathcal{S}_c(P_5) &= ([3], 12, 23, 13)\\
\mathcal{S}_{c'}(P_5) &= ([\mathbf{3}], \mathbf{12}, \underline{\mathbf{2}, \mathbf{3}}, \mathbf{13})\\[4pt]
\mathcal{S}_c(P_6) &= ([3], 23, 13, 13, 12)\\
\mathcal{S}_{c'}(P_6) &= ([\mathbf{3}], \mathbf{23}, \mathbf{3}, \mathbf{13}, \mathbf{1}, \mathbf{12})\\[4pt]
\mathcal{S}_c(P_{12}) &= ([4], 123, 124, 34, 134, [4], 23, 13, 134, 12, 24)\\
\mathcal{S}_{c'}(P_{12}) &= ([\mathbf{4}], \mathbf{123}, \underline{\mathbf{12}, \mathbf{4}}, \mathbf{34}, \mathbf{134}, \mathbf{23}, \mathbf{3}, \mathbf{13}, \mathbf{1}, \mathbf{2}, \mathbf{24})\\[4pt]
\mathcal{S}_c(P_{13}) &= (123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34)\\
\mathcal{S}_{c'}(P_{13}) &= (\mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34})\\[4pt]
\mathcal{S}_c(P_{14}) &= (123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34, 134)\\
\mathcal{S}_{c'}(P_{14}) &= (\mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34}, \mathbf{134})\\[4pt]
\mathcal{S}_c(P_{15}) &= ([4], 123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34, 134)\\
\mathcal{S}_{c'}(P_{15}) &= ([\mathbf{4}], \mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34}, \mathbf{134}).
\end{aligned}
$$

Figure 1. Showing that $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$ for $n \in [4, 6] \cup [12, 15]$

We now define a strong regal $(k+1)$-coloring $c_{H^\star} : E(H^\star) \to \mathcal{P}^*([k+1])$ of $H^\star$ by

$$c_{H^\star}(v_{i+1}v_i) = c_H(v_i v_{i+1}) \cup \{k+1\} \text{ for } 1 \le i \le n-1.$$

Let G be the path of order $2n$ obtained from H and $H^\star$ by joining the two vertices v_n in H and $H^\star$ by the edge f. The edge coloring $c_G : E(G) \to \mathcal{P}^*([k+1])$ is defined by

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ c_{H^\star}(e) & \text{if } e \in E(H^\star) \\ c_{H^\star}(v_n v_{n-1}) & \text{if } e = f. \end{cases}$$

The coloring c_G is illustrated in Figure 2 for $G = P_{10}$ when $n = 5$. Note that, in order for $[k+1]$ to be an induced color of a vertex in G, $[k]$ had to be an induced color of a vertex in H. By Figure 1, if such a vertex exists, it must be v_1, in which case $c_{H^\star}(v_1) = [k+1]$, which implies that $[k+1]$ is assigned to an end vertex of G. Since this edge coloring is a strong regal $(k+1)$-coloring of the path G of order $2n$, it follows that the desired result holds for order $n = 2\ell$, with $\ell \ge 2$.

