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Article

Signed Magic Rectangles With Two Filled Cells in Each Column

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Abstract: A *signed magic rectangle* $SMR(m, n; r, s)$ is an $m \times n$ array with entries from X, where $X = \{0, \pm 1, \pm 2, \ldots, \pm(ms-1)/2\}$ if mr is odd and $X = \{\pm 1, \pm 2, \ldots, \pm mr/2\}$ if mr is even, such that precisely r cells in every row and s cells in every column are filled, every integer from set X appears exactly once in the array and the sum of each row and of each column is zero. In this paper we prove that a signed magic rectangle $SMR(m, n; r, 2)$ exists if and only if $m = 2$ and $n = r \equiv 0, 3 \pmod 4$ or $m, r \geq 3$ and $mr = 2n$.

Keywords: Graph, Signed magic rectangle

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1. Introduction

A *magic rectangle* of order $m \times n$, $MR(m, n)$, is an arrangement of the numbers from 0 to $mn - 1$ in an $m \times n$ rectangle such that each number occurs exactly once in the rectangle and the sum of the entries of each row is the same and the sum of entries of each column is also the same. The following theorem, whose proof can be found in [1, 2] and [3], settles the existence of an $MR(m, n)$.

Theorem 1. *An $m \times n$ magic rectangle exists if and only if $m \equiv n \pmod 2$, $m + n > 5$, and $m, n > 1$.*

A *k-magic square* of order n is an arrangement of the numbers from 0 to $kn - 1$ in an $n \times n$ array such that each row and each column has exactly k filled cells, each number occurs exactly once in the array, and the sum of the entries of any row or any column is the same. The study of magic squares with empty cells was initiated in [4]. A magic square is called *k-diagonal* if its entries all belong to k consecutive diagonals (this includes broken diagonals as well).

Theorem 2. [4] *There exists a k-diagonal magic square of order n if and only if $n = k = 1$ or $3 \leq k \leq n$ and either n is odd or k is even.*

A *signed magic rectangle* $SMR(m, n; r, s)$ is an $m \times n$ array with entries from X, where $X = \{0, \pm 1, \pm 2, \ldots, \pm(ms-1)/2\}$ if mr is odd and $X = \{\pm 1, \pm 2, \ldots, \pm mr/2\}$ if mr is even, such that precisely r cells in every row and s cells in every column are filled, every integer from set X appears exactly once in the array and the sum of each row and of each column is zero. By the definition, $mr = ns$, $r \leq n$ and $s \leq m$. If $r = n$ or $s = m$, then the rectangle has no empty cell. In the case where $m = n$, we call the array a *signed magic square*. Signed magic squares represent a type of magic square where each number from the set X is used once.

The following two theorems can be found in [5].

Theorem 3. *An $SMR(m,n)$ exists precisely when $m=n=1$, or when $m=2$ and $n \equiv 0,3 \pmod 4$, or when $n=2$ and $m \equiv 0,3 \pmod 4$, or when $m,n>2$.*

In [5] the notation $SMS(n;k)$ is used for a signed magic square with k filled cells in each row and k filled cells in each column.

Theorem 4. *There exists an $SMS(n;k)$ precisely when $n=k=1$ or $3 \leq k \leq n$.*

In this paper we prove that a signed magic rectangle $SMR(m,\ n;r,2)$ exists if and only if $m=2$ and $n=r \equiv 0,3 \pmod 4$ or $m,r \geq 3$ and $mr=2n$.

Being the smallest poset that contains G, $\langle G \rangle$ is called the principle ideal generated G, which we refer to as a *graph ideal*. So, we can describe $\downarrow G$ as a union of graph ideals. However, the use of order theory here is not superficial. Our main method for determining the down-arrow Ramsey set relies on viewing red-blue colorings of a graph as unions of graph ideals.

2. Main Constructions

A rectangular array is *shiftable* if it contains the same number of positive entries as negative entries in every column and in every row. Figure 1 displays a shiftable $SMR(2,4;4,2)$. These arrays are called *shiftable* because they may be shifted to use different absolute values. By increasing the absolute value of each entry by k, we add k to each positive entry and $-k$ to each negative entry. If the number of entries in a row is 2ℓ, this means that we add $\ell k+\ell(-k)=0$ to each row, and the same argument applies to the columns. Thus, when shifted, the array retains the same row and column sums.

