Journal of Combinatorial Mathematics and Combinatorial Computing, 121: 107–112
DOI:10.61091/jcmcc121-11
http://www.combinatorialpress.com/jcmcc
Published 30 June 2024

Combinatorial Press

Article

On Vertex Euclidean Deficiency of One-Point Union and One-Edge Union of Complete Graphs

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Abstract: A (p, g)-graph G is Euclidean if there exists a bijection $f : V \to \{1, 2, \ldots, p\}$ such that for any induced C_3-subgraph $\{v_1, v_2, v_3\}$ in G with $f(v_1) < f(v_2) < f(v_3)$, we have that $f(v_1) + f(v_2) > f(v_3)$. The Euclidean Deficiency of a graph G is the smallest integer k such that $G \cup N_k$ is Euclidean. We study the Euclidean Deficiency of one-point union and one-edge union of complete graphs.

Keywords: Euclidean Deficiency, Complete graph

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1. Introduction

Definition 1. *A (p, g)-graph G is Euclidean if there exists a bijection $f : V \to \{1, 2, \ldots, p\}$ such that for any induced C_3-subgraph $\{v_1, v_2, v_3\}$ in G with $f(v_1) < f(v_2) < f(v_3)$, we have that*

$$f(v_1) + f(v_2) > f(v_3).$$

Let Euclid(0) be the set of all Euclidean graphs.

Example 1. *An Euclidean graph with 5 vertices:*

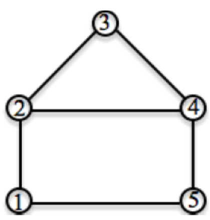

Figure 1

Example 2. *All triangle free graphs are Euclidean.*

Example 3. *Euclidean (p, g)-graphs with $p = 1, 2, \ldots, 6$*

An immediate observation is that C_3 is not Euclidean, for the vertices will be labeled with 1, 2, 3, but $1 + 2 \not> 3$. This observation can also be extended: the label 1 cannot be used on any vertices *contained in some C_3 subgraph. For the sake of contradiction, assume this C_3 subgraph is labeled $1, x, y$ where $1 < x < y$, then $1 + x \leq y$ and hence $1 + x \not> y$ for any integer $y > x$. It follows*

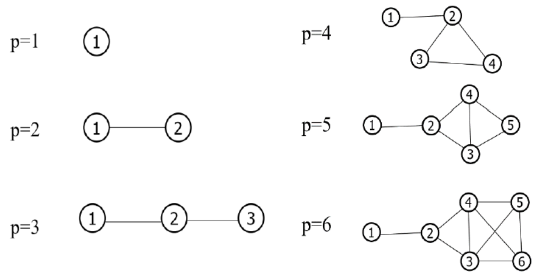

Figure 2

That if all vertices are part of some C_3 subgraph then, the graph must necessarily not be Euclidean.

Definition 2. *The Euclidean deficiency of a (p, g)-graph G is $\min\{k : G \cup N_k \in Euclid(0)\}$, where N_k is the null graph with k vertices, and we'll denote this number by $\mu(G)$. For a given $k > 0$, let $Euclid(k)$ be the set of all graphs with Euclidean deficiency k.*

Example 4. *Graphs with Euclidean deficiency 1:*

Theorem 1. *Let $n \geq 3$ and K_n be the complete graph of order n, then $\mu(K_n) = n - 2$.*

Proof. Observe letting the vertices having labels $n - 1, n, \ldots, 2n - 1$ works, since $n - 1, n$ are the smallest labels, it follows that, for label of any two vertices v_1, v_2,

$$f(v_1) + f(v_2) \geq n - 1 + n = 2n - 1,$$

greater than all other labels. This establishes that $\mu(K_n) \leq n - 2$. On the other hand, let $x, x + 1$ be the smallest two labels, then the largest label is $x + n$ and since the graph is complete, the vertices containing these labels form a C_3 subgraph, therefore

$$x + x + 1 > x + n,$$

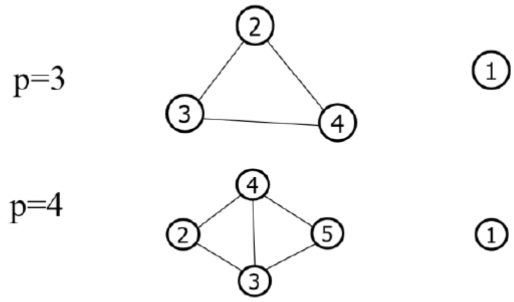

Figure 3

or equivalently, $x > n - 1$. This establishes that $\mu(K_n) \geq n - 2$, and therefore $\mu(K_n) = n - 2$. □

Theorem 2. *If $H \in Euclid(k)$ and $H \subseteq G$ such that*

1. *$G \setminus H$ is triangular free,*
2. *$|V(G \setminus H)| \geq k$,*

then $G \in Euclid(0)$.