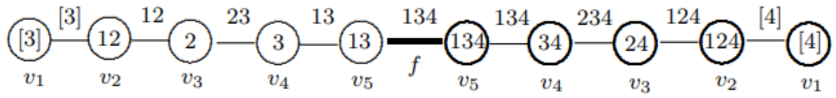

Figure 2. Constructing a Strong Regal 4-Coloring of P_{10}

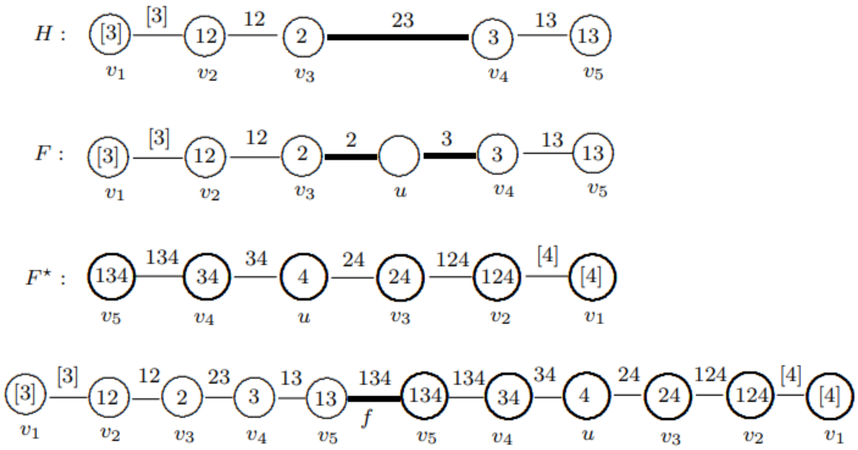

Figure 3. Constructing a Strong Regal 4-Coloring of P_{11}

Next, for each $n \in [4,5] \cup [12,15]$, let F be the path of $n+1$ obtained from H by subdividing an edge v_jv_{j+1} of H where $c'_H(v_j) \cap c'_H(v_{j+1}) = \emptyset$, obtaining the subpath (v_j, u, v_{j+1}). Define an edge coloring c_F of F by

$$c_F(e) = \begin{cases} c'_H(v_j) & \text{if } e = v_ju \\ c'_H(v_{j+1}) & \text{if } e = uv_{j+1} \\ c_H(e) & \text{if } e \neq v_ju, uv_{j+1}. \end{cases}$$

(The edge coloring c_F is not a regal edge coloring since $c'_F(u) = \emptyset$.) Let

$$F^\star = (v_n, v_{n-1}, \ldots, v_{j+1}, u, v_j, \ldots, v_1)$$

be the path F in reverse order. Define the edge coloring $c_{F^\star} : E(F^\star) \to \mathcal{P}^*([k+1])$ of $F^\star$ by

$$c_{F^\star}(e) = c_F(e) \cup \{k+1\} \text{ for each } e \in E(F^\star).$$

Then $c'_{F^\star}(v_i) = c'_H(v_i) \cup \{k+1\}$ for $1 \le i \le n$ and $c'_{F^\star}(u) = \{k+1\}$. In particular, $c'_{F^\star}(v_1) = [k+1]$. Since $c'_{F^\star}$ is vertex-distinguishing, it follows that $c_{F^\star}$ is a strong regal $(k+1)$-coloring of $F^\star$. The graphs H, F and $F^\star$ are shown in Figure 3 as well as the corresponding edge colorings.

Let G be the path of order $2n+1$ obtained from H and $F^\star$ by joining the vertex v_n in H and $F^\star$ by the edge f. The edge coloring $c_G : E(G) \to \mathcal{P}^*([k+1])$ is defined by

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ c_{F^\star}(e) & \text{if } e \in E(F^\star) \\ c_{F^\star}(v_nv_{n-1}) & \text{if } e = f. \end{cases}$$

The coloring c_G is illustrated in Figure 3 for $G = P_{11}$ when $n = 5$. This edge coloring is a strong regal $(k+1)$-coloring of the path G of order $2n+1$, where if $[k+1]$ appears as an induced color, it is induced on an end vertex.

The colorings defined above show, in particular, that if $n \in [4,31]$ where $n \neq 7$, then P_n has a strong regal k- coloring with $k = 1 + \lfloor \log_2 n \rfloor$ such that if $[k]$ appears as an induced color, it is induced on an end vertex. Furthermore, the induced vertex coloring of each such path P_n where $n \in [16,31]$ has the property that there exist two adjacent vertices whose colors are disjoint and $[5]$ is induced on an end vertex if it appears in the induced coloring. By proceeding as above, we see that if $n \in [32,63]$, then there is a strong regal $1 + \lfloor \log_2 n \rfloor$ coloring of P_n such that the induced vertex coloring of each such path has the property that there exist two adjacent vertices whose colors are disjoint and [6] is induced on an end vertex if it appears in the induced coloring. Consequently, for each integer $\ell \geq 2$ and each integer $n \in [2^\ell, 2^{\ell+1} - 1]$ except $n = 7$ the result holds. That is, for each integer $n \geq 4$ and $n \neq 7$, there exist a strong regal k-coloring of P_n, say c, with such that if $c'(u) = [k]$ for some vertex $u \in V(P_n)$, then u is a leaf of P_n . □ □

Proposition 9. *If T is a cubic caterpillar of order $n \geq 4$, then T is regal-zero, that is,* $\text{sreg}(T) = 1 + \lfloor \log_2 n \rfloor$.

