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Article

Sufficient Conditions for a Graph kG Admitting all $[1, k]$-factors

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Abstract: Let $g, f : V(G) \to \{0, 1, 2, 3, \cdots\}$ be two functions satisfying $g(x) \leq f(x)$ for every $x \in V(G)$. A (g, f)-factor of G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$. An (f, f)-factor is simply called an f-factor. Let φ be a nonnegative integer-valued function defined on $V(G)$. Set

$$D^{\text{even}}_{g,f} = \Big\{\varphi : g(x) \leq \varphi(x) \leq f(x) \text{ for every } x \in V(G) \text{ and } \sum_{x\in V(G)} \varphi(x) \text{ is even}\Big\}.$$

If for each $\varphi \in D^{even}_{g,f}$, G admits a φ-factor, then we say that G admits all (g, f)-factors. All (g, f)-factors are said to be all $[1, k]$-factors if $g(x) \equiv 1$ and $f(x) \equiv k$ for any $x \in V(G)$. In this paper, we verify that for a connected multigraph G satisfying $N_G(X) = V(G)$ *or* $|N_G(X)| > \left(1 + \frac{1}{k+1}\right)|X| - 1$ for every $X \subset V(G)$, kG admits all $[1, k]$-factors, where $k \geq 2$ is an integer and kG denotes the graph derived from G by replacing every edge of G with k parallel edges.

Keywords: Graph, Spanning subgraph, φ-factor, $[1, k]$-factors

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1. Introduction

The graphs discussed here are multigraphs, which may admit multiple edges but do not admit loops. A graph is called a simple graph if it admits neither multiple edges nor loops. For convenience, we simply call a multigraph a graph when we show notations and definitions. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. For $x \in V(G)$, we write $d_G(x)$ for the degree of x in G, and $N_G(x)$ for the set of the vertices adjacent to x in G. We write $N_G(X) = \bigcup_{x\in X} N_G(x)$ for any $X \subseteq V(G)$. We denote by $I(G)$ the set of isolated vertices of G, and by $\omega_{\geq k}(G)$ the number of components of G with order at least k. Specially, we write $i(G) = |I(G)|$ and $\omega(G) = \omega_{\geq 1}(G)$. Let kG denote the graph derived from G by replacing every edge of G with k parallel edges.

Let $g, f : V(G) \to \{0, 1, 2, 3, \cdots\}$ be two functions satisfying $g(x) \leq f(x)$ for every $x \in V(G)$. A (g, f)-factor of G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$. An (f, f)-factor is simply called an f-factor. If $g(x) \equiv 1$ and $f(x) \equiv k$ for any $x \in V(G)$, then a (g, f)-factor is called a $[1, k]$-factor, where $k \geq 1$ is a fixed integer. A $[1, 1]$-factor is simply called a 1-factor.

Let φ be a nonnegative integer-valued function defined on $V(G)$. In the following, we write

$$D_{g,f}^{\text{even}} = \Big\{\varphi : g(x) \le \varphi(x) \le f(x) \text{ for every } x \in V(G) \text{ and } \sum_{x\in V(G)} \varphi(x) \text{ is even}\Big\}.$$

If for each $\varphi \in D_{g,f}^{even}$, G admits a φ-factor, then we say that G admits all (g, f)-factors. All (g, f)-factors are said to be all $[1, k]$-factors if $g(x) \equiv 1$ and $f(x) \equiv k$ for any $x \in V(G)$.

Lots of authors studied factors [1–16] and all factors [17–20] of graphs. The neighborhood conditions for graphs having factors were derived by many authors [21–27]. Lu, Kano and Yu [18] characterized a graph G such that kG admits all $[1, k]$-factors. Lu, Kano and Yu's results generalized Tutte's 1-factor theorem [28].

Theorem 1. *([18]). Let $k \ge 2$ be an integer, and let G be a connected multigraph. Then kG admits all $[1, k]$-factors if and only if*

$$k \cdot i(G - X) + \omega_{\ge k+1}(G - X) \le |X| + 1$$

for any $X \subseteq V(G)$.

