Journal of Combinatorial Mathematics and Combinatorial Computing, 120: 91–100
DOI:10.61091/jcmcc120-008
http://www.combinatorialpress.com/jcmcc
Received 20 September 2019, Accepted 30 December 2020, Published 30 June 2024

*Article*

# The $t-$Fibonacci Sequences in Some Finite Groups With Center $Z(G) = G'$

**M. Hashemi**[1], **and M. Pirzadeh**[1,*]

[1] Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

* **Correspondence:** pirzadeh.math@gmail.com

**Abstract:** We consider finitely presented groups $G_{mn}$ as follows:

$$G_{mn} = \langle x, y | x^m = y^n = 1, [x, y]^x = [x, y], [x, y]^y = [x, y]\rangle \qquad m, n \geq 2.$$

In this paper, we first study the groups $G_{mn}$. Then by using the properties of $G_{mm}$ and $t-$Fibonacci sequences in finitely generated groups, we show that the period of $t-$Fibonacci sequences in $G_{mm}$ are a multiple of $K(t, m)$. In particular for $t \geq 3$ and $p = 2$, we prove $LEN_t(G_{pp}) = 2K(t, p)$.

**Keywords:** Fibonacci sequences, $t-$Fibonacci sequences, Group

---

## 1. Introduction

Fibonacci numbers and their generalizations have many applications in every field of science and art; see for example, [1–3]. Fibonacci numbers $F_n$ are defined by the recurrence relation $F_0 = 0, F_1 = 1; F_n = F_{n-2} + F_{n-1}, n \geq 2$. For any given integer $t \geq 2$, the $t-$step Fibonacci sequence $F_n(t)$ is defined [4] by the following recurrence formula:

$$F_n(t) = F_{n-1}(t) + F_{n-2}(t) + \cdots + F_{n-t}(t),$$

with initial conditions $F_0(t) = 0, F_1(t) = 0, \ldots, F_{t-2}(t) = 0$ and $F_{t-1}(t) = 1$. For $m \geq 2$, we consider $F_n(t, m) = F_n(t)\ (mod\ m)$. Following Wall [5] one may also prove that $F_n(t, m)$ is periodic sequences. We use $K(t, m)$ to denote the minimal length of the period of the sequence $F_n(t, m)$ and call it Wall number of m with respect to $t-$step Fibonacci sequence. For example, for

$$\{F_n(4)\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, 2, 4, 8, 15, 29, \ldots\},$$

by considering

$$\{F_n(4) \bmod 2\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, \underline{0}, 0, 0, 1, 1, \ldots\},$$

we get $K(4, 2) = 5$.

The Fibonacci sequences in finite groups have been studied by many authors; see for example, [6–11]. We now introduce a generalization of Fibonacci sequences in finite groups which first presented in [4] by Knox.

**Definition 1.** *Let $j \leq t$. A $t-$Fibonacci sequence in a finite group is a sequence of group elements $x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial set $\{x_1, x_2, \ldots, x_j\}$, each element is defined by*

$$x_n = \begin{cases} x_1 x_2 \ldots\ x_{n-1}, & j < n \leq t, \\ x_{n-t}\ x_{n-t+1} \ldots\ x_{n-1}, & n > t. \end{cases}$$

Note that the initial elements $x_1, x_2, \ldots, x_j$, generate the group. The $t-$Fibonacci sequence of $G$ with seed in $X = \{x_1, x_2, \ldots, x_j\}$ is denoted by $F_t(G; X)$ and its period is denoted by $LEN_t(G; X)$ (see [7, 12]). When it is clear which generating set is being investigated, we will write $LEN_t(G)$ for $LEN_t(G; X)$.

Now, we consider

$$G_m = G_{mm} = \langle x, y | x^m = y^m = 1, [x, y]^x = [x, y], [x, y]^y = [x, y] \rangle \quad m \geq 2.$$