Proof. By Proposition 1, it suffices to show that a strong regal k-coloring exists, where $k = 1+\lfloor \log_2 n \rfloor$. If $n = 4$, then T is a star and the result holds. Since every vertex in a cubic caterpillar has odd degree, n must be even. Assume $n = 2\ell$ where $\ell \geq 3$. Let $P_s = (v_1, v_2, v_3, \ldots, v_s)$ be the shortest path in T such that every vertex in T has distance at most one from a vertex in P_s. This implies that v_1 and v_s are each adjacent to at least one other vertex not in P_s, say v_0 and v_{s+1} respectively, for otherwise P_s would not be a minimum path with the designated property. Since each vertex $v \in V(P_s)$ has $\deg_T(v) \geq 2$, each vertex in P_s must have exactly 3 neighbors, so each vertex v_i in P_s is adjacent to another vertex u_i not in P_s, for $1 \leq i \leq s$. Note that since T is a caterpillar, $\deg(u_i) = 1$ for $1 \leq i \leq s$. This implies the order of T is $2s + 2$. Let $P = (v_0, v_1, v_2, \ldots, v_s)$. Since the order of T is even, $2^{k-1} \leq n \leq 2^k - 2$. If follows that $2^{k-2} \leq \frac{n}{2} \leq 2^{k-1} - 1$. Observe that the order of P is $s + 1 = \frac{n}{2}$. Then, by Lemma 1 it follows that P has a strong regal $k - 1$-coloring, say c_P, and we may assume that

$$c'_P(v_0) = \{1, 2, \ldots, k - 1\}.$$

Define a coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} c_P(e) & \text{if } e \in E(P) \\ c'_P(v_i) \cup \{k\} & \text{if } e = u_i v_i,\ 1 \leq i \leq s \\ [\text{k}] & \text{if } e = v_s v_{s+1}. \end{cases}$$

Then, the induced coloring $c' : V(T) \to \mathcal{P}^*([k])$ is given by

$$c'(v) = \begin{cases} c'_P(v) & \text{if } v \in V(P) \\ c'_P(v_i) \cup \{k\} & \text{if } v = u_i,\ 1 \leq i \leq s \\ [\text{k}] & \text{if } v = v_{s+1}. \end{cases}$$

It follows that c' is neighbor distinguishing since no neighbor of a vertex u_i is assigned the color $\{1, 2, \ldots, k - 1\}$ by c'_P. Thus, c is a strong regal k coloring of T and so $\text{sreg}(T) = 1 + \lfloor \log_2 n \rfloor$. □

4. Royal Chromatic Number of a Graph

For a connected graph G of order 3 or more with $\text{roy}(G) = k$, let c be a royal k-edge coloring of G. Recall that this means we only require the induced vertex coloring c' to be proper. If the induce vertex coloring c' is rainbow, then the number of vertex colors used in maximum; while if the vertex coloring c' is proper, the number of vertex colors used may not be minimum. Define the *royal chromatic number* of a connected graph G of order 3 or more with royal index k to be the minimum possible number of vertex colors used in a royal k-edge coloring G. This parameter is denoted by $\chi_{\text{sroy}}(G)$. A *majestic k-edge coloring* is a royal k-edge coloring with the added restriction that each edge color is a singleton. The majestic index $maj(G)$ of a graph G is the minimum value of k such that G has a majestic k-edge coloring. Similarly, the majestic chromatic number of a connected graph G of order 3 or more with majestic index k is the minimum possible number of vertex colors used in a majestic k-edge coloring G. This value is denoted by $\chi_{\text{maj}}(G)$. Majestic colorings were introduced by Györi et al. in [5] and the majestic chromatic number was introduced in [6].

