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Article

On Vertex-Disjoint Chorded Cycles and Degree Sum Conditions

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Abstract: In this paper, we consider a degree sum condition sufficient to imply the existence of k vertex-disjoint chorded cycles in a graph G. Let $\sigma_4(G)$ be the minimum degree sum of four independent vertices of G. We prove that if G is a graph of order at least $11k + 7$ and $\sigma_4(G) \geq 12k - 3$ with $k \geq 1$, then G contains k vertex-disjoint chorded cycles. We also show that the degree sum condition on $\sigma_4(G)$ is sharp.

Keywords: Vertex-disjoint chorded cycles, Minimum degree sum, Degree sequence

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1. Introduction

The study of cycles in graphs is a rich and an important area. One question of particular interest is to find conditions that guarantee the existence of k vertex-disjoint cycles. Corrádi and Hajnal [1] first considered a minimum degree condition to imply a graph must contain k vertex-disjoint cycles, proving that if $|G| \geq 3k$ and the minimum degree $\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles. For an integer $t \geq 1$, let

$$\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \ : \ X \text{ is an independent vertex setof } G \text{ with } |X| = t. \right\},$$

and $\sigma_t(G) = \infty$ when the independence number $\alpha(G) < t$. Enomoto [2] and Wang [3] independently extended the Corrádi and Hajnal result, requiring a weaker condition on the minimum degree sum of any two non-adjacent vertices. They proved that if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k vertex-disjoint cycles. In 2006, Fujita et al. [4] proved that if $|G| \geq 3k + 2$ and $\sigma_3(G) \geq 6k - 2$, then G contains k vertex-disjoint cycles, and in [5], this result was extended to $\sigma_4(G) \geq 8k - 3$.

An extension of the study of vertex-disjoint cycles is that of vertex-disjoint chorded cycles. A *chord* of a cycle is an edge between two non-adjacent vertices of the cycle. We say a cycle is *chorded* if it contains at least one chord. In 2008, Finkel proved the following result on the existence of k vertex-disjoint chorded cycles.

Theorem 1. (Finkel [6]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles.*

In 2010, Chiba et al. proved Theorem 2. Since $\sigma_2(G) \geq 2\delta(G)$, Theorem 2 is stronger than Theorem 1.

Theorem 2 (Chiba, Fujita, Gao, Li [7]). *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\sigma_2(G) \geq 6k-1$, then G contains k vertex-disjoint chorded cycles.*

Recently, Theorem 2 was extended as follows. Since $\sigma_3(G) \geq 3\sigma_2(G)/2$, when the order of G is sufficiently large, Theorem 3 is stronger than Theorem 2.

Theorem 3 (Gould, Hirohata, Rorabaugh [8]). *Let $k \geq 1$ be an integer. If G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$, then G contains k vertex-disjoint chorded cycles.*

Remark 1. We note if $k = 1$ in Theorem 3, then Theorem 3 holds under the condition that $|G| \geq 7$.

In this paper, we consider a similar extension for chorded cycles, as, in [5], the existence of k vertex-disjoint cycles was proved under the condition $\sigma_4(G)$. In particular, we first show the following.

Theorem 4. *If G is a graph of order at least 15 and $\sigma_4(G) \geq 9$, then G contains a chorded cycle.*

Remark 2. We consider the following graph G of order 14. (See Fig. 1.) Then G satisfies the $\sigma_4(G)$ condition in Theorem 4. However, G does not contain a chorded cycle. Thus $|G| \geq 15$ is necessary.

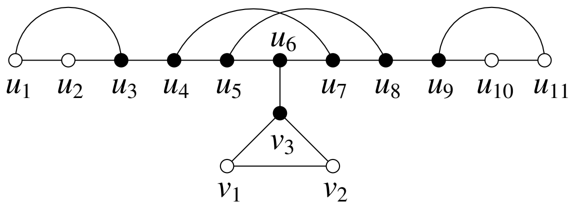

Figure 1. The Graph G of Order 14. The White Vertex (∘) Shows Degree 2, and the Black Vertex (•) Shows Degree 3

Theorem 5. *Let $k \geq 1$ be an integer. If G is a graph of order $n \geq 11k+7$ and $\sigma_4(G) \geq 12k-3$, then G contains k vertex-disjoint chorded cycles.*

Remark 3. Theorem 5 is sharp with respect to the degree sum condition. Consider the complete bipartite graph $G = K_{3k-1,n-3k+1}$, where large $n = |G|$. Then $\sigma_4(G) = 4(3k-1) = 12k-4$. However, G does not contain k vertex-disjoint chorded cycles, since any chorded cycle must contain at least three vertices from each partite set, in particular, from the $3k-1$ partite set. Thus $\sigma_4(G) \geq 12k-3$ is necessary.

For other related results on vertex-disjoint chorded cycles in graphs and bipartite graphs, we refer the reader to see [9–12].

Let G be a graph, H a subgraph of G and $X \subseteq V(G)$. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$. For $u \in V(G)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. Also we denote $d_H(X) = \sum_{u \in X} d_H(u)$. If $H = G$, then $d_G(X) = d_H(X)$. Furthermore, $N_G(X) = \cup_{u \in X} N_G(u)$ and $N_H(X) = N_G(X) \cap V(H)$. Let A, B be two vertex-disjoint subgraphs of G. Then $N_G(A) = N_G(V(A))$ and $N_B(A) = N_G(A) \cap V(B)$. The subgraph of G induced by X is denoted by $\langle X \rangle$. Let $G - X = \langle V(G) - X \rangle$ and $G - H = \langle V(G) - V(H) \rangle$. If $X = \{x\}$, then we write $G - x$ for $G - X$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For two disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the union of G_1 and G_2. Let Q be a path or a cycle with a given orientation and $x \in V(Q)$. Then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given direction. The reverse sequence of $Q[x, y]$ is denoted by $Q^-[y, x]$.

We also write $Q(x,y] = Q[x^+, y]$, $Q[x,y) = Q[x,y^-]$ and $Q(x,y) = Q[x^+, y^-]$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t(, x_1)$, then we assume an orientation of Q is given from x_1 to x_t (if Q is a cycle, then the orientation is clockwise). If P is a path connecting x and y of $V(G)$, then we denote the path P as $P[x,y]$. If G is one vertex, that is, $V(G) = \{x\}$, then we simply write x instead of G. For an integer $r \geq 1$ and two vertex-disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In this paper, since it is sufficient to consider the case of equality in the above inequality, when we write $(d_1, d_2, \ldots, d_r)$, we assume $d_B(v_i) = d_i$ for each $1 \leq i \leq r$. For two disjoint $X, Y \subseteq V(G)$, $E(X,Y)$ denotes the set of edges of G connecting a vertex in X and a vertex in Y. For a graph G, $comp(G)$ is the number of components of G. A cycle of length ℓ is called a *ℓ-cycle*. For terminology and notation not defined here, see [13].

2. Preliminaries

Definition 1. *Suppose $C_1, \ldots, C_r$ are r vertex-disjoint chorded cycles in a graph G. We say $\{C_1, \ldots, C_r\}$ is* minimal *if G does not contain r vertex-disjoint chorded cycles $C'_1, \ldots, C'_r$ such that $| \cup_{i=1}^{r} V(C'_i)| < | \cup_{i=1}^{r} V(C_i)|$.*

Definition 2. *Let $C = v_1, \ldots, v_t, v_1$ be a cycle with chord v_iv_j, $i < j$. We say a chord $vv' \neq v_iv_j$ is* parallel *to v_iv_j if either $v, v' \in C[v_i, v_j]$ or $v, v' \in C[v_j, v_i]$. Note if two distinct chords share an endpoint, then they are parallel. We say two distinct chords are* crossing *if they are not parallel.*

Definition 3. *Let u_iv_j and $u_\ell v_m$ be two distinct edges between two vertex-disjoint paths $P_1 = u_1, \ldots, u_s$ and $P_2 = v_1, \ldots, v_t$. We say u_iv_j and $u_\ell v_m$ are* parallel *if either $i \leq \ell$ and $j \leq m$, or $\ell \leq i$ and $m \leq j$. Note if two distinct edges between P_1 and P_2 share an endpoint, then they are parallel. We say two distinct edges between two vertex-disjoint paths are* crossing *if they are not parallel.*

Definition 4. *Let v_iv_j and $v_\ell v_m$ be two distinct edges between vertices of a path $P = v_1, \ldots, v_t$, with $j \geq i+2$ and $m \geq \ell+2$. We say v_iv_j and $v_\ell v_m$ are* nested *if either $i \leq \ell < m \leq j$ or $\ell \leq i < j \leq m$.*

Definition 5. *Let $P = v_1, \ldots, v_t$ be a path. We say a vertex v_i on P has a* left edge *if there exists an edge v_iv_j for some $j < i-1$, that is not an edge of the path. We also say v_i has a* right edge *if there exists an edge v_iv_j for some $j > i+1$, that is not an edge of the path.*

3. Lemmas

The following lemmas will be needed.

