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A class of constacyclic codes over $\frac{\mathbb{F}_{p^m}[u,v]}{\langle u^k, v^2, uv-vu\rangle}$

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ABSTRACT

Let p be a prime number, and let k and m be positive integers with $k \geq 2$. This paper studies the algebraic structure of λ-constacyclic codes of arbitrary length over the finite commutative ring $R = \frac{\mathbb{F}_{p^m}[u,v]}{\langle u^k, v^2, uv-vu\rangle}$, where λ is a unit in R given by $\lambda = \sum_{i=0}^{k-1} \lambda_i u^i + v \sum_{i=0}^{k-1} \lambda'_i u^i$, with $\lambda_i, \lambda'_i \in \mathbb{F}_{p^m}$ and $\lambda_0, \lambda_1 \neq 0$. We provide a complete classification of these constacyclic codes, determine their dual structures, and compute their Hamming distances when the code length is p^s.

Keywords: constacyclic codes, codes over rings, dual code, Hamming distance

1. Introduction

Constacyclic codes play a crucial role in the theory of error-correcting codes, serving as a natural extension of cyclic codes, which are among the most extensively studied code types. Let R be a finite commutative ring, and let λ be a unit in R. A λ-constacyclic code of length n over the ring R can be represented as an ideal in the quotient ring $\frac{R[x]}{\langle x^n - \lambda\rangle}$. This representation underscores the significance of constacyclic codes over rings as an important class of linear codes due to their rich algebraic structure.

In recent years, there has been a growing research interest in the study of constacyclic codes. However, the general classification of constacyclic codes remains a challenging task. To date, only a limited number of cases with specific lengths and defined over certain rings have been classified. Let R be a commutative ring. Dinh and López-Permouth [18] studied the structures of cyclic and negacyclic codes of length n when n is not divisible by the characteristic of the residue field $\bar{R}$. Moreover, Cao [4] investigated the algebraic structure of $(1 + w\gamma)$-constacyclic codes of arbitrary

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length over a finite commutative chain ring R, where $w \in R^\times$.

An important example of a finite ring is $R = \frac{\mathbb{F}_{p^m}[u]}{\langle u^k \rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{k-1}\mathbb{F}_{p^m}$. The structure of constacyclic codes over $\frac{\mathbb{F}_{p^m}[u]}{\langle u^k \rangle}$ has been extensively studied in the literature. For $k = 2$, significant research has been devoted to cyclic and constacyclic codes of length n over the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ for various prime numbers p and positive integers n (cf. [10, 11, 15, 8, 14, 2]). For $k \geq 3$, Kai et al. [20] investigated $(1 + \lambda u)$-constacyclic codes of arbitrary length over $\frac{\mathbb{F}_p[u]}{\langle u^k \rangle}$, where $\lambda \in \mathbb{F}_p^\times$. Subsequently, Cao et al. [6, 5] studied $(\delta + \alpha u^2)$-constacyclic codes of arbitrary length over $\frac{\mathbb{F}_{p^m}[u]}{\langle u^k \rangle}$, focusing on the case where k is an even integer and $\delta, \alpha \in \mathbb{F}_{p^m}^\times$. Sobhani [23] determined the structure of $(\delta + \alpha u^2)$-constacyclic codes of length p^s over $\frac{\mathbb{F}_{p^m}[u]}{\langle u^3 \rangle}$, where $\delta, \alpha \in \mathbb{F}_{p^m}^\times$. Moreover, Mahmoodi and Sobhani [21] provided a complete classification of $(1 + \alpha u^2)$-constacyclic codes of length p^s over $\frac{\mathbb{F}_{p^m}[u]}{\langle u^4 \rangle}$, where $\alpha \in \mathbb{F}_{p^m}^\times$.

Another important ring is $R = \frac{\mathbb{F}_{p^m}[u,v]}{\langle u^2, v^2, uv - vu \rangle}$, which is local but not a chain ring, with units of the form

$$\lambda_1 = \alpha, \quad \lambda_2 = \alpha + \delta_1 uv, \quad \lambda_3 = \alpha + \gamma v + \delta uv, \quad \lambda_4 = \alpha + \beta u + \delta uv, \quad \lambda_5 = \alpha + \beta u + \gamma v + \delta uv,$$

where $\alpha, \beta, \gamma, \delta_1 \in \mathbb{F}_{p^m}^*$ and $\delta \in \mathbb{F}_{p^m}$.