For every $t \geq 3$, to study the $t-$Fibonacci sequences of $G_m$, we define the sequence $\{g_n(t)\}_0^\infty$ of numbers as follows:

$$\begin{aligned}
g_0(3) =& g_1(3) = g_2(3) = g_3(3) = 0, g_4(3) = 1, g_5(3) = 6;\\
g_n(3) =& g_{n-3}(3) + g_{n-2}(3) + g_{n-1}(3) + (F_{n-3}(3))(F_{n-1}(3) - F_{n-2}(3))\\
&+ (F_{n-3}(3) + F_{n-2}(3))(F_n(3) - F_{n-1}(3))\\
g_0(t) =& g_1(t) = g_2(t) = 0, g_3(t) = g_3(t-1), \ldots, g_{t+1}(t) = g_{t+1}(t-1);\\
g_n(t) =& g_{n-t}(t) + g_{n-t+1}(t) + g_{n-t+2}(t) + \cdots + g_{n-1}(t)\\
&+ F_{n-3}(t)(F_{n-1}(t) - F_{n-2}(t))\\
&+ (F_{n-3}(t) + F_{n-2}(t))(F_n(t) - F_{n-1}(t))\\
&+ (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t))(F_{n+1}(t) - F_n(t))\\
&+ (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t) + F_n(t))(F_{n+2}(t) - F_{n+1}(t))\\
&\vdots\\
&+ (F_{n-3}(t) + \cdots + F_{n+t-5}(t))(F_{n+t-3}(t) - F_{n+t-4}(t)); \; n > t+1, t \geq 4.
\end{aligned}$$

The 2−Fibonacci length and 3−Fibonacci length of $G_m$ were investigated in [8, 12]. In this paper, we study the $t-$Fibonacci sequence of $G_m$. Section 2 is devoted to the proofs of some preliminary results that are needed for the main results of this paper. In Section 3, we generalize 3−Fibonacci sequences idea to $t-$Fibonacci sequences ($t \geq 4$). Also, we prove the Theorem 3, which show that for every $t \geq 3$ and $p = 2$, $LEN_t(G_p) = 2K(t, p)$.

## 2. Some Preliminaries

The aim of this section is to collect several facts and basic results that will be used in the rest of this paper. First for given integers $m \geq 2$ and $t \geq 4$, let $F_i = F_i(t, m), K(m) = K(t, m)$ then we prove the following results:

**Lemma 1.** *For integers $n, i$ and $m$ with $m \geq 2$, we have*

*1.*

$$F_{K(m)+i} \equiv F_i (mod\ m),$$

*2.*

$$F_{nK(m)+i} \equiv F_i (mod\ m).$$

*Proof.* Using of the definition of the Wall number of the $t-$step Fibonacci sequence one has:

$$F_{K(m)+i} \equiv F_i (mod\ m).$$

To prove ($ii$), according to the above relations, we have
$F_{nK(m)+i} \equiv F_{K(m)+((n-1)K(m)+i)} \equiv F_{(n-1)K(m)+i} \equiv \cdots \equiv F_i (mod\ m).$ □

**Corollary 1.** *For integers $n$ and $m \geq 2$, if*

$$\begin{cases} F_n & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{n+t-2} & \equiv 0 \pmod{m}, \\ F_{n+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

*Then $K(m)\,|n$.*

*Proof.* Let $n = aK(m) + i,\ 0 \leq i < K(m)$. Then by Lemma 1, we get

$$\begin{cases} F_i & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{i+t-2} & \equiv 0 \pmod{m}, \\ F_{i+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

Now, since $K(m)$ is the least integer such that the assumption holds, the proof is completed immediately. ◻

We need some results concerning the groups presented by

$$G_{mn} = \langle x, y | x^m = y^n = 1, [x,y]^x = [x,y], [x,y]^y = [x,y]\rangle \qquad m, n \geq 2.$$

First, we state a Lemma without proof that establishes some properties of groups of nilpotency class two.

**Lemma 2.** *If $G$ is a group and $G' \subseteq Z(G)$, then the following hold for every integer $k$ and $u, v, w \in G$*

1. $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$,
2. $[u^k, v] = [u, v^k] = [u, v]^k$,
3. $(uv)^k = u^k v^k [v, u]^{\frac{k(k-1)}{2}}$.

**Lemma 3.** *Let $m, n$ be positive integer numbers and $d = g.c.d(m, n)$. Then $|G_{mn}| = d \times mn$.*

*Proof.* Consider the subgroup $H = \langle a, [a, b]\rangle$ of $G_{mn}$. Obviously H is abelian and a simple coset enumeration by defining $n$ coset as $1 = H$ and $ib = i + 1, 1 \leq i \leq n - 1$ shows that $|G : H| = n$. Using the modified Todd-coxeter coset enumeration algorithm, yields the following presentation for H:

$$H = \langle h_1, h_2 | h_1^m = h_2^m = h_1^n = h_2^n = 1, [h_1, h_2] = 1\rangle.$$

So that $H \cong Z_m \times Z_d$ and $|G_{mn}| = |G : H| \times |H| = d \times mn$. ◻

The following proposition is of interest to consider and one may see the proof in [8].