Lemma 1 ([8]). *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if the C_i has two chords, then these chords must be crossing.*

Lemma 2 ([8]). *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. Then $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \cup_{i=1}^{r} V(C_i)$. Furthermore, for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^{r} V(C_i)$, if $d_C(x) = 4$, then $|C| = 4$, and if $d_C(x) = 3$, then $|C| \leq 6$.*

Lemma 3 ([8]). *Suppose there exist at least three mutually parallel edges or at least three mutually crossing edges connecting two vertex-disjoint paths P_1 and P_2. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$.*

Lemma 4 ([8]). *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2 with $|P_1 \cup P_2| \geq 7$. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$ not containing at least one vertex of $\langle P_1 \cup P_2 \rangle$.*

Lemma 5 ([8]). *Let P_1, P_2 be two vertex-disjoint paths, and let u_1, u_2 ($u_1 \neq u_2$) be in that order on P_1. Suppose $d_{P_2}(u_i) \geq 2$ for each $i \in \{1, 2\}$. Then there exists a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2 \rangle$.*

Lemma 6 ([8]). *Let H be a graph containing a path $P = v_1, \ldots, v_t$ ($t \geq 3$), and not containing a chorded cycle. If $v_1 v_i \in E(H)$ for some $i \geq 3$, then $d_P(v_j) \leq 3$ for any $j \leq i-1$ and in particular, $d_P(v_{i-1}) = 2$. And if $v_t v_i \in E(H)$ for some $i \leq t-2$, then $d_P(v_j) \leq 3$ for any $j \geq i+1$ and in particular, $d_P(v_{i+1}) = 2$.*

Lemma 7 ([8]). *Let H be a graph containing a path $P = v_1, \ldots, v_t$ ($t \geq 6$), and not containing a chorded cycle. If $d_P(v_1) = 1$, then $d_P(v_i) = 2$ for some $3 \leq i \leq 5$, and if $v_1 v_3 \in E(H)$, then $d_P(v_i) = 2$ for some $4 \leq i \leq 6$.*

Lemma 8 ([8]). *Let H be a graph containing a path $P = v_1, \ldots, v_t$ ($t \geq 6$), and not containing a chorded cycle. If $d_P(v_t) = 1$, then $d_P(v_i) = 2$ for some $t-4 \leq i \leq t-2$, and if $v_t v_{t-2} \in E(H)$, then $d_P(v_i) = 2$ for some $t-5 \leq i \leq t-3$.*

Lemma 9. *Let H be a connected graph of order at least 6. Suppose H contains neither a chorded cycle nor a Hamiltonian path. Let $H = \langle P_1 \cup P_2 \rangle$, where $P_1 = u_1, \ldots, u_s$ ($s \geq 5$) is a longest path in H and $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) is a longest path in $H - P_1$. If $u_i \in V(P_1)$ for some $2 \leq i \leq s-3$ is adjacent to an endpoint v of P_2 and $u_j \in V(P_1)$ for some $i+2 \leq j \leq s-1$ is adjacent to an endpoint v' of P_2 (possibly, $v = v'$), then $d_H(u_\ell) = 2$ for some $\ell \in \{i+1, j-1\}$.*

Proof. Let v, v' be as in the lemma, and we may assume $v = v_1$ and $v' = v_t$ (possibly, $v = v'$). Suppose $d_H(u_\ell) \geq 3$ for each $\ell \in \{i+1, j-1\}$. If u_{i+1} has a left edge, say $u_{i+1}u_h$ with $h < i$, then

$$P_1[u_h, u_i], v_1, P_2[v_1, v_t], u_j, P_1^-[u_j, u_{i+1}], u_h$$

is a cycle with chord $u_i u_{i+1}$, a contradiction. By symmetry, u_{j-1} does not have a right edge. Since $u_i v_1, u_j v_t \in E(H)$, $N_{P_2}(u_\ell) = \emptyset$ for each $\ell \in \{i+1, j-1\}$, otherwise, since consecutive vertices on P_1 each have adjacencies on P_2, there exists a longer path than P_1 in H, a contradiction. Note that even if $v = v'$, $N_{P_2}(u_\ell) = \emptyset$ for each $\ell \in \{i+1, j-1\}$. Since $d_H(u_\ell) \geq 3$ for each $\ell \in \{i+1, j-1\}$, u_{i+1} has a right edge and u_{j-1} has a left edge. No vertex in $P_1[u_i, u_j]$ can have an edge that does not lie on P_1 to some other vertex in $P_1[u_i, u_j]$, otherwise, this edge is a chord of the cycle $P_1[u_i, u_j], v_t, P_2^-[v_t, v_1], u_i$. Thus we have edges $u_{i+1}u_h$ with $h > j$, and $u_{j-1}u_{h'}$ with $h' < i$. Then

$$P_1[u_{h'}, u_i], v_1, P_2[v_1, v_t], u_j, P_1[u_j, u_h], u_{i+1}, P_1[u_{i+1}, u_{j-1}], u_{h'}$$

is a cycle with chord $u_i u_{i+1}$ (and $u_{j-1}u_j$), a contradiction. Thus the lemma holds. □

Lemma 10 ([8]). *Let H be a graph of order at least 13. Suppose H does not contain a chorded cycle. If H contains a Hamiltonian path, then there exists an independent set X of four vertices in H such that $d_H(X) \leq 8$.*

Lemma 11 ([8]). *Let H be a connected graph of order at least 4. Suppose H contains neither a chorded cycle nor a Hamiltonian path. Let $P_1 = u_1, \ldots, u_s$ ($s \geq 3$) be a longest path in H, and let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $H - P_1$. Then the following statements hold.*
(i) $N_{H-P_1}(u_i) = \emptyset$ *for each* $i \in \{1, s\}$.
(ii) $d_H(u_i) = d_{P_1}(u_i) \leq 2$ *for each* $i \in \{1, s\}$.
(iii) $N_{H-(P_1 \cup P_2)}(v_j) = \emptyset$ *for each* $j \in \{1, t\}$.
(iv) $d_{P_2}(v_j) \leq 2$ *for each* $j \in \{1, t\}$.
(v) $d_{P_i}(z) \leq 2$ *for each* $z \in V(H) - V(P_i)$ *and each* $i \in \{1, 2\}$.
(vi) $d_{P_1}(\{v_1, v_t\}) \leq 3$ *for each* $t \geq 2$.

Proofs of (v) *and* (vi). Note parts (i) to (iv) are from [8], hence we only prove parts (v) and (vi). Since H does not contain a chorded cycle, (v) holds. Suppose $d_{P_1}(\{v_1, v_t\}) \geq 4$. By (v), $d_{P_1}(v_j) = 2$ for each $j \in \{1, t\}$. Then, by Lemma 5, H has a chorded cycle, a contradiction. Thus (vi) holds. □

Lemma 12. *Let H be a connected graph of order at least 15. Suppose H contains neither a chorded cycle nor a Hamiltonian path. Let $P_1 = u_1, \dots, u_s$ ($s \geq 3$) be a longest path in H, and let $P_2 = v_1, \dots, v_t$ ($t \geq 1$) be a longest path in $H - P_1$ such that $d_{P_1}(v_1) \leq d_{P_1}(v_t)$. Then there exists an independent set X of four vertices in H such that $\{u_1, u_s, v_1\} \subseteq X$ and $d_H(X) \leq 8$.*

Remark 4. Let H be a graph of order 14 shown in Fig. 1 (Remark 2, Theorem 4), $P_1 = u_1, \dots, u_{11}$, and $P_2 = v_1, v_2, v_3$. Then H satisfies all the conditions except for the order in Lemma 12. However, the conclusion does not hold. Thus $|H| \geq 15$ is necessary.

Proof. Suppose $u_1u_s \in E(H)$. Since H is connected and $V(H - P_1) \neq \emptyset$, there exists a longer path than P_1, a contradiction. Thus $u_1u_s \notin E(H)$. Let $R = H - (P_1 \cup P_2)$. If $t = 1$, that is, $v_1 = v_t$, then $d_{P_1}(v_1) \leq 2$ by Lemma 11 (v). If $t \geq 2$, then $d_{P_1}(\{v_1, v_t\}) \leq 3$ by Lemma 11 (vi). Then $d_{P_1}(v_1) \leq 1$ by the assumption $(d_{P_1}(v_1) \leq d_{P_1}(v_t))$, and $d_{P_1}(v_t) \leq 2$ by Lemma 11 (v).

Claim 6. *If $|P_2| \leq 3$, then $H = \langle P_1 \cup P_2 \rangle$.*

Proof. Suppose $H \neq \langle P_1 \cup P_2 \rangle$. Now we prove the following two subclaims.

Subclaim 7. *For any $v \in V(P_2)$, $N_R(v) = \emptyset$.*

Proof. By Lemma 11 (iii), $N_R(v_j) = \emptyset$ for each $j \in \{1, t\}$. If $|P_2| \leq 2$, then the subclaim holds. Thus we may assume $|P_2| = 3$. Suppose $N_R(v') \neq \emptyset$ for some $v' \in V(P_2)$. Then $v' = v_2$. Let $w_1 \in N_R(v_2)$. If $v_1v_3 \in E(H)$, then the subclaim holds, otherwise, there exists a longer path than P_2 in $H - P_1$, a contradiction. Thus $v_1v_3 \notin E(H)$. Since $d_{P_1}(v_1) \leq 1$ and $d_{P_1}(v_3) \leq 2$, we have $d_H(v_1) \leq 2$ and $d_H(v_3) \leq 3$. Suppose a vertex on P_2 has a neighbor w_1 in R. Then $v_2w_1 \in E(H)$. Recall $u_1u_s \notin E(H)$, and note $u_iv_j \notin E(H)$ for any $i \in \{1, s\}$ and any $j \in \{1, 3\}$ by Lemma 11 (i). We also note $d_H(u_i) \leq 2$ for any $i \in \{1, s\}$ by Lemma 11 (ii). If $d_H(\{v_1, v_3\}) \leq 4$, then $X = \{u_1, u_s, v_1, v_3\}$ is an independent set in H and $d_H(X) \leq 8$, and X is the desired set. Thus we may assume $d_H(\{v_1, v_3\}) = 5$, that is, $d_H(v_1) = 2$ and $d_H(v_3) = 3$. Then $d_{P_1}(v_1) = 1$ and $d_{P_1}(v_3) = 2$. Recall $w_1 \in N_R(v_2)$. Clearly, $N_R(w_1) = \emptyset$, otherwise, there exists a longer path than P_2 in $H - P_1$, a contradiction. If $d_H(w_1) \leq 2$, then $X = \{u_1, u_s, v_1, w_1\}$ is the desired set. Thus $d_H(w_1) \geq 3$, that is, $d_{P_1}(w_1) \geq 2$. Note w_1 and v_3 lie on a path $P = w_1, v_2, v_3$, and w_1, v_3 send at least two edges each to P_1. By Lemma 5, there exists a chorded cycle in $\langle P_1 \cup P \rangle$, a contradiction. □

Subclaim 8. *For any $u \in V(P_1)$, $N_R(u) = \emptyset$.*

Proof. We first prove $d_H(v_1) \leq 2$. Suppose not, that is, $d_H(v_1) \geq 3$. Recall $d_{P_1}(v_1) \leq 1$. By Subclaim 7 and Lemma 11 (iv), $d_{P_1}(v_1) = 1$ and $d_{P_2}(v_1) = 2$. Thus $|P_2| = 3$ and $v_1v_3 \in E(H)$. Since $d_{P_1}(v_1) \leq d_{P_1}(v_3)$ by the assumption, $d_{P_1}(v_3) \geq 1$. Then $\langle P_1 \cup P_2 \rangle$ contains a cycle with chord v_1v_3, a contradiction. Thus $d_H(v_1) \leq 2$. Suppose there exists a vertex in P_1 with a neighbor w_1 in R. If $d_H(w_1) \leq 2$, then $X = \{u_1, u_s, v_1, w_1\}$ is the desired set. Thus $d_H(w_1) \geq 3$.