Dinh et al. [16] determined the algebraic structures of all constacyclic codes of length p^s over R, except for the λ_2-constacyclic codes, later studied in [12]. For $p = 2$, Dinh et al. [17] classified all self-dual λ-constacyclic codes of length 2^s over R corresponding to the units λ_3, λ_4, and λ_5. In our recent paper [1], we investigated the structure and duals of λ-constacyclic codes of length $2p^s$ over R for the units λ_3, λ_4, and λ_5. Motivated by this, we aim to extend our previous results by classifying λ-constacyclic codes of arbitrary length over

$$R_{u,v} = \frac{\mathbb{F}_{p^m}[u, v]}{\langle u^k, v^2, uv - vu \rangle},$$

with $k \geq 2$, and λ given by

$$\lambda = \sum_{i=0}^{k-1} \lambda_i u^i + v \sum_{i=0}^{k-1} \lambda'_i u^i,$$

where $\lambda_i, \lambda'_i \in \mathbb{F}_{p^m}$ and $\lambda_0, \lambda_1 \neq 0$.

The paper is structured as follows. Section 2 introduces essential preliminaries on finite rings and fundamental concepts in constacyclic codes. Section 3 establishes a decomposition of each λ-constacyclic code of arbitrary length over $R_{u,v}$ as a direct sum of ideals over certain rings. These rings are further examined in Section 4, where their ideals are classified into four types. Section 5 investigates the structure of dual λ-constacyclic codes of arbitrary length over over $R_{u,v}$. Finally, Section 6 determines their Hamming distance when the code length is p^s.

2. Preliminaries

In this paper, all rings considered are associative and commutative. An ideal I of a ring R is called principal if it is generated by a single element. A ring R is a principal ideal ring if each of its ideals is principal. A ring R is called local if it has a unique maximal ideal. A chain ring is a ring whose ideals are totally ordered by inclusion. A ring R is said to be Frobenius if $\frac{R}{J(R)}$ is isomorphic (as an R-module) to $\mathrm{soc}(R)$, where $J(R)$ denotes the Jacobson radical of R and $\mathrm{soc}(R)$ its socle.

Let R be a finite local commutative ring with maximal ideal M and residue field $\overline{R} = \frac{R}{M}$. For any polynomial $f(x) \in R[x]$, we denote by $\overline{f}(x)$ the polynomial in $\overline{R}[x]$ obtained by reducing each coefficient of $f(x)$ modulo M. Two polynomials $f_1(x), f_2(x) \in R[x]$ are said to be coprime if there exist polynomials $g_1(x), g_2(x) \in R[x]$ such that

$$f_1(x)g_1(x) + f_2(x)g_2(x) = 1.$$

A polynomial $f(x) \in R[x]$ is said to be regular if it is not a zero divisor; equivalently, $\overline{f}(x) \neq 0$ in $\overline{R}[x]$ (see [22]).

The following result is a well-known property of finite commutative chain rings.

Proposition 2.1. [18] *Let R be a finite commutative ring. The following conditions are equivalent:*

(a) R is a local ring whose maximal ideal M is principal (i.e., $M = \langle \Gamma \rangle$ for some element $\Gamma \in R$).

(b) R is a local principal ideal ring.

(c) R is a chain ring whose ideals are $\langle \Gamma^i \rangle$ for $0 \leq i \leq t$, where t is the nilpotency index of Γ. Moreover, for each $0 \leq i \leq t$, the cardinality of the ideal $\langle \Gamma^i \rangle$ is given by

$$|\langle \Gamma^i \rangle| = \left| \frac{R}{M} \right|^{t-i}.$$

A linear code C of length n over a ring R is an R-submodule of R^n. Let λ be a unit in R. A linear code C of length n over R is called a λ-constacyclic code if it satisfies the following condition:

$$(c_0, c_1, \ldots, c_{n-1}) \in C \implies (\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in C.$$

Each codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ can be represented as a polynomial $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. This identification leads us to the following proposition.