Using Theorem 1, we verify some results related to all $[1, k]$-factors in graphs. Our main results will be shown in Section 2.

2. Next Section

In this section, we discuss the relationship between neighborhood and all $[1, k]$-factors, and verify two results related to all $[1, k]$-factors.

Theorem 2. *Let k be an integer with $k \ge 2$, and let G be a connected multigraph. If G satisfies*

$$N_G(X) = V(G) \text{ or } |N_G(X)| > \Big(1 + \frac{1}{k+1}\Big)|X| - 1$$

for every $X \subset V(G)$, then kG admits all $[1, k]$-factors.

Proof. Assume that G satisfies the hypothesis of Theorem 2, but kG does not admit all $[1, k]$-factors. Then by Theorem 1, we derive

$$k \cdot i(G - X) + \omega_{\ge k+1}(G - X) \ge |X| + 2 \tag{1}$$

for some subset $X \subset V(G)$. The following proof will be divided into two cases by the value of $i(G-X)$.

Case 1. $i(G - X) = 0$.

According to (1), we get

$$\omega_{\ge k+1}(G - X) \ge |X| + 2,$$

which implies that $G - X$ has at least $|X| + 2$ components with order at least $k + 1$. We denote by Y the set of vertices of any $|X| + 1$ components of $G - X$ with order at least $k + 1$. It is clear that $N_G(Y) \ne V(G)$. Thus, we derive

$$|N_G(Y)| > \Big(1 + \frac{1}{k+1}\Big)|Y| - 1. \tag{2}$$

On the other hand, we easily see that $|N_G(Y)| \le |X| + |Y|$. Combining this with (2), we have

$$|X| + |Y| \ge |N_G(Y)| > \Big(1 + \frac{1}{k+1}\Big)|Y| - 1,$$

namely,

$$|X| > \frac{1}{k+1}|Y| - 1. \tag{3}$$

Note that $|Y| \geq (k+1)(|X|+1)$. Then using (3), we admit

$$|X| > \frac{1}{k+1}|Y| - 1 \geq (|X|+1) - 1 = |X|,$$

this is a contradiction.

Case 2. $i(G - X) > 0$.

Obviously, $N_G(V(G) \setminus X) \neq V(G)$. Thus, we have

$$|N_G(V(G) \setminus X)| > \Big(1 + \frac{1}{k+1}\Big)|V(G) \setminus X| - 1 = \Big(1 + \frac{1}{k+1}\Big)(|V(G)| - |X|) - 1. \tag{4}$$

Using (4) and $|N_G(V(G) \setminus X)| \leq |V(G)| - i(G - X)$, we obtain

$$|V(G)| - i(G - X) \geq |N_G(V(G) \setminus X)| > \Big(1 + \frac{1}{k+1}\Big)(|V(G)| - |X|) - 1,$$

that is,

$$|V(G)| + (k+1) \cdot i(G - X) < (k+2) \cdot |X| + k + 1. \tag{5}$$

On the other hand, we easily see that

$$|V(G)| \geq i(G - X) + |X| + (k+1) \cdot (|X| + 2) = i(G - X) + (k+2) \cdot |X| + 2(k+1). \tag{6}$$

It follows from (5) and (6) that

$$\begin{aligned}(k+2) \cdot |X| + k + 1 &> |V(G)| + (k+1) \cdot i(G - X) \\ &\geq i(G - X) + (k+2) \cdot |X| + 2(k+1) + (k+1) \cdot i(G - X) \\ &= (k+2) \cdot i(G - X) + (k+2) \cdot |X| + 2(k+1),\end{aligned}$$

which implies

$$(k+2) \cdot i(G - X) + k + 1 < 0,$$

which is a contradiction. Theorem 2 is proved. □

Theorem 3. *For any positive integer k with $k \geq 2$, there exist infinitely many graphs G that satisfy*

$$N_G(Y) = V(G) \text{ or } |N_G(Y)| \geq \Big(1 + \frac{1}{k+1}\Big)|Y| - 1$$

for every $Y \subset V(G)$, but kG does not admit all $[1, k]$-factors.