Proposition 10. *If G is a connected graph of order $n \geq 3$ with* $\text{roy}(G) = k$*, then*

1. $\max\{3, \chi(G)\} \leq \chi_{\text{roy}}(G) \leq n$,
2. $\text{roy}(G) \geq \lceil \log_2(\chi_{\text{roy}}(G) + 1) \rceil$,
3. *If* $\text{maj}(G) = \text{roy}(G) = k$*, then* $\chi_{\text{roy}}(G) \leq \chi_{\text{maj}}(G)$.

Proof. Consider a royal coloring c of G. Since the induced vertex coloring c' must be proper, it must use at least $\chi(G)$ colors, however no vertex coloring can use more colors than the number of vertices in the graph. Consequently, $\chi(G) \leq \chi_{\text{roy}}(G) \leq n$. Next, we obtain that $\chi_{\text{roy}}(G) \geq 3$. Since G is nonempty, $\chi(G) \geq 2$, so $\chi_{\text{roy}}(G) \geq 2$. It suffices to show that $\chi_{\text{roy}}(G) \neq 2$. Assume, to the contrary, that $\chi_{\text{roy}}(G) = 2$. Then, there exists a royal k-edge coloring c of G such that the induced vertex coloring c' uses exactly two distinct vertex colors, say sets A and B. Let u be a vertex in G such that $c'(u) = A$. Then, for any vertex $w \in N(u)$, $c'(w) = B$, since c' is proper. It follows that $c(e) \subseteq B$ for every edge incident to u, since e is incident to a neighbor of u. This implies that $A \subseteq B$. Considering a symmetric argument, $B \subseteq A$, which demonstrates that $A = B$, which is impossible because of our assumption that A and B were distinct colors. Next, a royal k-edge coloring of G can use at most $2^k - 1$ vertex colors, given $|\mathcal{P}^*([k])| = 2^k - 1$, which implies that $\chi_{\text{roy}}\,\text{roy}(G) \leq 2^k - 1$ for any graph G with $\text{roy}(G) = k$. It follows that $\text{roy}(G) \geq \lceil \log_2(\chi_{\text{roy}}(G) + 1) \rceil$. Lastly, suppose that $\text{maj}(G) = \text{roy}(G) = k$ and let $\chi_{\text{maj}}(G) = \ell$. Then, there exists a majestic k-edge coloring of G that uses exactly ℓ distinct vertex colors. However, every majestic coloring is also a royal coloring. Thus, $\chi_{\text{roy}}(G) \leq \ell$. □Note that $\chi_{\text{roy}}(K_n) = n$, which demonstrates that the upper bound on the royal chromatic number given in Proposition 10 is sharp. Recall an observation involving the majestic chromatic number. □

Observation 5. [6] *If G is a connected graph with* $\text{maj}(G) = 2$*, then*

$$\chi_{\text{maj}}(G) = 3.$$

The connection between majestic and royal colorings leads to the following result.

Proposition 11. *If* $\text{roy}(G) = 2$ *for a connected graph G with order $n \geq 3$, then* $\text{maj}(G) = \text{roy}(G) = 2$ *and* $\chi_{\text{roy}}(G) = \chi_{\text{maj}}(G) = 3$.

Proof. Let G be a graph with $\text{roy}(G) = 2$. Let $c : E(G) \to \mathcal{P}^*([2])$ be a royal 2-edge coloring of G. Then, c induces a proper vertex coloring $c' : V(G) \to \mathcal{P}^*([2])$. Observe that $c(e)$ must be a singleton set for all $e \in E(G)$. If there was an edge $e = uv$, such that $c(e) = \{1, 2\}$, then it would follow that $c'(u) = c'(v)$, which is impossible since c' is proper. Since c assigns only singleton sets to the edges of G, it can also be viewed as a majestic 2-edge coloring of G, which implies $\text{maj}(G) = 2$. By Observation 5, $\chi_{\text{maj}}(G) = 3$. Applying Proposition 10 yields the following string of inequalities:

$$3 \leq \chi_{\text{roy}}(G) \leq \chi_{\text{maj}}(G) = 3.$$

Hence $\chi_{\text{roy}}(G) = 3$. □ □

Next, we determine the royal chromatic number of paths and cycles. The majestic chromatic number of odd length paths of order $n \neq 6$ is 4. The royal chromatic number for paths has been determined and differs from the majestic chromatic number for paths of even order.