Proposition 2.2. [19] *A code C of length n over R is a λ-constacyclic code if and only if C is an ideal of $\frac{R[x]}{\langle x^n - \lambda \rangle}$.*

For n-tuples $a = (a_0, a_1, \ldots, a_{n-1})$ and $b = (b_0, b_1, \ldots, b_{n-1})$ in R^n, their inner product is defined as

$$a \cdot b = a_0b_0 + a_1b_1 + \cdots + a_{n-1}b_{n-1}.$$

Two n-tuples a and b are considered orthogonal if $a \cdot b = 0$. The dual code $C^\perp$ of a linear code C of length n over a ring R is defined as

$$C^\perp = \{a \in R^n \mid a \cdot b = 0 \text{ for all } b \in C\}.$$

The following proposition is well known.

Proposition 2.3. [24] *Consider a linear code C of length n over a finite Frobenius ring R. Then,*

$$|C| \cdot |C^\perp| = |R|^n.$$

In general, the dual of a λ-constacyclic code satisfies the following proposition.

Proposition 2.4. [10] *The dual of a λ-constacyclic code is a λ^{-1}-constacyclic code.*

The annihilator of an ideal I in a ring R, denoted as $\mathcal{A}(I)$, is defined as

$$\mathcal{A}(I) = \{a \in R \mid ab = 0 \text{ for all } b \in I\}.$$

For any polynomial $f(x) = a_0 + a_1x + \ldots + a_lx^l \in R[x]$ with $a_l \neq 0$, the reciprocal polynomial $f^*(x)$ is defined as

$$f^*(x) = x^l f(x^{-1}) = a_l + a_{l-1}x + \ldots + a_0x^l.$$

The following proposition establishes a relationship between the dual and the annihilator of a λ-constacyclic code.

Proposition 2.5. [13] *Let C be a λ-constacyclic code of length n over R, i.e., an ideal of $\frac{R[x]}{\langle x^n-\lambda\rangle}$. Then,*

$$C^\perp = \mathcal{A}(C)^* := \{f^*(x) \mid f(x) \in \mathcal{A}(C)\}.$$

Throughout this paper, we consider the rings:

$$R_{u,v} = \frac{\mathbb{F}_{p^m}[u,v]}{\langle u^k, v^2, uv - vu\rangle}, \quad T_u = \frac{\mathbb{F}_{p^m}[u]}{\langle u^k\rangle},$$

where p is a prime number, and m, k are positive integers. The Jacobson radical and the socle of $R_{u,v}$ are given by:

$$J(R_{u,v}) = \langle u, v\rangle, \quad \mathrm{soc}(R_{u,v}) = \langle uv\rangle.$$

Moreover, the quotient ring $\frac{R_{u,v}}{J(R_{u,v})}$ is isomorphic to $\mathrm{soc}(R_{u,v})$ as an $R_{u,v}$-module via the natural isomorphism:

$$r + J(R_{u,v}) \longmapsto ruv,$$

for any $r \in R_{u,v}$. Thus, $R_{u,v}$ is a Frobenius ring.

Each unit in $R_{u,v}$ has the form:

$$\lambda = \sum_{i=0}^{k-1} \lambda_i u^i + v\sum_{i=0}^{k-1} \lambda_i' u^i,$$

which can be rewritten as:

$$\lambda = \lambda_0 + \gamma u + \mu v,$$

where $\lambda_i, \lambda_i' \in \mathbb{F}_{p^m}$ and $\lambda_0, \lambda_1 \neq 0$, with:

$$\gamma = \lambda_1 + \lambda_2 u + \cdots + \lambda_{k-1}u^{k-2}, \quad \mu = \lambda_0' + \lambda_1' u + \cdots + \lambda_{k-1}' u^{k-1}.$$

Throughout this article, we assume $\lambda_1 \neq 0$, which ensures that γ is a unit in T_u.

Let s be a positive integer. Using the division algorithm, we write $s = q_s m + r_s$ with $0 \leq r_s < m$. Since $\lambda_0^{p^m} = \lambda_0$, we define:

$$\theta = \lambda_0^{p^{m-r_s}} = \lambda_0^{p^{(q_s+1)m-s}}.$$

It follows that $\theta^{p^s} = \lambda_0$.