**Proposition 1.** *Let $G = G_{mn}$. Then*

1. $G' = \langle [a, b]\rangle$.
2. *Every element of $G$ is in the form $x^i y^j g$ where $0 \leq i \leq m - 1, 0 \leq j \leq n - 1$ and $g \in G'$.*
3. $Z(G) = \langle x, y, z | x^{m/d} = y^{n/d} = z^d = [x, y] = [x, z] = [y, z] = 1\rangle$.

For the particular case, consider $m = n$ then for $m \geq 2$ we get

$$G_m = G_{mm} = \langle x, y | x^m = y^m = 1, [x, y]^x = [x, y], [x, y]^y = [x, y]\rangle.$$

**Corollary 2.** *With the above facts, we have*

1. $|G_m| = m^3, Z(G_m) = G'_m, |Z(G_m)| = m$.
2. *Every element of $G_m$ can be written uniquely in the form $x^r y^s [y, x]^t$ where $0 \leq r, s, t \leq m - 1$.*

### 3. The $t-$Fibonacci Sequences of $G_m$

In this section, we examine the $t-$Fibonacci sequence of $G_m$ with respect to the ordered generating set $X = \{x, y\}$. First, we show that every element of $F_4(G_m; X)$ has a standard form.

**Lemma 4.** *For every $n, (n \geq 3)$ every element $x_n$ of the $4-$Fibonacci sequences of group $G_m$ can be written in the form $x^{F_{n+2}-F_{n+1}}y^{F_{n+1}}[y, x]^{g_n}$.*

*Proof.* Let $x_r = x_r(4)$, $F_r = F_r(4)$ and $g_r = g_r(4)$. We proceed by induction on $n$, for $n = 3, 4$ we have

$x_3 = x_1x_2 = xy[y, x]^0 = x^{F_5-F_4}y^{F_4}[y, x]^{g_3}$

$x_4 = x_1x_2x_3 = x^2y^2[y, x] = x^{F_6-F_5}y^{F_5}[y, x]^{g_4}$

and if $x_k = x^{F_{k+2}-F_{k+1}}y^{F_{k+1}}[y, x]^{g_k}$ $(4 \leq k \leq n-1)$, then by the relation $x_n = x_{n-4}x_{n-3}x_{n-2}x_{n-1}$ we get

$$\begin{aligned}
x_n =& x_{n-4}\ x_{n-3}\ x_{n-2}\ x_{n-1}\\
=& x^{F_{n-2}-F_{n-3}}y^{F_{n-3}}[y, x]^{g_{n-4}}x^{F_{n-1}-F_{n-2}}y^{F_{n-2}}[y, x]^{g_{n-3}}\\
&\times x^{F_n-F_{n-1}}y^{F_{n-1}}[x, y]^{g_{n-2}}\ x^{F_{n+1}-F_n}y^{F_n}[y, x]^{g_{n-1}}\\
=& x^{F_{n-1}-F_{n-3}}\ y^{F_{n-3}+F_{n-2}}[y, x]^{g_{n-4}+g_{n-3}+F_{n-3}(F_{n-1}-F_{n-2})}x^{F_n-F_{n-1}}y^{F_{n-1}}\\
&\times [y, x]^{g_{n-2}}x^{F_{n+1}-F_n}y^{F_n}[y, x]^{g_{n-1}}\\
=& x^{F_n-F_{n-3}}\ y^{F_{n-3}+F_{n-2}+F_{n-1}}\\
&\times [y, x]^{g_{n-4}+g_{n-3}+g_{n-2}+F_{n-3}(F_{n-1}-F_{n-2})+(F_{n-3}+F_{n-2})(F_n-F_{n-1})}\\
&\times x^{F_{n+1}-F_n}y^{F_n}[y, x]^{g_{n-1}}\\
=& x^{F_{n+1}-F_{n-3}}y^{F_{n-3}+F_{n-2}+F_{n-1}+F_n}\\
&\times [y, x]^{g_{n-4}+g_{n-3}+g_{n-2}+g_{n-1}+F_{n-3}(F_{n-1}-F_{n-2})+(F_{n-3}+F_{n-2})(F_n-F_{n-1})}\\
&\times [y, x]^{(F_{n-3}+F_{n-2}+F_{n-1})(F_{n+1}-F_n)}\\
=& x^{F_{n+2}-F_{n+1}}y^{F_{n+1}}[y, x]^{g_n}.
\end{aligned}$$

Thus the assertion holds. □

Now, we are ready to generalize the idea of $4-$Fibonacci sequence of $G_m$ to $t-$Fibonacci sequence of these groups.