Proposition 12. *If $n = 4$, then* $\chi_{\text{roy}}(P_n) = 4$.

Proof. First note that in [6], it is shown that $\text{maj}(P_4) = 3$. Also, $\text{roy}(G) \leq \text{maj}(G)$. By Proposition 5, it follows that $\text{roy}(P_4) = 3$. Since $\chi_{\text{roy}}(P_4) \geq 3$ and $\text{roy}(P_4) = 3$, we must show no royal 3-edge coloring exists that uses only 3 induced vertex colors. Assume, to the contrary, that there is a royal 3-edge coloring $c : E(P_4) \to \mathcal{P}^*([3])$ such that the induced vertex coloring $c' : V(P_4) \to \mathcal{P}^*([3])$ uses only 3 colors. Let $P_4 = (u_1, u_2, u_3, u_4)$. In order to properly color P_4 using less than 4 colors, two vertices must be assigned the same color by c'. If $c'(u_1) = c'(u_4) = A$, then $c(u_1u_2) = A$ and $c(u_3u_4) = A$. Thus, for any set $c(u_2u_3)$, $c'(u_2) = c'(u_3)$, a contradiction. We may assume $c'(u_1) = c'(u_3) = A$, with $A \subseteq [3]$. Since c' is proper, we must have that $c'(u_2) = B$, with $B \subseteq [3]$ and $A \neq B$. It follows that $c(u_1u_2) = A$, $c(u_1u_2) \subseteq B$ and $c(u_2u_3) \subseteq A$, so $c'(u_2) = B \subseteq A$ and $c'(u_1) = A \subseteq B$. However, this implies that $A = B$, which is impossible, since c' is proper. Define a royal 3-edge coloring c of P_4 by the sequence $s = (\{1\}, \{2\}, \{3\})$. Then, the induced vertex coloring c' is given by the color sequence $s' = (\{1\}, \{1, 2\}, \{2, 3\}, \{3\})$, which is a proper vertex coloring of C_5 using exactly 4 colors. □ □

Theorem 6. *If P_n is a path of order $n \geq 3$ with $n \neq 4$, then*

$$\chi_{\mathrm{roy}}(P_n) = 3.$$

Proof. By Proposition 10, $\chi_{\mathrm{roy}}(P_n) \geq 3$. It suffices to show that a royal k-edge coloring of P_n exists using exactly 3 vertex colors, where $k = \mathrm{roy}(P_n)$. We consider two cases *Case* 1. *n is odd.* If n is odd, then $\mathrm{roy}(P_n) = 2$, since $\mathrm{maj}(P_n) = 2$ when n is odd (see [6]). The desired result follows from Proposition 11. *Case* 2. *n is even.* Since $\mathrm{maj}(P_n) = 3$ for even n (see [6]), it follows that $\mathrm{roy}(P_n) = 3$ by Proposition 11. Let $P_n = (u_1, u_2, \ldots, u_n)$. Define a royal 3-edge coloring $c : E(G) \to \mathcal{P}^*([3])$ of P_n by

$$c(u_iu_{i+1}) = \begin{cases} 1 & \text{if } i = 3 \\ 2 & \text{if } i = 4 \\ 3 & \text{if } i = 2 \\ \{1,2\} & \text{if } i = 1 \text{ or } i \equiv 0, 3 \text{ (mod 4 for } 7 \leq i \leq n-1 \\ \{1,3\} & \text{if } i \equiv 1, 2 \text{ (mod 4 for } 5 \leq i \leq n-1. \end{cases}$$