Our objective is to study λ-constacyclic codes of length ηp^s over $R_{u,v}$, where η is an integer coprime to p. By Proposition 2.2, these codes correspond to the ideals of the ring:

$$\mathcal{R}_\lambda := \frac{R_{u,v}[x]}{\langle x^{\eta p^s} - \lambda\rangle}.$$

The following ring homomorphisms will be used in subsequent sections:

$$\Phi : R_{u,v}[x] \to T_u[x], \quad f(x) \mapsto f(x) \mod v,$$

$$\bar{\cdot} : R_{u,v}[x] \to \mathbb{F}_{p^m}[x], \quad f(x) \mapsto f(x) \mod (u,v).$$

3. Direct sum decomposition of λ-constacyclic codes over $R_{u,v}$

Since η is coprime with p, the polynomial $x^\eta - \theta$ has a unique factorization in $\mathbb{F}_{p^m}[x]$:

$$x^\eta - \theta = z_1(x)z_2(x)\dots z_r(x),$$

where the factors are monic, irreducible, and pairwise coprime. Raising both sides to p^s gives:

$$x^{\eta p^s} - \lambda_0 = z_1(x)^{p^s} z_2(x)^{p^s} \dots z_r(x)^{p^s}.$$

Since $R_{u,v}$ is a finite local ring and $x^{\eta p^s} - \lambda$ is a regular element in $R_{u,v}[x]$, Hensel's Lemma [22] ensures the existence of a unique factorization

$$x^{\eta p^s} - \lambda = \omega_1(x)\omega_2(x)\dots\omega_r(x),$$

where $\omega_1(x), \dots, \omega_r(x)$ are monic and pairwise coprime polynomials in $R_{u,v}[x]$. Moreover, for each $i = 1, \dots, r$,

$$\overline{\omega_i}(x) = z_i(x)^{p^s}.$$

Defining $\widehat{\omega}_i(x) = \frac{x^{\eta p^s}-\lambda}{\omega_i(x)}$, it follows that $\widehat{\omega}_i(x)$ and $\omega_i(x)$ are coprime. Hence, there exist polynomials $l_i(x), l_i'(x) \in R_{u,v}[x]$ such that:

$$l_i(x)\widehat{\omega}_i(x) + l_i'(x)\omega_i(x) = 1.$$

Setting $e_i(x) = l_i(x)\widehat{\omega}_i(x)$, it follows from [7, Theorem 3.2] that the following properties hold:

- The set $\{e_1(x), e_2(x), \dots, e_r(x)\}$ forms a complete system of primitive pairwise orthogonal idempotents, meaning that

$$\sum_{i=1}^{r} e_i(x) = 1, \quad e_i(x)e_j(x) = 0 \quad \text{for } i \neq j, \quad e_i(x)^2 = e_i(x). \tag{1}$$

- The ring $\mathcal{R}_\lambda$ decomposes as

$$\mathcal{R}_\lambda = \bigoplus_{i=1}^{r} e_i(x)\mathcal{R}_\lambda.$$

- For each i, the mapping

$$\pi_i : \frac{R_{u,v}[x]}{\langle \omega_i(x) \rangle} \to e_i(x)\mathcal{R}_\lambda, \quad \pi_i(c(x)) = e_i(x)c(x),$$

  is a ring isomorphism.

As a consequence, we obtain the following theorem.

Theorem 3.1. *A subset C of $\mathcal{R}_\lambda$ is an ideal if and only if*

$$C = \bigoplus_{i=1}^{r} e_i(x)I_i,$$

where I_i is an ideal of $\frac{R_{u,v}[x]}{\langle \omega_i(x) \rangle}$.

Thus, to classify all λ-constacyclic codes over $R_{u,v}$, we need only to study the ideals of $\frac{R_{u,v}[x]}{\langle \omega_i(x) \rangle}$. To this end, for each $1 \leq i \leq t$, we define:

- $\psi_i(x) = \Phi(\omega_i(x))$,
- $\mathcal{R}_i = \frac{R_{u,v}[x]}{\langle \omega_i(x) \rangle}$,
- $\mathcal{S}_i = \frac{T_u[x]}{\langle \psi_i(x) \rangle}$,
- $\hat{z}_i(x) = \frac{x^\eta - \theta}{z_i(x)}$.