**Theorem 1.** *For every $t \geq 4$ and $n \geq 3$, each element $x_n$ of the $t-$Fibonacci sequences of group $G_m$ can be written in the form*

$$x_n(t) = x^{F_{n+t-2}(t)-F_{n+t-3}(t)}y^{F_{n+t-3}(t)}[y, x]^{g_n(t)}.$$

*Proof.* Use an induction method on $t$. We have for $t = 4$:

$$x_n(4) = x^{F_{n+2}(4)-F_{n+1}(4)}y^{F_{n+1}(4)}[y, x]^{g_n(4)}$$

and if for every $k$, $(5 \leq k \leq t)$, we have

$$x_n(k) = x^{F_{n+2}(k)-F_{n+1}(k)}y^{F_{n+1}(k)}[y, x]^{g_n(k)}.$$

It is sufficient to show that

$$x_n(t+1) = x^{F_{n+t-1}(t+1)-F_{n+t-2}(t+1)}y^{F_{n+t-2}(t+1)}[y, x]^{g_n(t+1)}.$$

For this, we use an induction method on $n$:
If $3 \leq s \leq t$, by definitions of $F_n$ and $g_n$ we get $F_s(t+1) = F_s(t)$ and $g_s(t+1) = g_s(t)$, then $x_s(t+1) = x_s(t)$. By this and the inductive hypothesis $t$ we have

$$x_s(t+1) = x^{F_{s+t-1}(t+1)-F_{s+t-2}(t+1)}y^{F_{s+t-2}(t+1)}[y, x]^{g_s(t+1)}.$$

Now we suppose that the hypothesis of induction holds for all $s \leq n-1$. By definition of $x_n(t+1)$ and Corollary 2-(ii), we get;

$$\begin{aligned}
x_n(t+1) =& x_{n-(t+1)}(t+1)x_{n-(t+1)+1}(t+1) \times \cdots \times x_{n-1}(t+1) \\
=& x^{F_{n-2}(t+1)-F_{n-3}(t+1)}\, y^{F_{n-3}(t+1)}[y,x]^{g_{n-t-1}(t+1)} \\
& \times x^{F_{n-1}(t+1)-F_{n-2}(t+1)} y^{F_{n-2}(t+1)}[y,x]^{g_{n-t}(t+1)} \\
& \times x^{F_n(t+1)-F_{n-1}(t+1)}\, y^{F_{n-1}(t+1)}[y,x]^{g_{n-t+1}(t+1)} \\
& \times \cdots \times x^{F_{n+t-2}(t+1)-F_{n+t-3}(t+1)}\, y^{F_{n+t-3}(t+1)}[y,x]^{g_{n-1}(t+1)} \\
=& x^{F_{n-1}(t+1)-F_{n-3}(t+1)}\, y^{F_{n-3}(t+1)+F_{n-2}(t+1)} \\
& \times [y,x]^{g_{n-t-1}(t+1)+g_{n-t}(t+1)+F_{n-3}(t+1)(F_{n-1}(t+1)-F_{n-2}(t+1))} \\
& \times x^{F_n(t+1)-F_{n-1}(t+1)}\, y^{F_{n-1}(t+1)}[y,x]^{g_{n-t+1}(t+1)} \\
& \times \cdots \times x^{F_{n+t-2}(t+1)-F_{n+t-3}(t+1)} y^{F_{n+t-3}(t+1)}[y,x]^{g_{n-1}(t+1)} \\
=& x^{F_n(t+1)-F_{n-3}(t+1)}\, y^{F_{n-3}(t+1)+F_{n-2}(t+1)+F_{n-1}(t+1)} \\
& \times [y,x]^{g_{n-t-1}(t+1)+g_{n-t}(t+1)+g_{n-t+1}(t+1)} \\
& \times [y,x]^{F_{n-3}(t+1)((F_{n-1}(t+1)-F_{n-2}(t+1))} \\
& \times [y,x]^{(F_{n-3}(t+1)+F_{n-2}(t+1))(F_n(t+1)-F_{n-1}(t+1))} \\
& \times \cdots \times x^{F_{n+t-2}(t+1)-F_{n+t-3}(t+1)}\, y^{F_{n+t-3}(t+1)}[y,x]^{g_{n-1}(t+1)}.
\end{aligned}$$