Then, the induced coloring $c' : V(P_n) \to \mathcal{P}^*([3])$ is given by

$$c'(u_i) = \begin{cases} \{1,2\} & \text{if } i = 1 \text{ or } i \equiv 0 \text{ (mod 4 for } 4 \leq i \leq n \\ \{1,3\} & \text{if } i = 3 \text{ or } i \equiv 2 \text{ (mod 4 for } 6 \leq i \leq n \\ [3] & \text{if } i = 2 \text{ or } i \equiv 1, 3 \text{ (mod 4 for } 5 \leq i \leq n-1. \end{cases}$$

Since c' is a proper 3-coloring of G, it follows that $\chi_{\mathrm{roy}}(P_n) = 3$ for $n \geq 3$, with $n \neq 4$. □For a cycle C_n of order $n \geq 3$, $\chi_{\mathrm{maj}}(C_n) = 3$ if and only if $n \equiv 0$ (mod 4 or $n \equiv 0$ (mod 3 by results in [6]. Otherwise $\chi_{\mathrm{maj}}(C_n) \geq 4$. As with paths, the royal chromatic number of cycles contrasts that of the majestic chromatic number by achieving the lower bound given in Proposition 10 in all but one case. □

Proposition 13. *If $n = 5$, then $\chi_{\mathrm{roy}}(C_n) = 4$.*

Proof. First, $\mathrm{maj}(C_5) = 3$ (see [6]). By Proposition 11, it follows that $\mathrm{roy}(C_5) = 3$. Since $\chi_{\mathrm{roy}}(C_5) \geq 3$ and $\mathrm{roy}(C_5) = 3$, we must show no royal 3-edge coloring exists that uses only 3 induced vertex colors and exhibit a royal 3-edge coloring that uses exactly 4 induced vertex colors. First, assume to the contrary that a royal 3-edge coloring $c : E(C_5) \to \mathcal{P}^*([3])$ exists such that the induced vertex coloring $c' : V(C_5) \to \mathcal{P}^*([3])$ uses only 3 colors. Let $C_5 = (u_0, u_1, u_2, u_3, u_4, u_5 = u_0)$. Observe that a proper coloring of C_5 may assign the same color to at most two vertices in C_5. It follows that two pairs of vertices in C_5 must be assigned the same color by c'. We may assume that $c'(u_0) = c'(u_2) = A$, for some $A \subseteq [3]$. Since c' is proper $c'(u_3) \neq c'(u_4)$, so either $c'(u_1) = c'(u_3)$ or $c'(u_1) = c'(u_4)$. Without loss of generality, let $c'(u_1) = c'(u_3) = B$, with $B \subseteq [3]$. Then, $A \neq B$. However, since u_0u_1 and u_1u_2 are both incident to a vertex colored A, $c(u_0u_1) \subseteq A$ and $c(u_1u_2) \subseteq A$. These are the only edges incident to u_1, hence $c'(u_1) = B \subseteq A$. Similarly, since u_1u_2 and u_2u_3 are both incident to a vertex colored B, $c(u_1u_2) \subseteq B$ and $c(u_2u_3) \subseteq B$. These are the only edges incident to u_2, hence $c'(u_2) = A \subseteq B$. This implies that $A = B$, which is impossible since c' must be proper.