1	−2	−3	4
−1	2	3	−4

Figure 1. A Shiftable $SMR(2,4;4,2)$

Theorem 5. *Let there exist a shiftable $SMR(m,n;r,s)$. Then for every $k \geq 1$*

1. *there exists a shiftable $SMR(m,kn;kr,s)$ and*
2. *there exists a shiftable $SMR(km,kn;r,s)$.*

Proof. Let A be a shiftable $SMR(m,n;r,s)$. Note that since A is shiftable, it follows that r and s are both even. Partition an empty $m \times kn$ rectangle, say B, into k empty rectangles of size $m \times n$, say P_ℓ, where $0 \leq \ell \leq k-1$. For each $(i,j;e) \in A$ we fill the cell (i,j) of P_ℓ with $e+\ell(mr/2)$ if e is positive or with $e-\ell(mr/2)$ if e is negative. The resulting rectangle is a shiftable $SMR(m,kn;kr,s)$. See Figure 2.

1	−2	−3	4	5	−6	−7	8	9	−10	−11	12
−1	2	3	−4	−5	6	7	−8	−9	10	11	−12

Figure 2. A Shiftable $SMR(2,12;12,2)$

□

Theorem 6. *Let there exist a shiftable $SMR(m,n;r,s)$ and a (shiftable) $SMR(m,n';r',s)$ with mr' even. Then there exists a (shiftable) $SMR(m,kn+n';kr+r',s)$ for $k \geq 1$.*

1	-2	-3	4								
-1	2	3	-4								
				5	-6	-7	8				
				-5	6	7	-8				
								9	-10	-11	12
								-9	10	11	-12

Figure 3. A Shiftable $SMR(6, 12; 4, 2)$

Proof. Apply Part 1 of Theorem 5 with a shiftable $SMR(m, n; r, s)$ to obtain a shiftable $SMR(m, kn; kr, s)$, say A, for $k \geq 1$. Let B be a (shiftable) $SMR(m, n'; r', s)$ and let C be the $m \times kn$ rectangle obtained from A by adding $mr'/2$ to each positive entry of A and subtracting $mr'/2$ from each negative entry of A. Finally, let D be the $m \times (kn + n')$ rectangle obtained from B and C as follows: if $(i, j; e) \in B$, then $(i, j; e) \in D$ and if $(i, j; e) \in C$, then $(i, j + n'; e) \in D$. It is easy to see that D is a (shiftable) $SMR(m, kn + n'; kr + r', s)$. □

Figure 4 displays an $SMR(2, 11; 11, 2)$ obtained by the construction given in the proof of Theorem 6 using the shiftable $SMR(2, 4; 4, 2)$ given in Figure 1, an $SMR(2, 3; 3, 2)$ and $k = 2$.

-1	-2	3	-4	5	6	-7	-8	9	10	-11
1	2	-3	4	-5	-6	7	8	-9	-10	11

Figure 4. An $SMR(2, 11; 11, 2)$

Theorem 7. *Let there exist a shiftable $SMR(m, n; r, s)$ and a (shiftable) $SMR(m', n'; r, s)$ with $m'r$ even., then there exists a (shiftable) $SMR(km + m', kn + n'; r, s)$ for $k \geq 1$.*

Proof. Apply Part 2 of Theorem 5 with a shiftable $SMR(m, n;\ r, s)$ to obtain a shiftable $SMR(km, kn; r, s)$, say A, for $k \geq 1$. Let B be a (shiftable) $SMR(m', n'; r, s)$ and let C be the $m \times kn$ rectangle obtained from A by adding $m'r/2$ to each positive entry of A and subtracting $m'r/2$ from each negative entry of A. Finally, let D be the $(km + m') \times (kn + n')$ rectangle obtained from B and C as follows: if $(i, j; e) \in B$, then $(i, j; e) \in D$ and if $(i, j; e) \in C$, then $(i + m', j + n'; e) \in D$. It is easy to see that D is a (shiftable) $SMR(km + m', kn + n'; r, s)$. □

Figure 5 displays a shiftable $SMR(7, 14; 4, 2)$ obtained by the construction given in the proof of Theorem 7 using the shiftable $SMR(2, 4; 4, 2)$ given in Figure 1, the shiftable $SMR(3, 6; 4, 2)$ given in Figure 12, and $k = 2$.