Since $\psi_i(x) = \Phi(\omega_i(x))$, the universal property of quotient rings guarantees that Φ induces a unique homomorphism

$$\Phi_i : \mathcal{R}_i \to \mathcal{S}_i,$$

satisfying

$$\Phi_i(f(x) + \langle \omega_i(x) \rangle) = \Phi(f(x)) + \langle \psi_i(x) \rangle, \quad \forall f(x) \in R_{u,v}[x].$$

The following pairs of propositions will be used in the subsequent sections.

Proposition 3.2. *The polynomial $\hat{z}_i(x)$ is a unit in $\mathcal{S}_i$ and $\mathcal{R}_i$.*

Proof. It suffices to prove that $\hat{z}_i(x)$ is coprime to $\omega_i(x)$ in $R_{u,v}[x]$ and to $\psi_i(x)$ in $T_u[x]$.

Since $\hat{z}_i(x)$ and $z_i(x)^{p^s}$ are coprime in $\mathbb{F}_{p^m}[x]$, there exist polynomials $f_i(x), g_i(x) \in \mathbb{F}_{p^m}[x]$ such that

$$f_i(x) z_i(x)^{p^s} + g_i(x) \hat{z}_i(x) = 1.$$

Given that $\overline{\omega_i}(x) = z_i(x)^{p^s}$, we can write in $R_{u,v}[x]$:

$$\omega_i(x) = z_i(x)^{p^s} + u h_i(x) + v h_i'(x),$$

for some $h_i(x), h_i'(x) \in R_{u,v}[x]$. Substituting this into the previous relation, we obtain in $R_{u,v}[x]$:

$$f_i(x)\omega_i(x) + g_i(x)\hat{z}_i(x) = 1 + u h_i(x) f_i(x) + v h_i'(x) f_i(x). \tag{2}$$

Since u and v are nilpotent in $R_{u,v}[x]$, the element $1 + u h_i(x) f_i(x) + v h_i'(x) f_i(x)$ is a unit in $R_{u,v}[x]$. Consequently, $\omega_i(x)$ and $\hat{z}_i(x)$ are coprime in $R_{u,v}[x]$.

Now, applying Φ to Eq. (2), we get:

$$\Phi(f_i(x))\Phi(\omega_i(x)) + \Phi(g_i(x))\Phi(\hat{z}_i(x)) = 1 + u\Phi(h(x) f_i(x)).$$

By definition of Φ, we have $\Phi(\omega_i(x)) = \psi_i(x)$ and $\Phi(\hat{z}_i(x)) = \hat{z}_i(x)$, leading to:

$$\Phi(f_i(x))\psi_i(x) + \Phi(g_i(x))\hat{z}_i(x) = 1 + u\Phi(h(x) f_i(x)).$$

Since u is nilpotent in $T_u[x]$, the element $1 + u\Phi(h_i'(x) f_i(x))$ is a unit in $T_u[x]$. We conclude that $\psi_i(x)$ and $\hat{z}_i(x)$ are coprime in $T_u[x]$. This proves that $\hat{z}_i(x)$ is a unit in both $\mathcal{S}_i$ and $\mathcal{R}_i$, completing the proof. □

Proposition 3.3. [3] *In $\mathcal{S}_i$, we have the following properties:*

- $\langle z_i(x)^{p^s} \rangle = \langle u \rangle$, *and thus $z_i(x)$ is nilpotent with nilpotency index kp^s.*
- *The ring $\mathcal{S}_i$ is a chain ring with the following ideal chain:*

$$\mathcal{S}_i = \langle 1 \rangle \supsetneq \langle z_i(x) \rangle \supsetneq \cdots \supsetneq \langle z_i(x)^{kp^s-1} \rangle \supsetneq \langle z_i(x)^{kp^s} \rangle = \langle 0 \rangle. \tag{3}$$

- *Each ideal $\langle z_i(x)^j \rangle$ contains $p^{\deg z_i m(kp^s - j)}$ elements, for all $0 \leq j \leq kp^s$.*

4. The ring $\mathcal{R}_i$ and its ideals

In this section, we classify the ideals of the ring $\mathcal{R}_i$ and determine their cardinalities. We start with the following proposition.