First suppose $d_{P_1}(w_1) \geq 2$. Then $d_{P_1}(w_1) = 2$ by Lemma 11 (v), and $d_R(w_1) \geq 1$ by Subclaim 7. Let $w_2 \in N_R(w_1)$. If $d_H(w_2) \leq 2$, then $X = \{u_1, u_s, v_1, w_2\}$ is the desired set. Thus $d_H(w_2) \geq 3$. If $d_{P_1}(w_2) \geq 2$, then we have two vertices on a path $P = w_1, w_2$, each sending at least two edges to another path P_1, and by Lemma 5, a chorded cycle exists in $\langle P_1 \cup P \rangle$, a contradiction. Thus $d_{P_1}(w_2) \leq 1$, and by Subclaim 7, $d_R(w_2) \geq 2$. Let $w_3 \in N_{R-w_1}(w_2)$. If $d_H(w_3) \leq 2$, then $X = \{u_1, u_s, v_1, w_3\}$ is the desired set. Thus $d_H(w_3) \geq 3$. Suppose $d_{P_1}(w_3) \geq 2$. Then consider the path $P = w_1, w_2, w_3$. Since w_1 and w_3 send at least two edges to another path P_1, a chorded cycle exists in $\langle P_1 \cup P \rangle$ by Lemma 5, a contradiction. Thus $d_{P_1}(w_3) \leq 1$. Also, $N_{R-\{w_1,w_2\}}(w_3) = \emptyset$, otherwise, there exists a longer path than P_2 in $H - P_1$, a contradiction. By Subclaim 7, $N_{P_2}(w_3) = \emptyset$. Thus $d_{P_1}(w_3) = 1$ and $w_1, w_2 \in N_H(w_3)$. Then $\langle P_1 \cup P \rangle$ contains a cycle with chord w_1w_3, a contradiction.

Next suppose $d_{P_1}(w_1) = 1$. Then $d_R(w_1) \geq 2$ by Subclaim 7. Let $w_2, w_3 \in N_R(w_1)$. If $d_H(w_i) \leq 2$ for some $i \in \{2, 3\}$, then $X = \{u_1, u_s, v_1, w_i\}$ is the desired set. Thus $d_H(w_i) \geq 3$ for each $i \in \{2, 3\}$.

Suppose $d_R(w_i) \geq 3$ for some $i \in \{2,3\}$. Without loss of generality, we may assume $i = 2$. Then w_2 has a neighbor w_4 in R distinct from w_1 and w_3, and hence w_3, w_1, w_2, w_4 is a longer path than P_2 in $H - P_1$, a contradiction. Thus for each $i \in \{2,3\}$, $d_R(w_i) \leq 2$, and then $d_{P_1}(w_i) \geq 1$ by Subclaim 7. Note w_i for each $i \in \{2,3\}$ does not have a neighbor in R distinct from w_1, w_2, w_3, otherwise, there exists a longer path than P_2 in $H - P_1$, a contradiction. Now suppose $d_R(w_i) = 2$ for some $i \in \{2,3\}$. Then $w_2w_3 \in E(H)$. Let $P = w_2, w_1, w_3$. Since $d_{P_1}(w_i) \geq 1$ for each $i \in \{2,3\}$, there exists a cycle with chord w_2w_3 in $\langle P_1 \cup P \rangle$, a contradiction. Thus $d_R(w_i) \leq 1$ for each $i \in \{2,3\}$, and then $d_{P_1}(w_i) \geq 2$ by Subclaim 7. By Lemma 5, a chorded cycle exists in $\langle P_1 \cup P \rangle$, a contradiction. □

Since H is connected, we get a contradiction by Subclaims 7 and 8. Thus Claim 6 holds. □

Claim 9. *We have* $d_{P_1}(v_t) \geq 1$.

Proof. Suppose $d_{P_1}(v_t) = 0$. By the assumption $(d_{P_1}(v_1) \leq d_{P_1}(v_t))$, we have $d_{P_1}(v_1) = 0$. Then we may assume $|P_2| = t \geq 3$, otherwise, we get a contradiction by Claim 6 and the connectedness of H. Recall $u_1u_s \notin E(H)$. By Lemmas 11 (iii) and (iv), $d_H(v_j) \leq 2$ for each $j \in \{1, t\}$. If $v_1v_t \notin E(H)$, then $X = \{u_1, u_s, v_1, v_t\}$ is the desired set. Thus $v_1v_t \in E(H)$.

First suppose $|P_2| = t = 3$. By Claim 6, $H = \langle P_1 \cup P_2 \rangle$. Since $v_1v_3 \in E(H)$, consider $P_2' = v_2, v_1, v_3$. Then v_2 can be regarded as an endpoint of P_2'. Since $d_{P_1}(v_1) = 0$, we may assume $d_{P_1}(v_2) = 0$ by considering v_2 instead of v_1. Since $N_{P_1}(P_2) = \emptyset$, this contradicts the connectedness of H.

Next suppose $|P_2| = t \geq 4$. Recall $u_1u_s \notin E(H)$ and $v_1v_t \in E(H)$. Consider $P_2' = P_2^-[v_{t-1}, v_1], v_t$. Then v_{t-1} can be regarded as an endpoint of P_2'. Thus $N_R(v_{t-1}) = \emptyset$ by Lemma 11 (iii), and $d_{P_2}(v_{t-1}) \leq 2$ by Lemma 11 (iv). Since $d_{P_1}(v_1) = 0$, we may assume $d_{P_1}(v_{t-1}) = 0$ by considering v_{t-1} instead of v_1. Thus $d_H(v_{t-1}) = 2$. Hence $X = \{u_1, u_s, v_1, v_{t-1}\}$ is the desired set, and Claim 9 holds. □

Now we consider the following three cases based on $|P_2|$.

Case 1. Suppose $|P_2| = t = 1$.

Then $P_2 = v_1$. By Claim 6, $H = \langle P_1 \cup P_2 \rangle$. Since $|H| \geq 15$, $|P_1| \geq 14$. Recall $d_{P_1}(v_1) \leq 2$ when $t = 1$. By Claim 9, $d_{P_1}(v_1) \in \{1,2\}$. Note $d_H(v_1) = d_{P_1}(v_1)$.

First suppose $d_{P_1}(v_1) = 2$. Let $u_i, u_j \in N_{P_1}(v_1)$ with $i < j$. Note $i \geq 2$ and $j \leq s - 1$ by Lemma 11 (i). If $j = i + 1$, then H contains a Hamiltonian path, a contradiction. Thus $j \geq i + 2$. By Lemma 9, $d_H(u_\ell) = 2$ for some $\ell \in \{i+1, j-1\}$. Note $u_\ell u_1, u_\ell u_s \notin E(H)$. Then $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set.

Next suppose $d_{P_1}(v_1) = 1$. Note $d_{P_1}(u_1) \leq 2$. Assume $u_1u_i \in E(H)$ for some $4 \leq i \leq s - 1$. By Lemma 6, $d_{P_1}(u_{i-1}) = 2$. If $v_1u_{i-1} \in E(H)$, then $v_1, u_{i-1}, P_1^-[u_{i-1}, u_1], u_i, P_1[u_i, u_s]$ is a Hamiltonian path, a contradiction. Thus $v_1u_{i-1} \notin E(H)$ and $d_H(u_{i-1}) = 2$. Then $X = \{u_1, u_{i-1}, u_s, v_1\}$ is the desired set. Thus if $d_{P_1}(u_1) = 2$, then $u_1u_3 \in E(H)$. Then $d_{P_1}(u_i) = 2$ for some $3 \leq i \leq 6$ by Lemma 7. Similarly, either $d_{P_1}(u_s) = 1$ or $u_su_{s-2} \in E(H)$ by symmetry. Then $d_{P_1}(u_j) = 2$ for some $s-5 \leq j \leq s-2$ by Lemma 8. Note $|P_1| = s \geq 14$. Since $d_{P_1}(v_1) = 1$ by our assumption, $v_1u_\ell \notin E(H)$ for some $\ell \in \{i, j\}$, and $d_H(u_\ell) = 2$. Thus $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set.

Case 2. Suppose $|P_2| = t \in \{2,3\}$.

By Claim 6, $H = \langle P_1 \cup P_2 \rangle$. Recall $d_{P_1}(\{v_1, v_t\}) \leq 3$, $d_{P_1}(v_1) \leq 1$, and $d_{P_1}(v_t) \leq 2$. We also note $d_{P_1}(\{v_1, v_t\}) \geq 1$ by Claim 9. Since $|H| \geq 15$, $|P_1| = s \geq 12$.

First suppose $|N_{P_1}(\{v_1, v_t\})| \in \{2,3\}$. Let $u_i, u_j \in N_{P_1}(\{v_1, v_t\})$ with $i < j$. Assume $j = i + 1$. Then H contains a longer path than P_1, a contradiction. Thus $j \geq i + 2$. Note $i \geq 2$ and $j \leq s - 1$ by Lemma 11 (i). By Lemma 9, $d_H(u_\ell) = 2$ for some $\ell \in \{i+1, j-1\}$. Note $u_\ell u_1 \notin E(H)$ and $u_\ell u_s \notin E(H)$. If $d_H(v_1) \leq 2$, then $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set. Thus we may assume that $d_H(v_1) \geq 3$. Since $d_{P_1}(v_1) \leq 1$ and $d_{P_2}(v_1) \leq 2$, we have $d_{P_1}(v_1) = 1$ and $d_{P_2}(v_1) = 2$. Then $t = 3$ and $v_1v_3 \in E(H)$. Since $d_{P_1}(v_1) \leq d_{P_1}(v_t) = d_{P_1}(v_3)$ by the assumption, we have $d_{P_1}(v_3) \geq 1$. Thus $\langle P_1 \cup P_2 \rangle$ contains a cycle with chord v_1v_3, a contradiction.