1		-3	-4		6								
-1	2		4	-5									
	-2	3		5	-6								
						-7	8	9	-10				
						7	-8	-9	10				
										-11	12	13	-14
										11	-12	-13	14

Figure 5. A Shiftable $SMR(7, 14; 4, 2)$

3. The Existence of an $SMR(m, 3m/2; 3,\ 2)$ and an $SMR(m, 5m/2; 5, 2)$

In this section we present direct constructions for the existence of an $SMR(m, 3m/2; 3, 2)$, where $m \geq 2$ and even, and an $SMR(m, 5m/2; 5, 2)$, where $m \geq 4$ and even. We will make use of these results in Section 4. Note that if m is odd there is no $SMR(m, 3m/2; 3,\ 2)$ because $3m$ is odd and there is no $SMR(m, 5m/2; 5, 2)$ because $5m$ is odd.

Proposition 1. *There exists an $SMR(m, 3m/2; 3, 2)$ for m even and $m \geq 2$.*

Proof. Define an $m \times 3$ rectangle A as follows.

Column 1: $\begin{cases} (i, 1; i) \in A \text{ for } 1 \leq i \leq m/2, \\ (i, 1; (m/2) - i) \in A \text{ for } (m/2) + 1 \leq i \leq m. \end{cases}$

Column 2: $\begin{cases} (i, 2; (3m/2) - 2i + 1) \in A \text{ for } 1 \leq i \leq m/2, \\ (i, 2; -i) \in A \text{ for } (m/2) + 1 \leq i \leq m. \end{cases}$

Column 3: $\begin{cases} (i, 3; (-3m/2) + i - 1) \in A \text{ for } 1 \leq i \leq m/2, \\ (i, 3; (-m/2) + 2i) \in A \text{ for } (m/2) + 1 \leq i \leq m. \end{cases}$

By construction, it is easy to see that the entries in A consist of $\{\pm 1, \pm 2, \ldots, \pm 3m/2\}$, which are the numbers in an $SMR(m,\ 3m/2; 3, 2)$. Figure 6 displays the rectangle A when $m = 8, 10$. We now prove that the sum of each row of A is zero. The row sum for row i of A, where $1 \leq i \leq m/2$, is

$$i + ((3m/2) - 2i + 1) + ((-3m/2) + i - 1) = 0.$$

Similarly, the row sum for row i of A, where $(m/2) + 1 \leq i \leq m$, is

$$((m/2) - i) + (-i) + ((-m/2) + 2i) = 0.$$

Let a, k and $-k$ be the numbers in a row of A. Then $a + k + (-k) = 0$, which implies that $a = 0$. Since zero does not appear in A, it follows that the numbers k and $-k$ do appear in the same row of A.

Now let B be an empty $m \times 3m/2$ rectangle. For each $(i, j; k) \in A$ let $(i, |k|; k) \in B$. By construction, the numbers in row i of B are precisely the numbers in row i of A. Therefore the row sum for each row of B is also zero. Since $\pm k$ are entries of A for each $1 \leq k \leq 3m/2$, it follows that column k of B contains only k and $-k$. Hence, B is an $SMR(m, 3m/2; 3, 2)$ for m even and $m \geq 2$. □

Figure 7 displays an $SMR(8, 12; 3, 2)$ obtained by the construction given in Proposition 1.

1	11	−12
2	9	−11
3	7	−10
4	5	−9
−1	−5	6
−2	−6	8
−3	−7	10
−4	−8	12

Array A when $m = 8$

1	14	−15
2	12	−14
3	10	−13
4	8	−12
5	6	−11
−1	−6	7
−2	−7	9
−3	−8	11
−4	−9	13
−5	−10	15

Array A when $m = 10$

Figure 6. Array A Given in Proposition 1

It is an easy exercise to see that there is no $SMR(2, 5; 5, 2)$. The following proposition shows how to build an $SMR(m, 5m/2; 5, 2)$ for m even and $m \geq 4$.