Proposition 4.1. *Let I be an ideal of the ring $\mathcal{R}_i$. Then, it can be expressed as*

$$I = \langle z_i(x)^\alpha + vf(x)\rangle + vJ, \tag{4}$$

where $0 \leq \alpha \leq kp^s$, $\Phi_i(I) = \langle z_i(x)^\alpha\rangle$, and $f(x)$ is a polynomial satisfying $z_i(x)^\alpha + vf(x) \in I$. Moreover, the ideal J is defined as

$$J = \{a(x) \in \mathcal{R}_i \mid va(x) \in I\}.$$

Proof. Let $0 \leq \alpha \leq kp^s$ and $f(x) \in \mathcal{R}_i$ such that $\Phi_i(I) = \langle z_i(x)^\alpha\rangle$ and $z_i(x)^\alpha + vf(x) \in I$. Then, applying Φ_i,

$$\Phi_i(z_i(x)^\alpha + vf(x)) = z_i(x)^\alpha.$$

For any $g(x) \in I$, there exists $h(x) \in \mathcal{S}_i$ such that $\Phi_i(g(x)) = h(x)z_i(x)^\alpha$. By surjectivity of Φ_i, there exists $h'(x) \in \mathcal{R}_i$ satisfying $\Phi_i(h'(x)) = h(x)$. Thus, we obtain $\Phi_i(g(x)) = \Phi_i(h'(x))\Phi_i(z_i(x)^\alpha + vf(x))$, which implies that

$$g(x) - h'(x)(z_i(x)^\alpha + vf(x)) \in \ker(\Phi_i) \cap I.$$

Consequently, we deduce

$$I \subseteq \langle z_i(x)^\alpha + vf(x)\rangle + \ker(\Phi_i) \cap I.$$

Finally, since $\ker(\Phi_i) = v\mathcal{R}_i$, it follows that

$$I = \langle z_i(x)^\alpha + vf(x)\rangle + vJ.$$

□

The following theorem provides a complete classification of the distinct ideals of the ring $\mathcal{R}_i$.

Theorem 4.2. *The distinct ideals of $\mathcal{R}_i$ are classified as follows:*

- *Type 1: The trivial ideals $\langle 0\rangle$ and $\langle 1\rangle$.*
- *Type 2: Ideals of the form $\langle vz_i(x)^\alpha\rangle$, where $0 \leq \alpha \leq kp^s - 1$.*
- *Type 3: Ideals of the form $\langle z_i(x)^\alpha + vz_i(x)^\beta g(x)\rangle$, where $1 \leq \alpha \leq kp^s - 1$, $0 \leq \beta < \Upsilon$, and $g(x)$ is either 0 or a unit in $\mathcal{S}_i$. The parameter Υ is given by:*

$$\Upsilon = \min\left\{t \mid vz_i(x)^t \in \langle z_i(x)^\alpha + vz_i(x)^\beta g(x)\rangle\right\}. \tag{5}$$

- *Type 4: Ideals of the form $\langle z_i(x)^\alpha + vz_i(x)^\beta g(x), vz_i(x)^\delta\rangle$, where $0 \leq \beta < \delta < \Upsilon$, and $g(x)$ is either 0 or a unit in $\mathcal{S}_i$. The parameter Υ is the same as given in (5).*

Proof. The Type 1 ideals are trivial. Let I be a non-trivial ideal of $\mathcal{R}_i$. We consider two cases:

- If $\Phi_i(I) = \{0\}$, then by Proposition 4.1, I can be expressed as $I = vJ$, where

$$J = \{a(x) \in \mathcal{R}_i \mid va(x) \in I\}.$$

The set $\Phi_i(J)$ forms an ideal of $\mathcal{S}_i$, implying that $\Phi_i(J) = \langle z_i(x)^\alpha \rangle$ for some $0 \leq \alpha \leq kp^s - 1$. Applying Proposition 4.1 again, we obtain:

$$J = \langle z_i(x)^\alpha + vf(x) \rangle + vK,$$

where $K = \{a(x) \in \mathcal{R}_i \mid va(x) \in J\}$ and $f(x) \in \mathcal{R}_i$. Given that $v^2 = 0$, it follows that

$$I = \langle vz_i(x)^\alpha \rangle.$$

This classifies I as Type 2.