Now, define a royal 3-edge coloring c of C_5 by the sequence

$$s = (\{1\}, \{1\}, \{1,2\}, \{3\}, \{2\}).$$

Then, the induced vertex coloring c' is given by the color sequence

$$s' = (\{1,2\}, \{1\}, \{1,2\}, [3], \{2,3\}),$$

which is a proper vertex coloring of C_5 using exactly 4 colors. □

Theorem 7. *If C_n is a cycle of order $n \geq 3$ with $n \neq 5$, then*

$$\chi_{\text{roy}}(C_n) = 3.$$

Proof. By Proposition 10, $\chi_{\text{roy}}(C_n) \geq 3$. It suffices to show that there exists a royal k-edge coloring C_n using exactly 3 vertex colors, where $k = \text{roy}(C_n)$. Let $C_n = (u_0, u_2, \ldots, u_n = u_0)$. We consider four cases. *Case* 1. $n \equiv 0$ (mod 4. Recall that if $n \equiv 0$ (mod 4, then $\text{maj}(G) = 2$ and so $\text{roy}(G) = 2$ by Proposition 11 (see [6]). It follows that from Proposition 5 that $\chi_{\text{roy}}(C_n) = 3$ for $n \equiv 0$ (mod 4. *Case* 2. $n \equiv 1$ (mod 4. Then $\text{maj}(C_n) = 3$ and so $\text{roy}(C_n) = 3$ by Proposition 11 (see [6]). For the case of $n = 9$, let a royal 3-edge coloring be given by the color sequence

$$s = (\{1\}, \{3\}, \{1,2\}, \{1\}, \{3\}, \{1,2\}, \{1\}, \{3\}, \{1,2\}).$$

The vertex color sequence induced by s is given by

$$s' = (\{1,2\}, \{1,3\}, [3], \{1,2\}, \{1,3\}, [3], \{1,2\}, \{1,3\}, [3]),$$

as desired. We may assume $n \geq 13$. Define a royal 3-edge coloring $c : E(C_n) \to \mathcal{P}^*([3])$ of C_n by

$$c(u_iu_{i+1}) = \begin{cases} \{1\} & \text{if } i \in \{0,3,6\} \\ \{3\} & \text{if } i \in \{1,4,7\} \\ \{1,2\} & \text{if } i \in \{2,5\} \text{ or } i \equiv 0,1 \text{ (mod 4 for } 8 \leq i \leq n-1 \\ \{1,3\} & \text{if } i \equiv 3,4 \text{ (mod 4 for } 10 \leq i \leq n-2. \end{cases}$$

Then, the induced coloring $c' : V(C_n) \to \mathcal{P}^*([3])$ is given by

$$c'(u_i) = \begin{cases} \{1,2\} & \text{if } i \in \{0,3,6\} \text{ or } i \equiv 1 \text{ (mod 4 for } 9 \leq i \leq n-4 \\ \{1,3\} & \text{if } i \in \{1,4\} \text{ or } i \equiv 3 \text{ (mod 4 for } 7 \leq i \leq n-2 \\ [3] & \text{if } i \in \{2,5\} \text{ or } i \equiv 0,2 \text{ (mod 4 for } 8 \leq i \leq n-1. \end{cases}$$

Since c' is a proper 3-coloring of C_n, it follows that $\chi_{\text{roy}}(P_n) = 3$ for $n \equiv 1$ (mod 4, with $n \geq 9$. *Case* 3. $n \equiv 2$ (mod 4. Then $\text{maj}(C_n) = 3$ and so $\text{roy}(C_n) = 3$ by Proposition 11 (see [6]). We may assume $n \geq 6$. Define a royal 3-edge coloring $c : E(C_n) \to \mathcal{P}^*([3])$ of C_n by

$$c(u_iu_{i+1}) = \begin{cases} \{1\} & \text{if } i \in \{0,3\} \\ \{3\} & \text{if } i = 1 \\ \{1,2\} & \text{if } i = 2 \text{ or } i \equiv 1,2 \text{ (mod 4 for } 5 \leq i \leq n-1 \\ \{1,3\} & \text{if } i \equiv 0,3 \text{ (mod 4 for } 4 \leq i \leq n-2. \end{cases}$$

Then, the induced coloring $c' : V(C_n) \to \mathcal{P}^*([3])$ is given by

$$c'(u_i) = \begin{cases} \{1,2\} & \text{if } i \in \{0,3\} \text{ or } i \equiv 2 \text{ (mod 4 for } 6 \leq i \leq n-4 \\ \{1,3\} & \text{if } i = 1 \text{ or } i \equiv 0 \text{ (mod 4 for } 4 \leq i \leq n-2 \\ [3] & \text{if } i \in \{2,5\} \text{ or } i \equiv 0,2 \text{ (mod 4 for } 5 \leq i \leq n-1. \end{cases}$$