1										11	−12
	2							9		−11	
		3				7			−10		
			4	5				−9			
−1				−5	6						
	−2				−6		8				
		−3				−7			10		
			−4				−8				12

Figure 7. An $SMR(8, 12; 3, 2)$

Proposition 2. *There exists an $SMR(m, 5m/2; 5, 2)$ for m even and $m \geq 4$.*

Proof. Define a $m \times 5$ rectangle C as follows.

Column 1: $\begin{cases} (i, 1; i) \in C \text{ for } 1 \leq i \leq m/2, \\ (i, 1; (m/2) - i) \in C \text{ for } \frac{m+2}{2} \leq i \leq m. \end{cases}$

Column 2: $\begin{cases} (i, 2; (m/2) + 2i - 1) \in C \text{ for } 1 \leq i \leq m/2, \\ (i, 2; (-3m/2) + i - 1) \in C \text{ for } \frac{m+2}{2} \leq i \leq m. \end{cases}$

Column 3: $\begin{cases} (i, 3; (-m) - i) \in C \text{ for } 1 \leq i \leq m/2, \\ (i, 3; (5m/2) - 2i + 2) \in C \text{ for } \frac{m+2}{2} \leq i \leq m. \end{cases}$

Column 4: $\begin{cases} (i, 4; (-3m/2) - i) \in C \text{ for } 1 \leq i \leq (m/2), \\ (i, 4; (3m/2) + i) \in C \text{ for } \frac{m+2}{2} \leq i \leq m. \end{cases}$

Column 5: $\begin{cases} (i, 5; 2m - i + 1) \in C \text{ for } 1 \leq i \leq m/2, \\ (i, 5; -3m + i - 1) \in C \text{ for } \frac{m+2}{2} \leq i \leq m. \end{cases}$

By construction, the entries in C consist of $\{\pm 1, \ldots, \pm 5m/2\}$, which are the numbers in an $SMR(m, 5m/2; 5, 2)$. Figure 8 displays the rectangle C when $m = 8$. We now prove that the sum of each row of C is zero. The row sum for row i of C, where $1 \leq i \leq m/2$, is

$$i + ((m/2) + 2i - 1) + (-m - i) + ((-3m/2) - i) + (2m - i + 1) = 0.$$

Similarly, the row sum for row i of C, where $(m/2) + 1 \leq i \leq m$, is

$$\begin{aligned} ((m/2) - i) + ((-3m/2) + i - 1) + (5m/2) - 2i + 2) \\ +((3m/2) + i) + (-3m + i - 1) = 0. \end{aligned}$$

Let a, b, c, d, e be the numbers in row i and columns $1, 2, 3, 4, 5$ of C, respectively. It is straightforward to see that if $x, y \in \{a, b, c\}$ and $z \in \{d, e\}$, then $x + y \neq 0$ and $x + z \neq 0$. Now let $d + e = 0$. If $1 \leq i \leq m/2$, then

$$d + e = ((-3m/2) - i) + (2m - i + 1) = (m/2) - 2i + 1 = 0.$$

This implies that $i = (m + 2)/4$.
If $(m/2) + 1 \leq i \leq m$, then

$$d + e = ((3m/2) + i) + (-3m + i - 1) = (-3m/2) + 2i - 1 = 0.$$

This implies that $i = (3m + 2)/4$.

Therefore if $m \equiv 0 \pmod 4$, then the numbers k and $-k$ do not appear in the same row of C. If $m \equiv 2 \pmod 4$ and $i \neq (m+2)/2, (3m+2)/4$, then the numbers k and $-k$ do not appear in row i of C.

When $m \equiv 2 \pmod 4$ we construct an $m \times 5$ array C' by rearranging the eight entries of C which are in the intersection of columns 1 and 2 with rows $(m-2)/2, (m+2)/2, (3m-2)/4$ and $(3m+2)/4$ as follows. Switch

$((m-2)/4, 1; (m-2)/4)$ and $(m+2)/4, 1; (m+2)/4)$,
$((m-2)/4, 5; (7m+6)/4)$ and $((m+2)/4, 5; (7m+2)/4)$,
$((3m-2)/4, 1; (-m+2)/4)$ and $(3m+2)/4, 1; (-m-2)/4)$,
and $((3m-2)/4, 5; (-9m-6)/4)$ and $((3m+2)/4, 5;$
$(-9m-2)/4)$.

Figure 8 displays the rectangle C' when $m = 10$. It is easy to see that the sum of each row of C' is zero and k and $-k$ do not appear in any row of C'.