We continue this process and find that

$$\begin{aligned}
x_n(t+1) =& x^{F_{n+t-1}(t+1)-F_{n+t-2}(t+1)} y^{F_{n-3}(t+1)+\cdots+F_{n+t-3}(t+1)} \\
& \times [y,x]^{g_{n-t-1}(t+1)+\cdots+g_{n-1}(t+1)+F_{n-3}(t+1)(F_{n-1}(t+1)-F_{n-2}(t+1))} \\
& \times [y,x]^{(F_{n-3}(t+1)+F_{n-2}(t+1))(F_n(t+1)-F_{n-1}(t+1))} \\
& \times [y,x]^{(F_{n-3}(t+1)+\cdots+F_{n+t-4}(t+1))(F_{n+t-2}(t+1)-F_{n+t-3}(t+1))} \\
=& x^{F_{n+t-1}(t+1)-F_{n+t-2}(t+1)} y^{F_{n+t-2}(t+1)}[y,x]^{g_n(t+1)}.
\end{aligned}$$

The theorem is proved. □

**Example 1.** *For integer $m = 2$, by using above Theorem and relations of $G_m$, we obtain the $4-$Fibonacci sequence of $G_m$ ($i.e. F_4(G_m; X)$) as follows:*

$$\begin{aligned}
x_1 =& x, x_2 = y, x_3 = x^{F_5-F_4}y^{F_4}[y,x]^{g_3} = x^3y = xy, \\
x_4 =& x^{F_6-F_5}y^{F_5}[y,x]^{g_4} = x^2y^2[y,x] = [y,x], \\
x_5 =& x^{F_7-F_6}y^{F_6}[y,x]^{g_5} = x^4y^4[y,x]^6 = e, \\
x_6 =& x^{F_8-F_7}y^{F_7}[y,x]^{g_6} = x^7y^8[y,x]^{28} = x, \\
x_7 =& x^{14}y^{15}[y,x]^{98} = y, x_8 = x^{27}y^{29}[y,x]^{379} = yx, \\
x_9 =& x^{52}y^{56}[y,x]^{1436} = e, x_{10} = x^{100}y^{108}[y,x]^{5378} = e, \\
x_{11} =& x^{193}y^{208}[y,x]^{19984} = x, x_{12} = x^{372}y^{401}[y,x]^{74434} = y, \\
x_{13} =& x^{717}y^{773}[y,x]^{276868} = xy, x_{14} = x^{1382}y^{1490}[y,x]^{1029149} = [y,x], \ldots
\end{aligned}$$

*Consequently*

$$\begin{aligned}
x_{11} =& x_{10+1} = x = x_1, x_{12} = x_{10+2} = y = x_2, x_{13} = x_{10+3} = xy = x_3, \\
x_{14} =& x_{10+4} = [y,x] = x_4.
\end{aligned}$$

*Then $LEN_4(G_m) = 10 = 2K(4,2)$.*

**Theorem 2.** *If $LEN_t(G_m;X) = P$ then the equations*

$$\begin{cases} F_P & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{P+t-2} & \equiv 0 \pmod{m}, \\ F_{P+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

*hold. Moreover, $K(t,m)$ divides $P$.*

*Proof.* For a constant t, let $K(t,m) = K(m)$ and $x_n(t) = x_n$. Since $LEN_t(G_m;X) = P$, for $1 \leq i \leq t$, we have $x_{P+i} = x_i$. Then by the Theorem 1 and Corollary 2-(ii), we obtain

$$\begin{cases} F_{P+t-2} & \equiv F_{t-2} \pmod{m}, \\ F_{P+t-1} & \equiv F_{t-1} \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{P+t+t-3} & \equiv F_{t+t-3} \pmod{m}. \end{cases}$$

Now by definition of $F_n$, this equivalent to

$$\begin{cases} F_{P+t-3} & \equiv F_{t-3} \pmod{m}, \\ F_{P+t-2} & \equiv F_{t-2} \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{P+t+t-4} & \equiv F_{t+t-4} \pmod{m}. \end{cases}$$