Next suppose $|N_{P_1}(\{v_1, v_t\})| = 1$. Assume $u_1u_i \in E(H)$ for some $4 \leq i \leq s-1$. By Lemma 6, $d_{P_1}(u_{i-1}) = 2$. Let $P_1' = P_1^-[u_{i-1}, u_1], u_i, P_1[u_i, u_s]$. Then $|P_1'| = |P_1|$ and u_{i-1} can be regarded as an endpoint of P_1'. By Lemma 11 (i), $d_{P_2}(u_{i-1}) = 0$. Then $d_H(u_{i-1}) = d_{P_1}(u_{i-1}) = 2$. If $d_H(v_1) \leq 2$, then $X = \{u_1, u_{i-1}, u_s, v_1\}$ is the desired set. Thus we may assume that $d_H(v_1) \geq 3$. Then $d_{P_1}(v_1) = 1$, and $d_{P_2}(v_1) = 2$, that is, $t = 3$ and $v_1v_3 \in E(H)$. Also, $d_{P_1}(v_3) \geq 1$. Thus $\langle P_1 \cup P_2 \rangle$ contains a cycle with chord v_1v_3, a contradiction. Hence, either $d_{P_1}(u_1) = 1$ or $u_1u_3 \in E(H)$. Then $d_{P_1}(u_i) = 2$ for some $3 \leq i \leq 6$ by Lemma 7. Similarly, either $d_{P_1}(u_s) = 1$ or $u_su_{s-2} \in E(H)$ by symmetry. Then $d_{P_1}(u_j) = 2$ for some $s-5 \leq j \leq s-2$ by Lemma 8. Since $|N_{P_1}(\{v_1, v_t\})| = 1$ by our assumption, $u_\ell \notin N_{P_1}(\{v_1, v_t\})$ for some $\ell \in \{i, j\}$. Suppose $t = 2$. Then $d_H(v_1) \leq 2$ and $d_H(u_\ell) = d_{P_1}(u_\ell) = 2$. Thus $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set. Hence $t = 3$. If $v_1v_3 \notin E(H)$, then $d_H(v_1) \leq 2$ and $d_H(v_3) \leq 2$. Thus $X = \{u_1, u_s, v_1, v_3\}$ is the desired set. Hence we may assume that $v_1v_3 \in E(H)$. Note $d_{P_1}(v_1) \leq 1$. Suppose $d_{P_1}(v_1) = 1$. Since $d_{P_1}(v_3) \geq 1$, $\langle P_1 \cup P_2 \rangle$ contains a cycle with chord v_1v_3, a contradiction. Suppose $d_{P_1}(v_1) = 0$. Then $d_H(v_1) = 2$. If $d_H(u_\ell) = 2$, then $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set. Thus we may assume that $d_H(u_\ell) \geq 3$. Then $u_\ell v_2 \in E(H)$. Since $d_{P_1}(v_3) \geq 1$, $\langle P_1 \cup P_2 \rangle$ contains a cycle with chord v_2v_3, a contradiction.

Case 3. Suppose $|P_2| = t \geq 4$.

Recall $d_{P_1}(v_1) \leq 1$ and $d_{P_1}(v_t) \leq 2$. We consider two subcases as follows.

Subcase 1. Suppose $d_{P_1}(v_1) = 1$.

By Claim 9, $d_{P_1}(v_t) \geq 1$. Then $d_{P_2}(v_1) = d_{P_2}(v_t) = 1$, otherwise, there exists a cycle in $\langle P_1 \cup P_2 \rangle$ with chord adjacent to v_1 or v_t, a contradiction. Thus $d_H(v_1) = 2$ by Lemma 11 (iii). If $d_{P_1}(v_t) = 1$, then $d_H(v_t) = 2$ by Lemma 11 (iii). Then $X = \{u_1, u_s, v_1, v_t\}$ is the desired set. Thus $d_{P_1}(v_t) = 2$. Let $u_i, u_j \in N_{P_1}(v_t)$ with $i < j$. Consider the vertex v_{t-1}. If $d_H(v_{t-1}) = 2$, then $X = \{u_1, u_s, v_1, v_{t-1}\}$ is the desired set. Thus $d_H(v_{t-1}) \geq 3$. If $d_{P_2}(v_{t-1}) \geq 3$, then there exists a cycle in $\langle P_1 \cup P_2 \rangle$ with chord adjacent to v_{t-1}, a contradiction. Thus $d_{P_2}(v_{t-1}) = 2$, and then $N_{P_1}(v_{t-1}) \neq \emptyset$ or $N_R(v_{t-1}) \neq \emptyset$.

First suppose $N_{P_1}(v_{t-1}) \neq \emptyset$. If v_1 or v_{t-1} has a neighbor in $P_1[u_1, u_i] \cup P_1[u_j, u_s]$, then there exist three parallel edges between P_1 and P_2, and by Lemma 3, a chorded cycle exists in $\langle P_1 \cup P_2 \rangle$, a contradiction. Thus $N_{P_1(u_i,u_j)}(v_\ell) \neq \emptyset$ for each $\ell \in \{1, t-1\}$. Then we again have three parallel edges or three crossing edges, and by Lemma 3, a chorded cycle exists in $\langle P_1 \cup P_2 \rangle$, a contradiction.

Next suppose $N_R(v_{t-1}) \neq \emptyset$. Let $w \in N_R(v_{t-1})$. If $d_H(w) \leq 2$, then $X = \{u_1, u_s, v_1, w\}$ is the desired set. Thus $d_H(w) \geq 3$. Then $d_{P_1}(w) \leq 1$, otherwise, since $d_{P_1}(v_t) = 2$, there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$ by Lemma 5, a contradiction. Since P_2 is a longest path in $H - P_1$, $N_R(w) = \emptyset$. Thus $d_{P_1}(w) = 1$ and $d_{P_2}(w) = 2$. Let $u_p \in N_{P_1}(v_1)$ and $u_q \in N_{P_1}(w)$. Without loss of generality, we may assume $p \leq q$. By Lemma 11 (iii), $wv_1, wv_t \notin E(H)$. Thus $wv_\ell \in E(H)$ for some $2 \leq \ell \leq t-2$. Then $w, v_{t-1}, P_2^-[v_{t-1}, v_1], u_p, P_1[u_p, u_q], w$ is a cycle with chord wv_ℓ, a contradiction.

Subcase 2. Suppose $d_{P_1}(v_1) = 0$.

Suppose $v_1v_t \in E(H)$. Then note $d_H(v_1) = 2$. Now we consider the path $P_2' = P_2^-[v_{t-1}, v_1], v_t$. Then v_{t-1} can be regarded as an endpoint of P_2'. Since $d_{P_1}(v_1) = 0$ by the assumption, we may assume $d_{P_1}(v_{t-1}) = 0$ by considering v_{t-1} instead of v_1. Thus $d_H(v_{t-1}) = 2$. Recall $u_1u_s \notin E(H)$. Then $X = \{u_1, u_s, v_1, v_{t-1}\}$ is the desired set. Thus $v_1v_t \notin E(H)$. If $d_H(v_t) \leq 2$, then $X = \{u_1, u_s, v_1, v_t\}$ is the desired set. Thus $d_H(v_t) \geq 3$. By Lemma 11 (iii), (iv), and (v), we have $d_H(v_t) \leq 4$ and $d_{P_1}(v_t) \in \{1, 2\}$.

First suppose $d_{P_1}(v_t) = 2$. Let $u_i, u_j \in N_{P_1}(v_t)$ with $i < j$. Note $i \geq 2$ and $j \leq s-1$ by Lemma 11 (i), and $|P_1| \geq |P_2| \geq 4$. If $j = i+1$, then there exists a longer path than P_1, a contradiction. Thus $j \geq i+2$. Therefore, $|P_1| \geq 5$. If $d_H(u_\ell) = 2$ for some $\ell \in \{i+1, j-1\}$, then $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set. Thus $d_H(u_\ell) \geq 3$ for each $\ell \in \{i+1, j-1\}$. By Lemma 9, we may assume $H \neq \langle P_1 \cup P_2 \rangle$. Now we claim $N_R(u_\ell) \neq \emptyset$ for some $\ell \in \{i+1, j-1\}$. Assume not. Note $N_{P_2}(u_\ell) = \emptyset$ since P_1 is a longest path in H. Since H does not contain a chorded cycle, there exist edges $u_{i+1}u_h$ with $h > j$ and $u_{j-1}u_{h'}$ with $h' < i$. Then

$$P_1[u_{h'}, u_i], v_t, u_j, P_1[u_j, u_h], u_{i+1}, P_1[u_{i+1}, u_{j-1}], u_{h'}$$

is a cycle with chord u_iu_{i+1} (and $u_{j-1}u_j$), a contradiction. Thus the claim holds. If $j \geq i+3$, then we may assume $\ell = j-1$, that is, $N_R(u_{j-1}) \neq \emptyset$, otherwise, consider $P^-[u_s, u_1]$. Let $w_1 \in N_R(u_{j-1})$, and let $P_3 = w_1, \ldots, w_p$ $(p \geq 1)$ be a longest path starting from w_1 in R. If $d_H(w_p) \leq 2$, then $X = \{u_1, u_s, v_1, w_p\}$ is the desired set. Thus $d_H(w_p) \geq 3$. If $N_{P_2}(w) \neq \emptyset$ for some $w \in V(P_3)$, that is, $v_\ell \in N_{P_2}(w)$ for some $1 \leq \ell \leq t$, then

$$P_1[u_1, u_{j-1}], w_1, P_3[w_1, w], v_\ell, P_2[v_\ell, v_t], u_j, P_1[u_j, u_s]$$

is a longer path than P_1, a contradiction. Thus $N_{P_2}(w) = \emptyset$ for any $w \in V(P_3)$. Since P_3 is a longest path starting from w_1 in R, $N_{R-P_3}(w_p) = \emptyset$. Suppose $|P_3| = p = 1$. Since $N_R(w_1) = \emptyset$ and $d_H(w_p) \geq 3$, $d_{P_1}(w_1) \geq 3$. This contradicts Lemma 11 (v). Suppose $|P_3| = p = 2$. Then $d_H(w_2) \geq 3$, and by Lemma 11 (v), $d_{P_1}(w_2) = 2$. If $u_\ell \in N_{P_1}(w_2)$ for some $j \leq \ell \leq s$, then

$$P_1[u_i, u_{j-1}], w_1, P_3[w_1, w_2], u_\ell, P_1^-[u_\ell, u_j], v_t, u_i$$

is a cycle with chord $u_{j-1}u_j$, a contradiction. Thus $u_\ell, u_{\ell'} \in N_{P_1}(w_2)$ for some $1 \leq \ell < \ell' \leq j-1$. Then $P_1[u_\ell, u_{j-1}], w_1, P_3[w_1, w_2], u_\ell$ is a cycle with chord $w_2u_{\ell'}$, a contradiction. Suppose $|P_3| = p \geq 3$. Then $d_{P_3}(w_p) \leq 2$. Assume $d_{P_3}(w_p) = 2$. Since $d_{P_1}(w_p) \geq 1$, there exists a cycle in $\langle P_1 \cup P_3 \rangle$ with chord adjacent to w_p, a contradiction. Thus $d_{P_3}(w_p) = 1$, and $d_{P_1}(w_p) = 2$. Then we have a chorded cycle in $\langle P_1 \cup P_3 \rangle$ as in the case where $|P_3| = 2$ by considering w_p instead of w_2, a contradiction.