Now let $m \equiv 0 \pmod 4$, $m \geq 4$, and let D be an empty $m \times 5m/2$ rectangle. For each $(i, j; k) \in C$ let $(i, |k|; k) \in D$. By construction, the numbers in row i of D are precisely the numbers in row i of C. Therefore the row sum for each row of D is also zero. Since $\pm k$ are entries of C for each $1 \leq k \leq 5m/2$, it follows that column k of D contains only k and $-k$. Hence, D is an $SMR(m, 5m/2; 5, 2)$.

Similarly, if $m \equiv 2 \pmod 4$ and $m \geq 6$, we use the array C' to build an $SMR(m, 5m/2; 5, 2)$. □

1	5	−9	−13	16
2	7	−10	−14	15
3	9	−11	−15	14
4	11	−12	−16	13
−1	−8	12	17	−20
−2	−7	10	18	−19
−3	−6	8	19	−18
−4	−5	6	20	−17

Array C when $m = 8$

1	6	−11	−16	20
3	8	−12	−17	18
2	10	−13	−18	19
4	12	−14	−19	17
5	14	−15	−20	16
−1	−10	15	21	−25
−3	−9	13	22	−23
−2	−8	11	23	−24
−4	−7	9	24	−22
−5	−6	7	25	−21

Array C' when $m = 10$

Figure 8. Arrays C and C' Constructed by Proposition 2

4. The Existence of an $SMR(m, n; r, 2)$ with m Even

Let there exist an $SMR(m, n; r, 2)$. If $m = 4b$ or $m = 4b + 2$, then $n = 2br$ or $n = (2b + 1)r$, respectively. We study the existence of an $SMR(4b, 2br; r, 2)$ and an $SMR(4b+2, (2b+1)r; r, 2)$ in the following two subsections, respectively.

4.1. The Existence of an $SMR(4b, 2br; r, 2)$

In this subsection we construct signed magic rectangles with parameters $(4b, 8ab; 4a, 2)$, $(4b, 2b(4a+2); 4a+2, 2)$, $(4b, 2b(4a+1); 4a+1, 2)$, and $(4b, 2b(4a+3); 4a+3, 2)$, where $a, b \geq 1$.

Lemma 1. *There exists a shiftable $SMR(2q, 4pq; 4p, 2)$ for positive integers $p, q \geq 1$.*

Proof. Figure 1 displays a shiftable $SMR(2, 4; 4, 2)$. So by Part 1 of Theorem 5, there exists a shiftable $SMR(2, 4p; 4p, 2)$ for $p \geq 1$. Now by Part 2 of Theorem 5 there exists a shiftable $SMR(2q, 4pq; 4p, 2)$ for $p, q \geq 1$. □

Lemma 2. *There exists a shiftable $SMR(4b, 8ab; 4a, 2)$ for a, $b \geq 1$.*

Proof. Apply Lemma 1 with $p = a$ and $q = 2b$ to obtain a shiftable $SMR(4b, 8ab; 4a, 2)$ for all $a, b \geq 1$. □

Lemma 3. *There exists a shiftable $SMR(4b, 2b(4a + 2); 4a + 2, 2)$ for $a, b \geq 1$.*

Proof. Figure 9 displays a shiftable $SMR(4, 12; 6, 2)$. So by Part 2 of Theorem 5, there exists a shiftable $SMR(4b, 12b; 6, 2)$, say A, for $b \geq 1$. On the other hand, by Lemma 2, there exists a shiftable $SMR(4b, 8(a-1)b; 4(a-1), 2)$, say B, for $a \geq 2$ and $b \geq 1$. Now apply Theorem 6 with A and B to obtain a shiftable $SMR(4b, 2b(4a + 2); 4a + 2, 2)$ for $a, b \geq 1$. □

−1	2			−5	6			9	−11		
1	−2			5	−6			−9	11		
		−3	4			−7	8			10	−12
		3	−4			7	−8			−10	12

Figure 9. A Shiftable $SMR(4, 12; 6, 2)$

Lemma 4. *There exists a shiftable $SMR(4b, 2b(4a + 1); 4a + 1, 2)$ for $a, b \geq 1$.*