Proof. Let $r \geq 1$ be an integer. We construct a graph $G = K_r \vee ((r+2)K_{k+1})$, where K_r denotes the complete graph of order r, K_{k+1} denotes the complete graph of order $k+1$ and $\vee$ means "join".

Next, we demonstrate that

$$N_G(Y) = V(G) \text{ or } |N_G(Y)| \geq \Big(1 + \frac{1}{k+1}\Big)|Y| - 1$$

for every $Y \subset V(G)$. We shall consider two cases by the value of $|Y|$.

Case 1. $|Y| = 1$.

In this case, we obviously have that $|N_G(Y)| \geq k + r > \frac{1}{k+1} = \Big(1 + \frac{1}{k+1}\Big)|Y| - 1$.

Case 2. $|Y| \geq 2$.

Case 2.1. $Y \cap V(K_r) \neq \emptyset$.

It is clear that $N_G(Y) = V(G)$.

Case 2.2. $Y \cap V(K_r) = \emptyset$.

Let $K_{k+1}(i)$, $1 \leq i \leq r+2$, denote the disjoint copies of K_{k+1} in $G - V(K_r)$. Write

$$b_1 = \#\{i : |Y \cap V(K_{k+1}(i))| = 1\},$$

$$b_2 = \#\{i : |Y \cap V(K_{k+1}(i))| = 2\}, \cdots,$$

and

$$b_{k+1} = \#\{i : |Y \cap V(K_{k+1}(i))| = k+1\}.$$

Thus, we derive $|Y| = b_1 + 2b_2 + \cdots + (k+1)b_{k+1}$ and $|N_G(Y)| = r + kb_1 + (k+1)(b_2 + \cdots + b_{k+1})$. If $|Y| \leq (k+1)(r+1+(k-1)b_1+(k-1)b_2+(k-2)b_3+\cdots+b_k)$, then we admit

$$|N_G(Y)| = r + |Y| + (k-1)b_1 + (k-1)b_2 + (k-2)b_3 + \cdots + b_k \geq \Big(1 + \frac{1}{k+1}\Big)|Y| - 1.$$

In the following, we may assume that $|Y| > (k+1)(r+1+(k-1)b_1+(k-1)b_2+(k-2)b_3+\cdots+b_k)$. Note that $Y \cap V(K_r) = \emptyset$. Thus, we have $|Y| \leq |V(G)| - |V(K_r)| = (r+2)(k+1)$. Hence, we gain

$$(k+1)(r+1+(k-1)b_1+(k-1)b_2+(k-2)b_3+\cdots+b_k) < |Y| \leq (r+2)(k+1),$$

namely,

$$(k-1)b_1 + (k-1)b_2 + (k-2)b_3 + \cdots + b_k < 1,$$

which implies $b_1 = b_2 = \cdots = b_k = 0$, and so $|Y| = (k+1)b_{k+1}$. If $b_{k+1} = r+2$, then we derive $Y = V((r+2)K_{k+1})$, and so $N_G(Y) = V(G)$. Next, we assume that $b_{k+1} \leq r+1$. Clearly, $|N_G(Y)| = r + (k+1)b_{k+1} = (k+2)b_{k+1} - 1 + r + 1 - b_{k+1} \geq (k+2)b_{k+1} - 1 = \big(1 + \frac{1}{k+1}\big)|Y| - 1$.

Consequently, we verify that

$$N_G(Y) = V(G) \textit{ or } |N_G(Y)| \geq \Big(1 + \frac{1}{k+1}\Big)|Y| - 1$$

for every $Y \subset V(G)$.

Set $X = V(K_r)$. Then $i(G-X) = 0$ and $\omega_{\geq k+1}(G-X) = r+2$. Thus, we deduce

$$k \cdot i(G-X) + \omega_{\geq k+1}(G-X) = r+2 = |X| + 2 > |X| + 1.$$

Therefore, kG does not admit all $[1,k]$-factors by Theorem 1. Theorem 3 is verified. □

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