By repeating this process, we obtain

$$\begin{cases} F_P & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{P+t-2} & \equiv 0 \pmod{m}, \\ F_{P+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

So, the Corollary 1 yields that $K(m)|P$. □

**Note.** Let $G$ be a finite 2-generated group of nilpotent class two. Then $G \cong \langle a, b|R\rangle$, where $\{a^m = b^n = 1, [a,b]^a = [a,b], [a,b]^b = [a,b]\} \subseteq R$. By these facts, we believe that the Theorem 2 holds for a finite 2-generated group of nilpotent class two.

A list of $K(4,n)$ for all $2 \leq n \leq 101$ is given in the Table 1.

In [8], for $t = 2$ and $X = \{x, y\}$, the $t-$Fibonacci length of $G_m$ was studied by H. Doostie and M. Hashemi. They show that for every prime number $p$

$$LEN_t(G_p) = \begin{cases} 2K(t,p), & p = 2, \\ K(t,p), & p \neq 2. \end{cases}$$

Note that this formula, may be generalized for $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$; i.e. in this case we have $LEN_t(G_n) = l.c.m\{LEN_t(G_{p_1^{\alpha_1}}), \dots, LEN_t(G_{p_1^{\alpha_s}})\}$.

Now, we prove the following important theorem which gives an explicit formula for $LEN_t(G_2)$.

**Theorem 3.** *For $t \geq 3$ and $p = 2$, $LEN_t(G_p) = 2K(t,p)$.*

*Proof.* First, we show that $K(t,2) = t+1$. For any $t-$Fibonacci recurrence $\{U_n\}_{n\geq 0}$, we have

$$\begin{aligned} U_n =& U_{n-1} + U_{n-2} + \cdots + U_{n-t} = U_{n-1} + (U_{n-2} + \cdots + U_{n-t-1}) - U_{n-t-1} \\ =& 2U_{n-1} - U_{n-t-1}, \end{aligned} \tag{1}$$

| $n$ | $K(4,n)$ | $n$ | $K(4,n)$ | $n$ | $K(4,n)$ | $n$ | $K(4,n)$ |
|---|---|---|---|---|---|---|---|
| 2 | 5 | 27 | 234 | 52 | 420 | 77 | 6840 |
| 3 | 26 | 28 | 1710 | 53 | 303480 | 78 | 5460 |
| 4 | 10 | 29 | 280 | 54 | 1170 | 79 | 998720 |
| 5 | 312 | 30 | 1560 | 55 | 1560 | 80 | 1560 |
| 6 | 130 | 31 | 61568 | 56 | 3420 | 81 | 702 |
| 7 | 342 | 32 | 80 | 57 | 89154 | 82 | 240 |
| 8 | 20 | 33 | 1560 | 58 | 280 | 83 | 1157520 |
| 9 | 78 | 34 | 24560 | 59 | 205378 | 84 | 22230 |
| 10 | 1560 | 35 | 17784 | 60 | 1560 | 85 | 191568 |
| 11 | 120 | 36 | 390 | 61 | 226980 | 86 | 162800 |
| 12 | 130 | 37 | 1368 | 62 | 307840 | 87 | 3640 |
| 13 | 84 | 38 | 34290 | 63 | 4446 | 88 | 120 |
| 14 | 1710 | 39 | 1092 | 64 | 160 | 89 | 9320 |
| 15 | 312 | 40 | 1560 | 65 | 2184 | 90 | 1560 |
| 16 | 40 | 41 | 240 | 66 | 1560 | 91 | 4788 |
| 17 | 4912 | 42 | 22230 | 67 | 100254 | 92 | 60830 |
| 18 | 390 | 43 | 162800 | 68 | 24560 | 93 | 61568 |
| 19 | 6858 | 44 | 120 | 69 | 158158 | 94 | 519110 |
| 20 | 1560 | 45 | 312 | 70 | 88920 | 95 | 356616 |
| 21 | 4446 | 46 | 60830 | 71 | 357910 | 96 | 1040 |
| 22 | 120 | 47 | 103822 | 72 | 780 | 97 | 368872 |
| 23 | 12166 | 48 | 520 | 73 | 2664 | 98 | 11970 |
| 24 | 260 | 49 | 2394 | 74 | 6840 | 99 | 1560 |
| 25 | 1560 | 50 | 1560 | 75 | 1560 | 100 | 1560 |
| 26 | 420 | 51 | 63856 | 76 | 34290 | 101 | 1030300 |