Proof. By Proposition 2, there exists an $SMR(4b, 10b; 5, 2)$, say A, for $b \geq 1$. On the other hand, by Lemma 2, there exists a shiftable $SMR(4b, 8(a-1)b; 4(a-1), 2)$, say B, for $a \geq 2$ and $b \geq 1$. Now apply Theorem 6 with A and B to obtain an $SMR(4b, 2b(4a + 1); 4a + 1, 2)$ for $a \geq 2$ and $b \geq 1$. When $a = 1$ we apply Proposition 2. □

Lemma 5. *There exists a shiftable $SMR(4b, 2b(4a + 3); 4a + 3, 2)$ for $a, b \geq 1$.*

Proof. By Proposition 1, there exists an $SMR(4b, 6b; 3, 2)$, say A, for $b \geq 1$. On the other hand, by Lemma 2, there exists a shiftable $SMR(4b, 8ab; 4a, 2)$, say B, for $a, b \geq 1$. Now apply Theorem 6 with A and B to obtain $SMR(4b, 2b(4a + 3); 4a + 3, 2)$ for $a, b \geq 1$. □

4.2. The Existence of an $SMR(4b + 2, (2b + 1)r; r, 2)$

In this subsection we construct signed magic rectangles with parameters $(4b + 2, 2a(4b + 2); 4a, 2)$, $(4b+2, (2a+1)(4b+2); 4a+2, 2)$, $(4b+2, (4a+1)(2b+1); 4a+1, 2)$, and $(4b+2, (4a+3)(2b + 1); 4a + 3, 2)$ for all $a, b \geq 1$.

Lemma 6. *Let $n \equiv 3 \pmod 4$. Then there exists an $SMR(2, n;\, n, 2)$.*

Proof. By Lemma 1, there exists a shiftable $SMR(2, 4k; 4k, 2)$, say A, for $k \geq 1$. Let B be a 2×3 array with first row $1, 2, -3$ and second row $-1, -2, 3$. Then B is an $SMR(2, 3; 3, 2)$. Now apply Theorem 6 with A and B to obtain an $SMR(2, 4k + 3; 4k + 3, 2)$. See Figure 4. □

Lemma 7. *There exists a shiftable $SMR(4b + 2, 2a(4b + 2); 4a, 2)$ for $a, b \geq 1$.*

Proof. Apply Lemma 1 with $p = a$ and q=$2b + 1$ to obtain a shiftable $SMR(4b + 2, 2a(4b + 2); 4a, 2)$ for $a, b \geq 1$. □

Lemma 8. *There exists a shiftable $SMR(4b + 2, 3(4b + 2); 6, 2)$ for $b \geq 1$*

Proof. Apply Part 2 of Theorem 5 with the shiftable $SMR(4, 12;\, 6, 2)$ displayed in Figure 9 to obtain a shiftable $SMR(4(b-1), 12(b-1); 6, 2)$, say A. Then apply Theorem 7 with A and the shiftable $SMR(6, 18; 6, 2)$ displayed in Figure 10 to obtain a shiftable $SMR(4b + 2, 3(4b + 2); 6, 2)$. □

−1		3				7	−8					13	−14				
	−2		4				8	−9					14	−15			
		−3		5				9	−10					15	−16		
			−4		6				10	−11					16	−17	
1				−5						11	−12	−13					18
	2				−6	−7					12					17	−18

Figure 10. A $SMR(6, 18; 6, 2)$

Lemma 9. *There exists a shiftable* $SMR(4b+2, (2a+1)(4b+2); 4a+2, 2)$ *for* $a, b \geq 1$.

Proof. By Lemma 8, there is a shiftable $SMR(4b+2, 3(4b+2); 6, 2)$ for $b \geq 1$, say A. Apply Lemma 1 with $p = a-1$ and $q = 2b+1$ to obtain a shiftable $SMR(2(2b+1), 4(a-1)(2b+1); 4(a-1), 2)$, say B, for $a \geq 2$ and $b \geq 1$. Finally, apply Theorem 6 with arrays A and B to obtain a shiftable $SMR(4b+2, (2a+1)(4b+2); 4a+2, 2)$ for $a \geq 2$ and $b \geq 1$. When $a = 1$ apply Lemma 8. □

Lemma 10. *There exists an* $SMR(4b+2, (4a+1)(2b+1); 4a+1, 2)$ *for* $a, b \geq 1$.