Proof. Instead of NK, we can use the vertices of $G \setminus H$ and the result follows. □

2. Construction of graphs

Definition 3. *Let v_1, v_2 be vertices of graphs G_1, G_2, respectively, then the one-point union,*

$$OU((G_1, \{v_1\}), (G_2, \{v_2\})),$$

is the disjoint union G_2 to G_1 then v_2 attaches to v_1.

Example 5. *We'll now look at the Euclid deficiency of one-point union of complete graphs. First, looking at $OU(K_3, K_n)$ for $n \geq 3$ we see that*

$$OU(K_3, K_n) \in \begin{cases} Euclid(1) & n-4, \quad 3 \leq n \leq 5 \\ Euclid(n-4) & n \geq 6. \end{cases}$$

By testing out $OU(K_4, K_n)$ for $n \geq 4$, we have that

$$OU(K_4, K_n) \in \begin{cases} Euclid(2) & n-5, \quad 4 \leq n \leq 7 \\ Euclid(n-5) & n \geq 8. \end{cases}$$

Testing out similar cases, we see that

$$OU(K_5, K_n) \in \begin{cases} Euclid(3) & n-6, \quad 5 \leq n \leq 9 \\ Euclid(n-6) & n \geq 10, \end{cases}$$

$$OU(K_6, K_n) \in \begin{cases} Euclid(4) & n-7, \quad 6 \leq n \leq 11 \\ Euclid(n-7) & n \geq 12, \end{cases}$$

$$OU(K_7, K_n) \in \begin{cases} Euclid(5) & n-8, \quad 7 \leq n \leq 13 \\ Euclid(n-8) & n \geq 14. \end{cases}$$

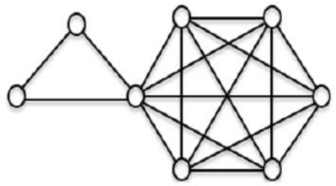

Figure 4

In general, we have that

Theorem 3. *For $m \leq n$,*

$$OU(K_m, K_n) \in \begin{cases} Euclid(m-2) & n-m-1, \quad n \leq 2m-1 \\ Euclid(n-m-1) & n \geq 2m. \end{cases}$$

Proof. Let $G = OU(K_m, K_n)$ and $a = \mu(G)$. We'll label the vertices on K_m with $a+1, a+2, \ldots, a+m$ with $a+m$ at the common vertex, and $a+m, a+m+1, \ldots, a+m+n-1$ on the K_n graph.

If $n \leq 2m-1$ then from the subgraph K_m consisting of $a+1, a+2, a+m$, we have that

$$a+1+a+2 > a+m$$

or $a \geq m-2$. Direct calculation of with $a = m-2$ on the C_3 subgraphs of G consisting of the smallest, second smallest and largest labels in K_n and K_m respectively:

$$a+1+a+2 = 2m-1 > 2m-2 = a+m$$

and

$$a + m + a + m + 1 = 4m - 3 = (2m - 1) + (2m - 2) \geq n + a + m > a + m + n - 1,$$

shows that $\mu(G) = m - 2$.

If $n \geq 2m$ then from the subgraph of K_n consisting of $a + m, a + m + 1, a + m + n - 1$, we have that

$$a + m + a + m + 1 > a + m + n - 1,$$

or equivalently $a \geq n - m - 1$. Direct calculation of with $a = n - m - 1$ on the C_3 subgraphs of G consisting of the smallest, second smallest and largest labels in K_n and K_m respectively:

$$a + 1 + a + 2 = 2n - 2m + 1 \geq n + 1 > n - 1 = a + m$$

and

$$a + m + a + m + 1 = 2n - 1 > 2n - 2 = a + m + n - 1,$$

shows that $\mu(G) = n - m - 1$. □

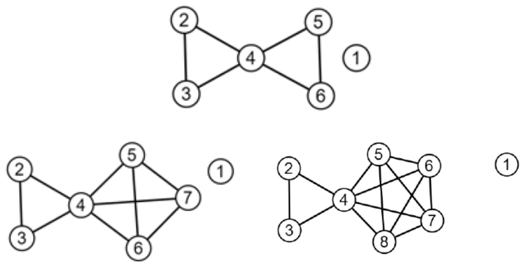

Figure 5

Definition 4. *Let e_1, e_2 be edges of graphs G_1, G_2, respectively, then the one-edge union,*

$$OE((G_1, \{v_1\}), (G_2, \{v_2\})),$$

is the disjoint union G_2 to G_1 then collapse e_2 to e_1.