Next suppose $d_{P_1}(v_t) = 1$. Let $u_i \in N_{P_1}(v_t)$ with $1 \leq i \leq s$. Note $i \notin \{1, s\}$ by Lemma 11 (i). Since $d_H(v_t) \geq 3$, $d_{P_2}(v_t) = 2$ by Lemmas 11 (iii) and (iv). Let $v_\ell \in N_{P_2}(v_t)$ with $\ell \leq t-2$. Now we consider the path $P_2' = P_2[v_1, v_\ell], v_t, P_2^-[v_t, v_{\ell+1}]$. Then $v_{\ell+1}$ can be regarded as an endpoint of P_2'. Since $d_{P_1}(v_t) = 1$, we may assume $d_{P_1}(v_{\ell+1}) = 1$. Let $u_j \in N_{P_1}(v_{\ell+1})$ with $1 \leq j \leq s$. Note $j \notin \{1, s\}$ by Lemma 11 (i). Then we may assume $j \leq i$, otherwise, consider $P^-[u_s, u_1]$. Suppose $\ell = t-2$, that is, $v_tv_{t-2} \in E(H)$. Then $P_1[u_j, u_i], v_t, v_{t-2}, v_{t-1}, u_j$ is a cycle with chord $v_{t-1}v_t$, a contradiction. Thus $\ell \leq t-3$. If $j = i-1$, then there exists a longer path than P_1, a contradiction.

Suppose $j = i$. Recall $v_tv_\ell \in E(H)$ with $\ell \leq t-3$. If $d_H(v_{t-1}) = 2$, then $X = \{u_1, u_s, v_1, v_{t-1}\}$ is the desired set. Thus $d_H(v_{t-1}) \geq 3$. Assume $u_j \in N_{P_1}(v_{t-1})$ for some $1 \leq j \leq s$. We may assume $j \leq i$, otherwise, consider $P^-[u_s, u_1]$. Then $P_1[u_j, u_i], v_t, P_2[v_\ell, v_{t-1}], u_j$ is a cycle with chord $v_{t-1}v_t$, a contradiction. Assume $v_{\ell'} \in N_{P_2}(v_{t-1})$ for some $\ell' \leq t-3$. Since $v_tv_\ell \in E(H)$, we may assume $\ell' < \ell$. Then $P_2[v_{\ell'}, v_\ell], v_t, u_i, P_2[v_{\ell+1}, v_{t-1}], v_{\ell'}$ is a cycle with chord $v_\ell v_{\ell+1}$ (and $v_{t-1}v_t$), a contradiction. Assume $N_R(v_{t-1}) \neq \emptyset$. Let $w \in N_R(v_{t-1})$. Now we consider the path $P_2' = P_2[v_1, v_{t-1}], w$. Then w can be regarded as an endpoint of P_2'. Since $d_{P_1}(v_t) = 1$, we may assume $d_{P_1}(w) = 1$. Let $u_j \in N_{P_1}(w)$ for some $1 \leq j \leq s$. We may assume $j \leq i$. Then $P_2[v_\ell, v_{t-1}], w, P_1[u_j, u_i], v_t, v_\ell$ is a cycle with chord $v_{t-1}v_t$, a contradiction.

Suppose $j \leq i-2$. If $d_H(u_h) = 2$ for some $h \in \{j+1, i-1\}$, then $X = \{u_1, u_h, u_s, v_1\}$ is the desired set. Thus $d_H(u_h) \geq 3$ for each $h \in \{j+1, i-1\}$. Now we claim $N_R(u_h) \neq \emptyset$ for some $h \in \{j+1, i-1\}$. Assume not. Note $N_{P_2}(u_h) = \emptyset$, since P_1 is a longest path in H. Since H does not contain a chorded cycle, there exist edges $u_{j+1}u_m$ with $m > i$ and $u_{i-1}u_{m'}$ with $m' < j$. Then

$$P_1[u_{m'}, u_j], v_{\ell+1}, P_2[v_{\ell+1}, v_t], u_i, P_1[u_i, u_m], u_{j+1}, P_1[u_{j+1}, u_{i-1}], u_{m'}$$

is a cycle with chord u_ju_{j+1} (and $u_{i-1}u_i$), a contradiction. Thus the claim holds. We also note that if $j \leq i-3$, then we may assume $N_R(u_{i-1}) \neq \emptyset$, otherwise, consider $P^-[u_s, u_1]$. Let $w_1 \in N_R(u_{i-1})$, and let $P_3 = w_1, \ldots, w_p$ $(p \geq 1)$ be a longest path in R. Then, as in the above case where $d_{P_1}(v_t) = 2$, there exists a chorded cycle in H, a contradiction. □

Lemma 13 ([8]). *Let $k \geq 2$ be an integer, and let G be a graph. Suppose G does not contain k vertex-disjoint chorded cycles. Let $\mathscr{C} = \{C_1, \ldots, C_{k-1}\}$ be a minimal set of $k-1$ vertex-disjoint chorded cycles in G, and let $H = G - \mathscr{C}$ and $X \subseteq V(H)$ with $|X| = 4$. Suppose H contains a Hamiltonian path. Then $d_{C_i}(X) \leq 12$ for each $1 \leq i \leq k-1$.*

4. Proof of Theorem 4

Suppose G does not contain a chorded cycle.

Claim 10. *G is connected.*

Proof. Suppose not, then $comp(G) \geq 2$. Let $G_1, G_2, \ldots, G_{comp(G)}$ be the components of G.

First suppose $comp(G) \geq 4$. By Theorem 1, there exists $x_i \in V(G_i)$ for each $1 \leq i \leq 4$ such that $d_{G_i}(x_i) \leq 2$. Let $X = \{x_1, x_2, x_3, x_4\}$. Then X is an independent set with $d_G(X) \leq 8$. This contradicts the $\sigma_4(G)$ condition.

Next suppose $comp(G) = 3$. Let $|G_1| \geq |G_2| \geq |G_3|$. Since $|G| \geq 15$ by the assumption, we have $|G_1| \geq 5$. If G_1 is complete, then G_1 contains a chorded cycle. Thus we may assume G_1 is not complete. By Theorem 2, there exist non-adjacent $x_0, x_1 \in V(G_1)$ such that $d_{G_1}(\{x_0, x_1\}) \leq 4$. Also, by Theorem 1, there exists $x_i \in V(G_i)$ for each $i \in \{2, 3\}$ such that $d_{G_i}(x_i) \leq 2$. Then $X = \{x_0, x_1, x_2, x_3\}$ is an independent set with $d_G(X) \leq 8$, a contradiction.

Finally, suppose $comp(G) = 2$. Let $|G_1| \geq |G_2|$. Since $|G| \geq 15$, $|G_1| \geq 8$. By Theorem 3 (Remark 1), G_1 contains an independent set X_0 of three vertices with $d_{G_1}(X_0) \leq 6$. Also, by Theorem 1, there exists $x \in V(G_2)$ such that $d_{G_2}(x) \leq 2$. Then $X = X_0 \cup \{x\}$ is an independent set with $d_G(X) \leq 8$, a contradiction. □

Let $P_1 = u_1, \ldots, u_s$ be a longest path in G. Note $s \geq 3$, since $|G| \geq 15$ and G is connected by Claim 10.

Claim 11. *G contains a Hamiltonian path.*

Proof. Suppose not, then P_1 is not a Hamiltonian path in G, and $V(G - P_1) \neq \emptyset$. Let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $G - P_1$ such that $d_{P_1}(v_1) \leq d_{P_1}(v_t)$. By Lemma 12, there exists an independent set X of four vertices in G such that $d_G(X) \leq 8$. This contradicts the $\sigma_4(G)$ condition. □

Since $|G| \geq 15$, by Claim 11 and Lemma 10, there exists an independent set X of four vertices in G such that $d_G(X) \leq 8$, a contradiction. This completes the proof of Theorem 4. □

5. Proof of Theorem 5

By Theorem 4, we may assume $k \geq 2$. Suppose Theorem 5 does not hold. Let G be an edge-maximal counter-example. If G is complete, then G contains k vertex-disjoint chorded cycles. Thus we may assume G is not complete. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. By the edge-maximality of G, G' is not a counter-example. Thus G' contains k vertex-disjoint chorded cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ vertex-disjoint chorded cycles. Over all sets of $k-1$ vertex-disjoint chorded cycles, choose $C_1, \ldots, C_{k-1}$ with $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, $H = G - \mathscr{C}$, and with P_1 a longest path in H, such that:

(A1) $|\mathscr{C}|$ is as small as possible,

(A2) subject to (A1), $comp(H)$ is as small as possible, and

(A3) subject to (A1) and (A2), $|P_1|$ is as large as possible.

We may also assume H does not contain a chorded cycle, otherwise, G contains k vertex-disjoint chorded cycles, a contradiction.

Claim 12. *H has an order at least* 18.