Since c' is a proper 3-coloring of C_n, it follows that $\chi_{\rm roy}(C_n) = 3$ for $n \equiv 2$ (mod 4, with $n \geq 6$. *Case* 4. $n \equiv 3$ (mod 4. Then $\text{maj}(C_n) = 3$ and so $\text{roy}(C_n) = 3$ by Proposition 11 (see [6]). For the case of $n = 3$, let a royal 3-edge coloring be given by the color sequence $s = (\{1\}, \{3\}, \{1, 2\})$. The vertex color sequence induced by s is given by $s' = (\{1, 2\}, \{1, 3\}, [3])$, as desired. We may assume $n \geq 7$. Define a royal 3-edge coloring $c : E(C_n) \to \mathcal{P}^*([3])$ of C_n by

$$c(u_iu_{i+1}) = \begin{cases} \{1\} & \text{if } i = 0 \\ \{3\} & \text{if } i = 1 \\ \{1,2\} & \text{if } i \equiv 2, 3 \text{ (mod 4 for } 2 \leq i \leq n-1 \\ \{1,3\} & \text{if } i \equiv 0, 1 \text{ (mod 44 for } 4 \leq i \leq n-2. \end{cases}$$

Then the induced coloring $c' : V(C_n) \to \mathcal{P}^*([3])$ is given by

$$c'(u_i) = \begin{cases} \{1,2\} & \text{if } i = 0 \text{ or } i \equiv 3 \text{ (mod 4 for } 3 \leq i \leq n-4 \\ \{1,3\} & \text{if or } i \equiv 1 \text{ (mod 4 for } 1 \leq i \leq n-2 \\ [3] & \text{if } i \equiv 0, 2 \text{ (mod 4 for } 2 \leq i \leq n-1. \end{cases}$$

Since c' is a proper 3-coloring of C_n, it follows that $\chi_{\rm roy}(C_n) = 3$ for $n \equiv 3$ (mod 4. □

5. Problems for Further Study

We have seen in Theorem 4 that if G is a connected graph of order at least 3, then $|\,\text{sroy}(G) - \text{sreg}(G)| \leq 1$. This gives rise to the following question.

Problem 1. *Let G be a connected graph of order at least* 3. *What conditions must G possess to guarantee that* $\text{sroy}(G) = \text{sreg}(G)$*?*

By Conjecture 1, if G is a connected graph of order $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$, then $\text{sroy}(G) \in \{k, k+1\}$ and $\text{sreg}(G) \in \{k, k+1\}$. There is another natural question here.

Problem 2. *Let G be a connected graph of order* $n \geq 3$ *where* $2^{k-1} \leq n \leq 2^k - 1$. *Is there a necessary and sufficient condition for which* $\text{sroy}(G) = k$*? Similarly, is there a necessary and sufficient condition for which* $\text{sreg}(G) = k$*?*

We saw that if G is a connected graph of order $n \geq 4$, then $\text{sroy}(\text{cor}(G)) \leq \text{sroy}(G) + 1$ and $\text{sreg}(\text{cor}(G)) \leq \text{sreg}(G) + 1$. There is a related question here.

Problem 3. *If G be a connected graph of order at least* 4*, is it true that* $\text{sroy}(\text{cor}(G)) \geq \text{sroy}(G)$ *and* $\text{sreg}(\text{cor}(G)) \geq \text{sreg}(G)$*?*

By Propositions 6 and 7, if G is a connected graph of order at least 3, then $\text{sroy}(G \,\square\, K_2) \leq \text{sroy}(G) + 1$ and $\text{sreg}(G \,\square\, K_2) \leq \text{sreg}(G) + 1$. There is also a related question here.

Problem 4. *If G be a connected graph of order at least* 3*, is it true that* $\text{sroy}(G \,\square\, K_2) \geq \text{sroy}(G)$ *and* $\text{sreg}(G \,\square\, K_2) \geq \text{sreg}(G)$*?*

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Conflict of Interest

The authors declare no conflict of interest.

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