Proof. Apply Lemma 1 with $p = a-1$ and $q = 2b+1$ to obtain a shiftable $SMR(2(2b+1), 4(a-1)(2b+1); 4(a-1), 2)$, say A, for $a \geq 2$. By Proposition 2 there is an $SMR(4b+2, 5(2b+1); 5, 2)$, say B, for $b \geq 1$. Finally, apply Theorem 6 with arrays A and B to obtain an $SMR(4b+2, (4a+1)(2b+1); 4a+1, 2)$ for $a, b \geq 1$. □

Lemma 11. *There exists an* $SMR(4b+2, (4a+3)(2b+1); 4a+3, 2)$ *for* $a, b \geq 0$.

Proof. Apply Lemma 1 with $p = a$ and $q = 2b+1$ to obtain a shiftable $SMR(2(2b+1), 4a(2b+1); 4a, 2)$, say A. By Proposition 1 there is an $SMR(4b+2, 3(2b+1); 3, 2)$, say B, for $b \geq 1$. Finally, apply Theorem 6 with arrays A and B to obtain an $SMR(4b+2, (4a+3)(2b+1); 4a+3, 2)$ for $a, b \geq 1$. □

We conclude this section with the following theorem.

Theorem 8. *Let* m *be even. There exists an* $SMR(m, n; r, 2)$ *if and only if* $m = 2$ *and* $n = r \equiv 0, 3 \pmod 4$ *or* $m \geq 4$, $r \geq 3$ *and* $mr = 2n$.

5. The existence of an $SMR(m, n; r, 2)$ with m odd and r even

In this section we investigate the existence of a signed magic rectangle $(m, n; r, 2)$ with m odd and r even. Note that if m and r are both odd, the is no $SMR(m, n; r, 2)$.

5.1. The existence of an $SMR(m, n; 4a, 2)$ with m odd

We consider two cases: $m = 4b+1$ and $m = 4b+3$.

Lemma 12. *There exists a shiftable* $SMR(4b+1, 2a(4b+1); 4a, 2)$ *for all* $a, b \geq 1$.

Proof. Apply Lemma 1 with $p = a = 1$ and $q = 2(b-1)$ to obtain a shiftable $SMR(4(b-1), 8(b-1); 4, 2)$ for $b \geq 2$.

Figure 11 displays a shiftable $SMR(5, 10; 4, 2)$. Therefore there is a shiftable $SMR(4b+1, 2(4b+1); 4, 2)$ by Theorem 7. Now apply Part 1 of Theorem 5 to obtain a shiftable $SMR(4b+1, 2a(4b+1); 4a, 2)$ for all $a, b \geq 1$. □

Lemma 13. *There exists a shiftable* $SMR(4b+3, 2a(4b+3); 4a, 2)$ *for all* $a, b \geq 1$.

Proof. Apply Lemma 1 with $p = 1$ and $q = 2b$ to obtain a shiftable $SMR(4b, 8b; 4, 2)$ for $b \geq 1$. Figure 12 displays a shiftable $SMR(3, 6; 4, 2)$. Therefore, by Theorem 7, there is a shiftable $SMR(4b+3, 2(4b+3); 4, 2)$. We now apply Part 1 of Theorem 5 to obtain a shiftable $SMR(4b+3, 2a(4b+3); 4a, 2)$ for all $a, b \geq 1$. □

1				−5	−6				10
−1	2				6	−7			
	−2	3				7	−8		
		−3	4				8	−9	
			−4	5				9	10

Figure 11. A Shiftable $SMR(5, 10; 4, 2)$

1		−3	−4		6
−1	2		4	−5	
	−2	3		5	−6

Figure 12. A Shiftable $SMR(3, 6; 4, 2)$

5.2. The existence of an $SMR(m, n; 4a + 2, 2)$ with m odd

We consider two cases: $m = 4b + 1$ and $m = 4b + 3$.