**Table 1**

and reduce modulo 2 to get that $U_n \equiv U_{n-(t+1)} \pmod 2$. Let $F_n := F_n(t)$. Then for $\{F_n\}_{n\geq 0}$, we already know that $F_0 = \cdots = F_{t-2} = 0$, $F_{t-1} = 1$ and clearly $F_t = F_{t-1} + \cdots + F_0 = 1$. Thus, modulo 2, $\{F_n\}_{n\geq 0}$ is simply the repeating block $0, 0, \ldots, 0, 1, 1$, where there are $t-1$ zeros. The above shows right away that $F_a F_{a+i} \equiv 0 \pmod 2$ if $i = 2, 3, \ldots, k-1$ and any $a$ (that is the product of any two members of $F_n$ with indices nonconsecutive but which differ by less than $k$ is zero modulo 2). Now, by the above remark, the recurrence relation for $g_n(t)$ and use the fact that $-x \equiv x \pmod 2$, we see that

$$\begin{aligned}
F_{n-3}(F_{n-1} - F_{n-2}) &\equiv F_{n-3}F_{n-2} \pmod 2;\\
(F_{n-3} + F_{n-2})(F_n - F_{n-1}) &\equiv F_{n-2}F_{n-1} \pmod 2;\\
(F_{n-3} + F_{n-2} + F_{n-1})(F_{n+1} - F_n) &\equiv F_{n-1}F_n \pmod 2;\\
&\cdots\ \cdots\ \cdots\\
(F_{n-3} + \cdots + F_{n+t-5})(F_{n+t-3} - F_{n+t-4}) &\equiv F_{n+t-5}F_{n+t-4} + F_{n+t-3}F_{n+t-2} \pmod 2.
\end{aligned}$$

The last one deserves an explanation. Indeed, note that by periodicity with period $t+1$ we have $F_{n-3} \equiv F_{n+t-2} \pmod 2$ and now $F_{n+t-2}$ and $F_{n+t-3}$ have consecutive indices. However, this is the only instance when this happens, in all the other relations only the product of the last term from the first (left) factor with the last term from the second (right) factor survive modulo 2. Thus,

$$g_n \equiv g_{n-1} + \cdots + g_{n-t} + (F_{n-3}F_{n-2} + \cdots + F_{n+t-5}F_{n+t-4} + F_{n+t-3}F_{n+t-2}) \pmod 2.$$

The last sum is "almost perfect". A perfect one would be if $F_{n+t-4}F_{n+t-3}$ would also be present in the above sum. If it were then this sum would be

$$F_mF_{m+1} + F_{m+1}F_{m+2} + \cdots + F_{m+t}F_{m+t+1} \qquad \text{for} \qquad m = n-3 \tag{2}$$

and one can check easily that this is constant 1 modulo 2 (it is enough to check it for the first $t+1$ values of $m = 0, 1, \ldots, t$ and then use periodicity). Since $2F_{n+t-4}F_{n+t-3} \equiv 0 \pmod 2$, by the above, we have

$$g_n \equiv g_{n-1} + \cdots + g_{n-t} + 1 + F_{n+t-4}F_{n+t-3} \pmod 2.$$

Now use the trick at (1). Namely, we have

$$\begin{aligned}
g_n &\equiv g_{n-1} + (g_{n-2} + \cdots + g_{n-t} + g_{n-t-1} + 1 + F_{n+t-5}F_{n+t-4})\\
&\quad + g_{n-t-1} + F_{n+t-4}F_{n+t-3} + F_{n+t-5}F_{n+t-4} \pmod 2\\
&\equiv 2g_{n-1} + g_{n-t-1} + F_{n+t-4}(F_{n+t-3} + F_{n+t-2}) \pmod 2\\
&\equiv g_{n-t-1} + F_{n+t-4}(F_{n+t-3} + F_{n+t-2}) \pmod 2.
\end{aligned}$$