Proof. Suppose to the contrary that $|H| \leq 17$. Next suppose $|C_i| \leq 11$ for each $1 \leq i \leq k-1$. Since $|G| \geq 11k + 7$ by assumption, it follows that $|H| \geq (11k + 7) - 11(k - 1) = 18$, a contradiction. Thus $|C_i| \geq 12$ for some $1 \leq i \leq k-1$. Without loss of generality, we may assume C_1 is a longest cycle in $\mathscr{C}$. Then $|C_1| \geq 12$. By Lemma 1, C_1 contains at most two chords, and if C_1 has two chords, then these chords must be crossing. For integers t and r, let $|C_1| = 4t + r$, where $t \geq 3$ and $0 \leq r \leq 3$.

Subclaim 13. *Let $t \geq 3$ be an integer. The cycle C_1 contains t vertex-disjoint sets $X_1, \ldots, X_t$ of four independent vertices each in G such that $d_{C_1}(\cup_{i=1}^{t} X_i) \leq 8t + 4$.*

Proof. For any $4t$ vertices of C_1, their degree sum in C_1 is at most $4t \times 2 + 4 = 8t + 4$, since C_1 has at most two chords. Thus it only remains to show that C_1 contains t vertex-disjoint sets of four independent vertices each. Recall $|C_1| = 4t + r \geq 4t$. Start anywhere on C_1 and label the first $4t$ vertices of C_1 with labels 1 through t in order, starting over again with 1 after using label t. If $r \geq 1$, then label the remaining r vertices of C_1 with the labels $t + 1, \ldots, t + r$. (See Fig. 2.) The labeling above yields t vertex-disjoint sets of four vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, since $t \geq 3$, any vertex in C_1 has a different label than the vertex that precedes it on C_1 and the vertex that succeeds it on C_1. Let C_0 be the cycle obtained from C_1 by removing all chords. Then the vertices in each of the sets are independent in C_0. Thus the only way vertices in the same set are not independent in C_1 is if the endpoints of a chord of C_1 were given the same label. Note any vertex labeled i is distance at least 3 in C_0 from any other vertex labeled i. Thus even if we exchange the label of x in C_0 for the one of x^- (or x^+), the vertices in each of the resulting t sets are still independent in C_0.

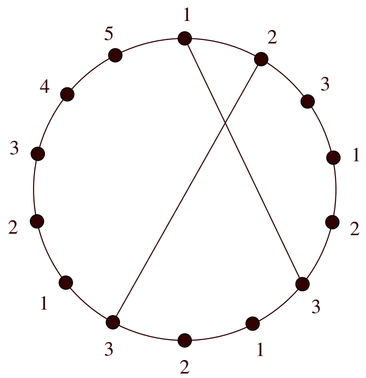

Figure 2. An Example When $t = 3, r = 2$

Case 1. No chord of C_1 has endpoints with the same label.

Then there exist t vertex-disjoint sets of four independent vertices each in C_1.

Case 2. Exactly one chord of C_1 has endpoints with the same label.

Recall C_1 contains at most two chords, and if C_1 contains two chords, then these chords must be crossing. Since $|C_1| \geq 12$, even if C_1 has two chords, each chord has an endpoint x such that there exists a vertex $x' \in \{x^-, x^+\}$ which is not an endpoint of the other chord. Choose such an endpoint x of the chord whose endpoints were assigned the same label, and exchange the label of x for the one of x'. The vertices in each of the resulting t sets are independent in C_1, and now no chord of C_1 has endpoints with the same label. Thus there exist t vertex-disjoint sets of four independent vertices each in C_1.

Case 3. Two chords of C_1 each have endpoints with the same label.

Then the two chords are crossing. Since endpoints of a chord have the same label in this case, recall these endpoints have distance at least 3. First suppose there exists an endpoint x of one chord of C_1 which is adjacent to an endpoint $y (= x^+)$ of the other chord on C_1. (See Fig. 3 (a).) Now we exchange the label of x for the one of y. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of four independent vertices each in C_1.

Next suppose no endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. (See Fig. 3 (b).) Let x_1x_2, y_1y_2 be the two distinct chords of C_1. Since the two chords are crossing, without loss of generality, we may assume x_1, y_1, x_2, y_2 are in that order on C_1. Now we exchange the labels of x_1 and x_1^+, and next the ones of y_2 and y_2^-. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of four independent vertices each in C_1. □

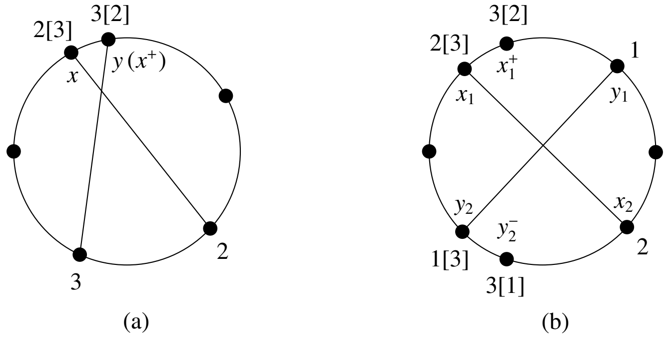

Figure 3. Examples: (a) The Labels of x and y are 2 and 3; (b) The Labels of x_1 and y_2 are 2 and 1. ($[i]$ Means i is a New Label for a Vertex after the Exchange)

Since $|C_1| \geq 12$, $d_{C_1}(v) \leq 2$ for any $v \in V(H)$ by Lemma 2 and (A1). Thus since $|H| \leq 17$ by our assumption, it follows that $|E(H, C_1)| \leq 34$. Let $\mathscr{X} = \cup_{i=1}^{t} X_i$ be as in Subclaim 13. By the $\sigma_4(G)$ condition, $d_G(\mathscr{X}) \geq t(12k-3)$. Suppose $k = 2$. Then $\mathscr{C}$ has only one cycle C_1. Since $k = 2$ and $t \geq 3$, $|E(C_1, H)| \geq d_H(\mathscr{X}) \geq t(12k-3) - (8t+4) = 13t - 4 \geq 35$, a contradiction. Thus $k \geq 3$. Then we have

$$\begin{aligned} |E(\mathscr{X}, \mathscr{C} - C_1)| &= d_G(\mathscr{X}) - d_{C_1}(\mathscr{X}) - d_H(\mathscr{X}) \\ &\geq t(12k-3) - (8t+4) - 34 \\ &= 12kt - 11t - 38, \end{aligned}$$

and since $t \geq 3$,

$$\begin{aligned} 12kt - 11t - 38 &= 12t(k-1) + t - 38 \geq 12t(k-1) - 35 \\ &> 12t(k-1) - 12t \\ &= 12t(k-2). \end{aligned}$$

Thus $|E(\mathscr{X}, C')| > 12t$ for some C' in $\mathscr{C} - C_1$, since $\mathscr{C} - C_1$ contains $k-2$ vertex-disjoint chorded cycles. Let $h = \max\{d_{C'}(v) | v \in \mathscr{X}\}$. Let v^* be a vertex of $\mathscr{X}$ such that $d_{C'}(v^*) = h$. Since $|E(\mathscr{X}, C')| > 12t$, if $h \leq 3$, then $|E(\mathscr{X}, C')| \leq 3 \times 4t = 12t$, a contradiction. Thus we may assume $h \geq 4$. By the maximality of C_1, $|C'| \leq |C_1| = 4t + r$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq 4t + r$. Recall $t \geq 3$ and $0 \leq r \leq 3$. Then

$$\begin{aligned} |E(\mathscr{X} - \{v^*\}, C')| &\geq (12t+1) - d_{C'}(v^*) \geq (12t+1) - (4t+r) \\ &= 8t - r + 1 \geq 22. \end{aligned} \tag{1}$$

Since $h = d_{C'}(v^*) \geq 4$, let v_1, v_2, v_3, v_4 be neighbors of v^* in that order on C'. Note v_1, v_2, v_3, v_4 partition C' into four intervals $C'[v_i, v_{i+1})$ for each $1 \leq i \leq 4$, where $v_5 = v_1$. By (1), there exist at least 22 edges from $C_1 - v^*$ to C'. Thus some interval $C'[v_i, v_{i+1})$ contains at least six of these edges. Without loss of generality, we may assume this interval is $C'[v_4, v_1)$. Then by Lemma 4, $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$ contains a chorded cycle not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$. Also, $v^*, C'[v_1, v_3], v^*$ is a cycle with chord $v^* v_2$, and it uses no vertices from $C'[v_4, v_1)$. Thus we have two shorter vertex-disjoint chorded cycles in $\langle C_1 \cup C' \rangle$, contradicting (A1). Hence Claim 12 holds. □

Claim 14. *H is connected.*

Proof. Suppose not, then $comp(H) \geq 2$. Let $H_1, H_2, \ldots, H_{comp(H)}$ be the components of H. First we prove the following subclaim.

Subclaim 15. *Suppose X is an independent set of four vertices in H such that $d_H(X) \leq 8$. Then there exists some C in $\mathscr{C}$ such that the degree sequences from four vertices of X to C are $(4,4,4,1)$, $(4,4,3,2)$ or $(4,3,3,3)$. Furthermore, then $|C| = 4$.*

Proof. By the $\sigma_4(G)$ condition, $d_{\mathscr{C}}(X) \geq (12k-3) - 8 = 12k - 11 > 12(k-1)$. Thus there exists some C in $\mathscr{C}$ such that $d_C(X) \geq 13$. By Lemma 2, $d_C(x) \leq 4$ for any $x \in X$. Now we consider degree sequences defined in Section 1 (Introduction) from four vertices of X to C. Recall that when we write (d_1, d_2, d_3, d_4), we assume $d_C(x_j) = d_j$ for each $1 \leq j \leq 4$, since it is sufficient to consider the case of equality. It follows that the degree sequences from four vertices of X to C are $(4,4,4,1)$, $(4,4,3,2)$ or $(4,3,3,3)$. Since each degree sequence contains a vertex with degree 4 in C, we have $|C| = 4$ by Lemma 2. Thus the subclaim holds. □

Now we consider the following three cases based on $comp(H)$.

Case 1. Suppose $comp(H) \geq 4$.