Lemma 14. *There exists a shiftable $SMR(4b + 1, 3(4b + 1); 6, 2)$ for all $b \geq 1$.*

Proof. Apply Part 2 of Theorem 5 with the shiftable $SMR(4, 12;\ 6, 2)$ given in Figure 9 to obtain a shiftable $SMR(4(b - 1), 12(b - 1); 6, 2)$ for $b \geq 1$. Figure 13 displays a shiftable $SMR(5, 15; 6, 2)$. Therefore there is a shiftable $SMR(4b + 1, 3(4b + 1); 6, 2)$ for $b \geq 1$ by Theorem 7. □

1	−2				−6				10		12			−15
	2	−3			6	−7						−13		15
		3	−4			7	−8			−11		13		
			4	−5			8	−9			−12		14	
−1				5				9	−10	11			−14	

Figure 13. A Shiftable $SMR(5, 15; 6, 2)$

Lemma 15. *There exists a shiftable $SMR(4b + 1, (2a + 1)(4b + 1); 4a + 2, 2)$ for all $a, b \geq 1$.*

Proof. Apply Lemma 1 with $p = 1$ and $q = 2b - 2$ to obtain a shiftable $SMR(2(2b - 2), 4(2b - 2); 4, 2)$ for $b \geq 2$. Figure 11 displays a shiftable $SMR(5, 10; 4, 2)$. Therefore there is a shiftable $SMR(4b + 1, 2(4b + 1); 4, 2)$ for $b \geq 1$ by Theorem 7. Now apply Part 1 of Theorem 5 to obtain a shiftable $SMR(4b + 1, 2(a - 1)(4b + 1); 4(a - 1), 2)$, say A_1, for all $a \geq 2$ and $b \geq 1$. By Lemma 14 there exists a shiftable $SMR(4b + 1, 3(4b + 1); 6, 2)$ for $b \geq 1$, say A_2. Now apply Theorem 6 with A_1 and A_2 to obtain a shiftable $SMR(4b + 1, (2a + 1)(4b + 1); 4a + 2, 2)$ for $a \geq 2$ and $b \geq 1$. When $a = 1$, we apply Lemma 14. □

Lemma 16. *There exists a shiftable $SMR(4b + 3, 3(4b + 3); 6, 2)$ for all $b \geq 1$.*

Proof. Apply Part 2 of Theorem 5 with the shiftable $SMR(4, 12;\ 6, 2)$ given in Figure 9 to obtain a shiftable $SMR(4b, 12b; 6, 2)$ for $b \geq 1$. Figure 14 displays a shiftable $SMR(3, 9; 6, 2)$. Therefore there is a shiftable $SMR(4b + 3, 3(4b + 3); 6, 2)$ by Theorem 7. □

Lemma 17. *There exists a shiftable $SMR(4b + 3, (2a + 1)(4b + 3); 4a + 2, 2)$ for all $a, b \geq 1$.*

1	−2		−4		6	7	−8	
	2	−3	4	−5		−7		9
−1		3		5	−6		8	−9

Figure 14. A Shiftable $SMR(3, 9; 6, 2)$

Proof. Apply Lemma 1 with $p = 1$ and $q = 2b$ to obtain a shiftable $SMR(2(2b), 4(2b); 4, 2)$ for $b \geq 1$. Figure 12 displays a shiftable $SMR(3, 6; 4, 2)$. Therefore there is a shiftable $SMR(4b + 3, 2(4b+3); 4, 2)$ by Theorem 7. Now apply Part 1 of Theorem 5 to obtain a shiftable $SMR(4b+3, 2(a-1)(4b+3); 4(a-1), 2)$, say A_1, for all $a \geq 2$ and $b \geq 1$. By Lemma 16 there exists a shiftable $SMR(4b+3, 3(4b+3); 6, 2)$, say A_2, for $b \geq 1$. Now apply Theorem 6 with A_1 and A_2 to obtain a shiftable $SMR(4b+3, (2a+1)(4b+3); 4a+2, 2)$ for $a \geq 2$ and $b \geq 1$. When $a = 1$ we apply Lemma 16. □

We summarise the results obtained in Lemmas 14-17 in the next theorem..

Theorem 9. *Let m be odd and r be even. Then there exists an $SMR(m, n; r, 2)$ if and only if $m \geq 3$, $r \geq 4$ and $mr = 2n$.*

We are now ready to state the main theorem of this paper.

Main Theorem. *There exists an $SMR(m, n; r, 2)$ if and only if $m = 2$ and $n = r \equiv 0, 3 \pmod 4$ or $m, r \geq 3$ and $mr = 2n$.*

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