The expression $F_{n+t-4}(F_{n+t-3} + F_{n+t-2})$ is most times 0 modulo 2 but not always since for $n = 3$ it becomes $F_{t-1}(F_t + F_{t-2}) \equiv 1 \pmod 2$. This shows that $t+1$ is not the period of $g_n$. Well, let's apply the above relation again with $n$ replaced by $n-t-1$. We get

$$\begin{aligned}
g_n &\equiv g_{n-t-1} + F_{n+t-4}(F_{n+t-3} + F_{n+t-2})\\
&\equiv (g_{n-(2t+1)} + F_{n-(t+1)+t-4}(F_{n-(t+1)+t-3} + F_{n-(t+1)+t-2}))\\
&\quad + F_{n+t-4}(F_{n+t-3} + F_{n+t-2}) \pmod 2,
\end{aligned}$$

and the expression involving the last four $t$-Fibonacci numbers is 0. Since $F_n$ is periodic modulo 2 with period $t+1$ (that is, that last expression is $2F_{n+t-4}(F_{n+t-3} + F_{n+t-2}) \equiv 0 \pmod 2$). Hence,

$$g_n \equiv g_{n-2(t+1)} \pmod 2,$$

which proves the Theorem. □

| $m$ | $LEN\_3(G\_m)$ | $K(3,m)$ | $LEN\_4(G\_m)$ | $K(4,m)$ |
|---|---|---|---|---|
| 2 | 8 | 4 | 10 | 5 |
| 3 | 13 | 13 | 26 | 26 |
| 4 | 16 | 8 | 20 | 10 |
| 8 | 32 | 16 | 40 | 20 |

**Table 2**

By using a computer program written in the computational algebra system GAP [13], we checked that the above formula holds for every $t = 3, 4$ and $2 \leq m \leq 10$. Some of these results are shown below. At the end of this section we state a conjecture about the $LEN_t(G_p)$, as follows:
**Conjecture.** For every $t \geq 3$ and prime number $p(p > 2)$

$$LEN_t(G_p) = K(t, p).$$

## Acknowledgments

The authors would like to thank the referee for the careful reading and the valuable suggestions about the results of this paper.

## References

1. Caspar, D.L. and Fontano, E., 1996. Five-fold symmetry in crystalline quasicrystal lattices. *Proceedings of the National Academy of Sciences, 93*(25), pp.14271-14278.
2. El Naschie, M.S., 2005. Deriving the essential features of the standard model from the general theory of relativity. *Chaos, Solitons & Fractals, 24*(4), pp.941-946.
3. El Naschie, M.S., 2005. Stability analysis of the two-slit experiment with quantum particles. *Chaos, Solitons & Fractals, 26*(2), pp.291-294.
4. Knox, S.W., 1990. Fibonacci sequences in finite groups. *Fibonacci Quarterly 30*(2), pp.116-120.
5. Wall, D.D., 1960. Fibonacci series modulo m. *The American Mathematical Monthly, 67*(6), pp.525-532.
6. Campbell, C.M. and Campbell, P.P., 2005. The Fibonacci length of certain centro-polyhedral groups. *Journal of Applied Mathematics and Computing, 19*, pp.231-240.
7. Deveci, Ö., Karaduman, E. and Campbell, C.M., 2011, December. On the k-nacci sequences in finite binary polyhedral groups. In *Algebra Colloquium* (Vol. 18, No. spec01, pp. 945-954). Academy of Mathematics and Systems Science, Chinese Academy of Sciences, and Suzhou University.
8. Doostie, H. and Hashemi, M., 2006. Fibonacci lengths involving the Wall number k (n). *Journal of Applied Mathematics and Computing, 20*, pp.171-180.
9. Golamie, R., Doostie, H. and Ahmadidelir, K., 2012. Fibonacci Length and Special Automorphisms of Finite (L, M— N, K)-Groups. *twms Journal of Pure and Applied Mathematics, 3*(2), Pp.182-189.
10. Hashemi, M. and Mehraban, E., 2018. On the generalized order 2-Pell sequence of some classes of groups. *Communications in Algebra, 46*(9), pp.4104-4119.
11. Hashemi, M. and Pirzadeh, M., 2018. T-Nacci Sequences in Some Special Groups of Order M3. *Mahematical Reports, 20*(4), pp.389-399.
12. Hashemi, M. and Mehraban, E., 2015. On The-Nacci Sequences Of Finitely Generated Groups. *Menemui Matematik (Discovering Mathematics), 37*(1), pp.20-27.

13. The GAP Group, GAP–Groups, Algorithms, and Programming, Version 4.4.12. `http://www.gap-system.org`.