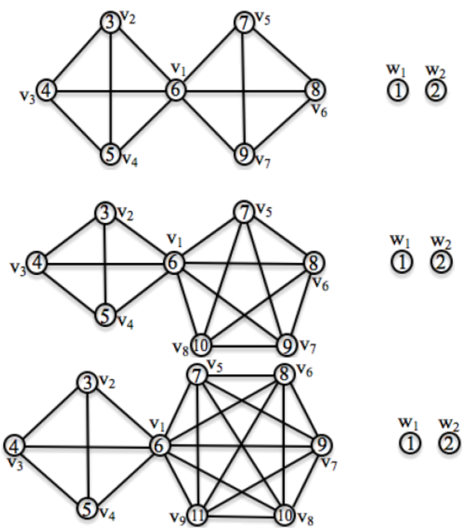

Figure 6

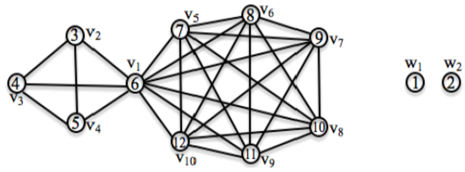

Figure 7

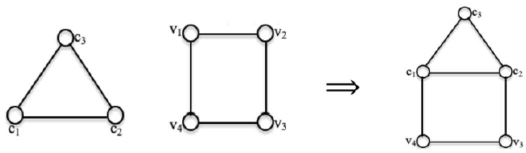

Figure 8

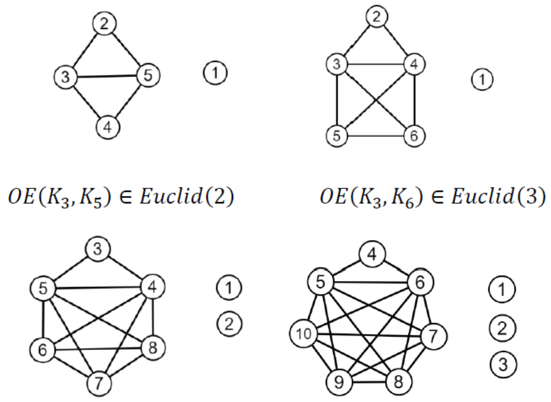

Figure 9

This construction is dual of one-point union of graphs.

Example 6.

$$OE((C_3,(c_1,c_2)),(C_4,(v_1,v_2))).$$

We'll now look at the Euclid deficiency of one-edge union of complete graphs. First, looking at $OE(K_3, K_n)$ for $n \geq 3$ we see that

$$OE(K_3, K_3) \in Euclid(1)$$

$$OE(K_3, K_5) \in Euclid(2)$$

or in general,

$$OE(K_3, K_4) \in Euclid(1)$$

$$OE(K_3, K_6) \in Euclid(3)$$

$$OE(K_3, K_n) \in \begin{cases} Euclid(1) & 3 \leq n \leq 4 \\ Euclid(n-3) & n \geq 5. \end{cases}$$

By testing out $OE(K_4, K_n)$ for $n \geq 4$, we have that

$$OE(K_4, K_4) \in Euclid(2)$$

$$OE(K_4, K_6) \in Euclid(2)$$

or in general,

$$OE(K_4, K_5) \in Euclid(2)$$

$$OE(K_4, K_7) \in Euclid(3)$$

$$OE(K_4, K_n) \in \begin{cases} Euclid(2) & 4 \leq n \leq 6 \\ Euclid(n-4) & n \geq 7. \end{cases}$$

Testing out similar cases, we see that

$$OE(K_5, K_n) \in \begin{cases} Euclid(3) & 5 \leq n \leq 8 \\ Euclid(n-5) & n \geq 9, \end{cases}$$

$$OE(K_6, K_n) \in \begin{cases} Euclid(4) & 6 \leq n \leq 10 \\ Euclid(n-6) & n \geq 12, \end{cases}$$

$$OE(K_7, K_n) \in \begin{cases} Euclid(5) & 7 \leq n \leq 12 \\ Euclid(n-7) & n \geq 13. \end{cases}$$

In general, we have that

Theorem 4. *For $m \leq n$,*

$$OU(K_m, K_n) \in \begin{cases} Euclid(m-2) & n \leq 2m-2 \\ Euclid(n-m) & n \geq 2m-1. \end{cases}$$

Proof. The proof is similar to that of Theorem 3. □

Acknowledgment

We appreciate Prof. Harris Kwong for his critical and helpful suggestions on this paper.

Conflict of interest

The author declares no conflict of interest.

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