By Theorem 1, there exists $x_i \in V(H_i)$ for each $1 \leq i \leq 4$ such that $d_{H_i}(x_i) \leq 2$. Let $X = \{x_1, x_2, x_3, x_4\}$. Then X is an independent set and $d_H(X) \leq 8$. By Subclaim 15, the degree sequences from four vertices of X to some C in $\mathscr{C}$ are $(4,4,4,1)$, $(4,4,3,2)$ or $(4,3,3,3)$ and $|C| = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$. Without loss of generality, we may assume $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3) \geq d_C(x_4)$. Then $d_C(x_1) = 4$. Since $|C| = 4$, for each degree sequence, x_2, x_3, x_4 must all have a common neighbor in C, say v_1. Since $d_C(x_1) = 4$, $C' = x_1, v_2, v_3, v_4, x_1$ is a 4-cycle with chord x_1v_3. If x_1 is not a cut-vertex of H_1, then $H_1 - x_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $comp(H') \leq comp(H) - 2$. This contradicts (A2). Thus we may assume x_1 is a cut-vertex of H_1. Since $d_{H_1}(x_1) \leq 2$, $d_{H_1}(x_1) = 2$. Thus $comp(H_1 - x_1) = 2$, and $comp(H') \leq comp(H) - 1$ for the new H'. This contradicts (A2).

Case 2. Suppose $comp(H) = 3$.

Without loss of generality, we may assume $|H_1| \geq |H_2| \geq |H_3|$. Since $|H| \geq 18$ by Claim 12, we have $|H_1| \geq 6$. Let $P_1 = u_1, \ldots, u_s$ be a longest path in H_1. Note $s \geq 3$. By Theorem 1, there exists $x_j \in V(H_j)$ for each $j \in \{2,3\}$ such that $d_{H_j}(x_j) \leq 2$.

First suppose $u_1u_s \in E(G)$. Then $P_1[u_1, u_s], u_1$ is a Hamiltonian cycle in H_1, otherwise, since H_1 is connected, there exists a longer path than P_1, a contradiction. Since H_1 does not contain a chorded cycle, we have $u_1u_3 \notin E(H_1)$. Note $d_{H_1}(u_i) = 2$ for each $i \in \{1,3\}$. Let $X = \{u_1, u_3, x_2, x_3\}$. Then X is an independent set and $d_H(X) \leq 8$. By Subclaim 15, the degree sequences from four vertices of X to some C in $\mathscr{C}$ are $(4,4,4,1)$, $(4,4,3,2)$ or $(4,3,3,3)$ and $|C| = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_3)$. Then $d_C(u_1) \geq 3$ and $N_C(u_3) \cap N_C(x_2) \cap N_C(x_3) \neq \emptyset$ by the degree sequences. Without loss of generality, we may assume $v_1 \in N_C(u_3) \cap N_C(x_2) \cap N_C(x_3)$. Suppose $d_C(u_1) = 4$. Then $C' = u_1, v_2, v_3, v_4, u_1$ is a 4-cycle with chord u_1v_3. Since H_1 contains a Hamiltonian cycle, u_1 is not a cut-vertex of H_1. Thus $H_1 - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $comp(H') \leq comp(H) - 2 = 3 - 2 = 1$. This contradicts (A2). Thus $d_C(u_1) = 3$ since $d_C(u_1) \geq 3$. Then the degree sequence is $(4,4,3,2)$ or $(4,3,3,3)$. Without loss of generality, we may assume $d_C(x_2) \leq d_C(x_3)$. In each degree sequence, it is sufficient to consider $d_C(u_1) = 3$, $d_C(u_3) = 2$, $d_C(x_2) = 3$ and $d_C(x_3) = 4$. Without loss of generality, we may assume $v_j \in N_C(u_1)$ for each $1 \leq j \leq 3$. Then $C' = u_1, v_1, v_2, v_3, u_1$ is a 4-cycle with chord u_1v_2. If $v_4 \in N_C(x_2)$, then $v_4 \in N_C(x_2) \cap N_C(x_3)$ since $d_C(x_3) = 4$. Then $comp(H') \leq comp(H) - 1 = 3 - 1 = 2$ for the new H', a contradiction. Thus $N_C(u_1) = N_C(x_2)$. Note C has a chord. Suppose $v_1v_3 \in E(G)$. Then $C' = u_1, v_1, v_4, v_3, u_1$ is a 4-cycle with chord v_1v_3. Since $d_C(x_3) = 4$, $v_2 \in N_C(x_2) \cap N_C(x_3)$. Then $comp(H') \leq comp(H) - 1 = 3 - 1 = 2$ for the new H', a contradiction. Suppose $v_2v_4 \in E(G)$. Then $C' = u_1, v_1, v_4, v_2, u_1$ is a 4-cycle with chord v_1v_2. Since $d_C(x_3) = 4$, $v_3 \in N_C(x_2) \cap N_C(x_3)$. Then $comp(H') \leq comp(H) - 1 = 3 - 1 = 2$ for the new H', a contradiction.

Next suppose $u_1u_s \notin E(G)$. Let $X = \{u_1, u_s, x_2, x_3\}$. Since H_1 does not contain a chorded cycle, $d_{H_1}(u_i) \le 2$ for each $i \in \{1, s\}$. Then X is an independent set and $d_H(X) \le 8$. Replacing u_3 by u_s in the above case where $u_1u_s \in E(G)$, we get a similar contradiction.

Case 3. Suppose $comp(H) = 2$.

Let $|H_1| \ge |H_2|$. Since $|H| \ge 18$ by Claim 12, $|H_1| \ge 9$. Let $P_1 = u_1, \ldots, u_s$ be a longest path in H_1. Note $s \ge 3$. By Theorem 1, there exists $x_2 \in V(H_2)$ such that $d_{H_2}(x_2) \le 2$.

First suppose $u_1u_s \in E(H_1)$. Note $P_1[u_1, u_s], u_1$ is a Hamiltonian cycle in H_1. Then $X_0 = \{u_1, u_3, u_5\}$ is an independent set and $d_{H_1}(X_0) = 6$, and $X = X_0 \cup \{x_2\}$ is an independent set and $d_H(X) \le 8$. By Subclaim 15, the degree sequences from four vertices of X to some C in $\mathscr{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$. Since X_0 is on the Hamiltonian cycle, we may assume $d_C(u_1) = \max\{d_C(u) \mid u \in \{u_1, u_3, u_5\}\}$. Then $d_C(u_1) \ge 3$ by the degree sequences. Suppose $d_C(u_1) = 4$. Since $N_C(u_3) \cap N_C(x_2) \ne \emptyset$ by the degree sequences, without loss of generality, we may assume $v_4 \in N_C(u_3) \cap N_C(x_2)$. Since $d_C(u_1) = 4$, $v_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, v_1, v_2, v_3, u_1$ is a 4-cycle with chord u_1v_2. Since H_1 contains a Hamiltonian cycle, u_1 is not a cut-vertex of H_1. Thus $H_1 - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H'. This contradicts (A2). Now suppose $d_C(u_1) = 3$. Then by the maximality of $d_C(u_1)$, we have only to consider the case where $d_C(u_i) = 3$ for each $i \in \{1, 3, 5\}$, and $d_C(x_2) = 4$. Let $v_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then we may assume $N_C(u_1) = N_C(u_3) = N_C(u_5)$, otherwise, we get a contradiction by the same arguments as the case where $d_C(u_1) = 4$. Note C has a chord. Suppose $v_1v_3 \in E(G)$. Then $C' = u_1, v_1, v_4, v_3, u_1$ is a 4-cycle with chord v_1v_3. Since $d_C(x_2) = 4$, $v_2 \in N_C(u_3) \cap N_C(x_2)$. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H', a contradiction. Suppose $v_2v_4 \in E(G)$. Then $C' = u_1, v_1, v_4, v_2, u_1$ is a 4-cycle with chord v_1v_2. Since $d_C(x_2) = 4$, $v_3 \in N_C(u_3) \cap N_C(x_2)$. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H', a contradiction.

Next suppose $u_1u_s \notin E(H_1)$. Without loss of generality, we may assume $d_C(u_1) \ge d_C(u_s)$. Assume P_1 is a Hamiltonian path in H_1. Note $s \ge 9$ since $|H_1| \ge 9$. Since P_1 is a Hamiltonian path in H_1, note $d_{P_1}(u) = d_{H_1}(u)$ for any $u \in V(P_1)$. We also note $d_{P_1}(u_i) \le 2$ for each $i \in \{1, s\}$. Suppose $d_{P_1}(u_1) = 1$. By Lemma 7, $d_{H_1}(u_i) = 2$ for some $3 \le i \le 5$. Since $s \ge 9$, $X_0 = \{u_1, u_i, u_s\}$ is an independent set and $d_{H_1}(X_0) \le 6$. Thus $X = X_0 \cup \{x_2\}$ is an independent set and $d_H(X) \le 8$. Then we get a contradiction by the same arguments as the case where $u_1u_s \in E(G)$. Next suppose $d_{P_1}(u_1) = 2$. Now assume $u_1u_3 \in E(H_1)$. By Lemma 7, $d_{H_1}(u_i) = 2$ for some $4 \le i \le 6$. Since $s \ge 9$, $X_0 = \{u_1, u_i, u_s\}$ is an independent set and $d_{H_1}(X_0) \le 6$, and we get a contradiction by considering $X = X_0 \cup \{x_2\}$ similar to the case where $u_1u_s \in E(H_1)$. Thus $u_1u_3 \notin E(H_1)$, that is, $u_1u_i \in E(H_1)$ for some $4 \le i \le s - 1$. By Lemma 6, $d_{H_1}(u_{i-1}) = 2$. Since $s \ge 9$, $X_0 = \{u_1, u_{i-1}, u_s\}$ is an independent set and $d_{H_1}(X_0) \le 6$, and we get a contradiction by considering $X = X_0 \cup \{x_2\}$.

Assume P_1 is not a Hamiltonian path in H_1. Then $V(H_1 - P_1) \ne \emptyset$. Let $P_2 = v_1, \ldots, v_t$ $(t \ge 1)$ be a longest path in $H_1 - P_1$. Without loss of generality, we may assume $d_{H_1}(v_1) \le d_{H_1}(v_t)$. If $u_1u_s \in E(H_1)$, then since there exists a longer path than P_1, we may assume $u_1u_s \notin E(H_1)$. Also we may assume $d_{H_1}(v_1) \le 2$, otherwise, since $d_{P_1}(v_i) \ge 1$ for each $i \in \{1, t\}$ by Lemma 11 (iii) and (iv), there exists a cycle in $\langle P_1 \cup P_2 \rangle$ with chord incident to v_1 or v_t, a contradiction. Thus $X_0 = \{u_1, u_s, v_1\}$ is an independent set and $d_{H_1}(X_0) \le 6$. Then $X = X_0 \cup \{x_2\}$ is an independent set and $d_H(X) \le 8$. By Subclaim 15, the degree sequences from four vertices of X to some C in $\mathscr{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = w_1, w_2, w_3, w_4, w_1$. Since $d_C(u_1) \ge d_C(u_s)$ by our assumption, $d_C(u_1) \ge 3$ by the degree sequences. First suppose $d_C(u_1) = 4$. Since $N_C(v_1) \cap N_C(x_2) \ne \emptyset$ by the degree sequences, without loss of generality, we may assume $w_4 \in N_C(v_1) \cap N_C(x_2)$. Since $d_C(u_1) = 4$, $w_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, w_1, w_2, w_3, u_1$ is a 4-cycle with chord u_1w_2. Since u_1 is an endpoint of the longest path P_1, u_1 is not a cut-vertex of H_1. Thus $H_1 - u_1$ is connected. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H'. This contradicts (A2). Suppose $d_C(u_1) = 3$. Then we may assume the degree sequence is $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. In each degree sequence, it is sufficient to consider $d_C(u_1) = 3$, $d_C(u_s) = 2$, and $\{d_C(v_1), d_C(x_2)\} = \{3, 4\}$. First assume $d_C(v_1) = 3$ and $d_C(x_2) = 4$. Without loss of generality, we may assume $w_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, w_1, w_2, w_3, u_1$ is a 4-cycle with chord u_1w_2. If $w_4 \in N_C(v_1)$, then $w_4 \in N_C(v_1) \cap N_C(x_2)$ since $d_C(x_2) = 4$. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H', a contradiction. Thus $N_C(u_1) = N_C(v_1)$. Note C has a chord. Suppose $w_1w_3 \in E(G)$. Then $C' = u_1, w_1, w_4, w_3, u_1$ is a 4-cycle with chord w_1w_3. Since $d_C(x_2) = 4$, $w_2 \in N_C(v_1) \cap N_C(x_2)$. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H', a contradiction. Suppose $w_2w_4 \in E(G)$. Then $C' = u_1, w_1, w_4, w_2, u_1$ is a 4-cycle with chord w_1w_2. Since $d_C(x_2) = 4$, $w_3 \in N_C(v_1) \cap N_C(x_2)$. Then $comp(H') \le comp(H) - 1 = 2 - 1 = 1$ for the new H', a contradiction. If $d_C(v_1) = 4$ and $d_C(x_2) = 3$, then we get a contradiction, similarly.

□

Claim 16. *H contains a Hamiltonian path.*

Proof. Suppose not, and let $P_1 = u_1, \ldots, u_s$ be a longest path in H. Note $s \ge 3$ since $|H| \ge 18$ and H is connected by Claim 14. Let $P_2 = v_1, \ldots, v_t$ $(t \ge 1)$ be a longest path in $G - P_1$ such that $d_{P_1}(v_1) \le d_{P_1}(v_t)$. By Lemma 12, there exists an independent set X of four vertices in H such that $\{u_1, u_s, v_1\} \subseteq X$ and $d_H(X) \le 8$. Then the degree sequences from four vertices of X to some C in $\mathscr{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = x_1, x_2, x_3, x_4, x_1$. We may assume $u_1u_s \notin E(H)$, otherwise, a path longer than P_1 exists, a contradiction. Without loss of generality, we may assume $d_C(u_1) \ge d_C(u_s)$. By the degree sequences, we have $d_C(u_1) \ge 3$.

Suppose $d_C(u_1) = 4$. Since $N_C(u_s) \cap N_C(v_1) \ne \emptyset$ by the degree sequences, without loss of generality, we may assume $x_4 \in N_C(u_s) \cap N_C(v_1)$. Since $d_C(u_1) = 4$, we have $x_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord u_1x_2. Since u_1 is an endpoint of the longest path P_1, u_1 is not a cut-vertex of H. Thus $H - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $P_1[u_2, u_s], x_4, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3).

Suppose $d_C(u_1) = 3$. Then we may assume the degree sequence is $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. First assume the degree sequence is $(4, 4, 3, 2)$. Since $d_C(u_1) \ge d_C(u_s)$, we have $d_C(u_1) = 3$, $d_C(u_s) = 2$ and $d_C(v_1) = 4$. Without loss of generality, we may assume $x_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord u_1x_2. Note u_1 is not a cut-vertex of H. If $x_4 \in N_C(u_s)$, then since $d_C(v_1) = 4$, there exists a longer path than P_1 in the new H', a contradiction. Thus we may assume $x_4 \notin N_C(u_s)$. Note C has a chord. Suppose $x_1x_3 \in E(G)$. Assume $x_2 \in N_C(u_s)$. Then $C' = u_1, x_3, x_4, x_1, u_1$ is a 4-cycle with chord x_1x_3. Since $d_C(v_1) = 4$, $x_2 \in N_C(u_s) \cap N_C(v_1)$, and there exists a longer path than P_1 in the new H', a contradiction. Thus $x_2 \notin N_C(u_s)$. Since $d_C(u_s) = 2$, $x_1, x_3 \in N_C(u_s)$. Then $C' = u_s, x_3, x_4, x_1, u_s$ is a 4-cycle with chord x_1x_3. Note u_s is not a cut-vertex of H. Since $d_C(v_1) = 4$, $x_2 \in N_C(u_1) \cap N_C(v_1)$. Then $P_1^-[u_{s-1}, u_1], x_2, P_2[v_1, v_t]$ is a longer path than P_1 in the new H', a contradiction. Suppose $x_2x_4 \in E(G)$. Assume $x_3 \in N_C(u_s)$. Then $C' = u_1, x_1, x_4, x_2, u_1$ is a 4-cycle with chord x_1x_2. Since $d_C(v_1) = 4$, $x_3 \in N_C(u_s) \cap N_C(v_1)$. Then there exists a longer path than P_1 in the new H', a contradiction. Thus $x_3 \notin N_C(u_s)$. By symmetry, $x_1 \notin N_C(u_s)$. Thus $d_C(u_s) \le 1$. This contradicts $d_C(u_s) = 2$.

Next assume the degree sequence is $(4, 3, 3, 3)$. Since $d_C(u_1) \ge d_C(u_s)$ and $d_C(u_1) = 3$ by our assumption, we have $d_C(u_s) = 3$. Thus $d_C(v_1) \ge 3$. First assume $d_C(v_1) = 4$. Let $x_i \in N_C(u_1)$ for each $1 \le i \le 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord u_1x_2. If $x_4 \in N_C(u_s)$, then since $d_C(v_1) = 4$, there exists a longer path than P_1 in the new H', a contradiction. Thus $N_C(u_1) = N_C(u_s)$. Suppose $x_1x_3 \in E(G)$. Then $C' = u_1, x_1, x_4, x_3, u_1$ is a 4-cycle with chord x_1x_3. Since $d_C(v_1) = 4$, $x_2 \in N_C(u_s) \cap N_C(v_1)$. Then there exists a longer path than P_1 in the new H', a contradiction. Suppose $x_2x_4 \in E(G)$. Then $C' = u_1, x_1, x_4, x_2, u_1$ is a 4-cycle with chord x_1x_2. Since $d_C(v_1) = 4$, $x_3 \in N_C(u_s) \cap N_C(v_1)$. Then there exists a longer path than P_1 in the new H', a contradiction.

Next assume $d_C(v_1) = 3$. Recall $d_C(u_1) = d_C(u_s) = 3$. Then $|N_C(u_s) \cap N_C(v_1)| \ge 2$. Let $x_i \in N_C(u_1)$ for each $1 \le i \le 3$. Suppose $x_1x_3 \in E(G)$. If $x_i \in N_C(u_s) \cap N_C(v_1)$ for some $i \in \{2, 4\}$, then there exists a longer path than P_1, a contradiction. Thus $x_1, x_3 \in N_C(u_s) \cap N_C(v_1)$. Suppose $x_4 \in N_C(u_s)$ and

$x_2 \in N_C(v_1)$. Then $C' = u_s, x_4, x_1, x_3, u_s$ is a 4-cycle with chord x_3x_4, and $P_1^-[u_{s-1}, u_1], x_2, P_2[v_1, v_t]$ is a longer path than P_1 in the new H', a contradiction. Suppose $x_2 \in N_C(u_s)$ and $x_4 \in N_C(v_1)$. Let $w \in X - \{u_1, u_s, v_1\}$. Since we assume the degree sequence is $(4, 3, 3, 3)$, we have $d_C(w) = 4$. Assume $w \in V(P_1)$. Then $P_1[u_1, u_s], x_2, u_1$ is a cycle with chord wx_2, and v_1, x_1, x_4, x_3, v_1 is the other cycle with chord x_1x_3. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C\rangle$, and G contains k vertex-disjoint chorded cycles, a contradiction. Assume $w \notin V(P_1)$. Then $C' = u_s, x_3, x_4, x_1, u_s$ is a 4-cycle with chord x_1x_3. Since $d_C(w) = 4$, $w, x_2, P_1[u_1, u_{s-1}]$ is a longer path than P_1 in the new H', a contradiction. Suppose $x_2x_4 \in E(G)$. Note $|N_C(u_s) \cap N_C(v_1)| \geq 2$. If $x_i \in N_C(u_s) \cap N_C(v_1)$ for some $i \in \{1, 3, 4\}$, then there exists a longer path than P_1, a contradiction. Thus $|N_C(u_s) \cap N_C(v_1)| \leq 1$, a contradiction. □

By Claims 10, 16 and Lemma 10, H contains an independent set X of four vertices such that $d_H(X) \leq 8$. By Claim 16 and Lemma 13,

$$d_G(X) = d_{\mathscr{C}}(X) + d_H(X) \leq 12(k-1) + 8 = 12k - 4.$$

This contradicts the $\sigma_4(G)$ condition. This completes the proof of Theorem 5. □

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