- If $\Phi_i(I) \neq \{0\}$, we note that $\Phi_i(I)$ is a non-trivial ideal of $\mathcal{S}_i$, then $\Phi_i(I) = \langle z_i(x)^\alpha \rangle$, for some $1 \leq \alpha \leq kp^s - 1$. By Proposition 4.1, the ideal I satisfies

$$I = \langle z_i(x)^\alpha + vf(x) \rangle + vJ,$$

where $f(x) \in \mathcal{R}_i$ and

$$J = \{a(x) \in \mathcal{R}_i \mid va(x) \in I\}.$$

Since $vJ \subseteq \langle v \rangle$, it follows that $\Phi_i(vJ) = \{0\}$. As in the previous case, we have

$$vJ = \langle vz_i(x)^\delta \rangle,$$

for some $0 \leq \delta \leq kp^s$. Then $I = \langle z_i(x)^\alpha + vf(x), vz_i(x)^\delta \rangle$.

Considering $\Phi_i(f(x)) \in \mathcal{S}_i$, if $\Phi_i(f(x)) \neq 0$, there exists a maximal β such that

$$\Phi_i(f(x)) \in \langle z_i(x)^\beta \rangle, \quad \Phi_i(f(x)) \notin \langle z_i(x)^{\beta+1} \rangle.$$

Thus, we write

$$\Phi_i(f(x)) = z_i(x)^\beta d(x),$$

where $d(x) \in \mathcal{S}_i$ is a unit. In $\mathcal{R}_i$, we have

$$f(x) = vh(x) \quad \text{or} \quad f(x) = z_i(x)^\beta d(x) + vh(x),$$

for some $h(x) \in \mathcal{R}_i$. Since $v^2 = 0$, we deduce

$$vf(x) = 0 \quad \text{or} \quad vf(x) = vz_i(x)^\beta d(x).$$

Consequently,

$$I = \langle z_i(x)^\alpha + vz_i(x)^\beta g(x), vz_i(x)^\delta \rangle,$$

where $g(x)$ is either 0 or a unit in $\mathcal{S}_i$. Let Υ be as defined in Eq. (5). If $\delta \geq \Upsilon$, then I simplifies to $\langle z_i(x)^\alpha + vz_i(x)^\beta g(x) \rangle$, corresponding to Type 3. Otherwise, if $\delta < \Upsilon$, it remains $I = \langle z_i(x)^\alpha + vz_i(x)^\beta g(x), vz_i(x)^\delta \rangle$, corresponding to Type 4.

□

Before computing the value of Υ, we first establish the following lemma.

Lemma 4.3. *In $\mathcal{R}_i$, we have*

$$z_i(x)^{kp^s} = k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s} v.$$

Proof. In $R_{v,v}[x]$, with all computations carried out modulo $x^{\eta p^s} - \lambda$, we obtain:

$$\begin{aligned} z_i(x)^{kp^s}\widehat{z}(x)^{kp^s} &= \left(x^{\eta p^s} - \lambda_0\right)^k \\ &= (\gamma u + \mu v)^k \\ &= \sum_{l=0}^{k} \binom{k}{l} (\gamma u)^{k-l} (\mu v)^l \\ &= (\gamma u)^k + k(\gamma u)^{k-1}(\mu v) \\ &= k\mu(\gamma u)^{k-1} v \\ &= k\mu\left(x^{\eta p^s} - \lambda_0\right)^{(k-1)} v. \end{aligned}$$

The last step is justified by the fact that the v-terms in $\left(x^{\eta p^s} - \lambda_0\right)^{(k-1)}$ vanish after multiplication by v, since $v^2 = 0$. On the other hand, by Proposition 3.2, $\widehat{z}(x)$ is a unit in $\mathcal{R}_i$. Thus, in $\mathcal{R}_i$, we obtain:

$$z_i(x)^{kp^s} = k\mu\left(\widehat{z}(x)^{kp^s}\right)^{-1}\left(x^{\eta p^s} - \lambda_0\right)^{(k-1)} v = k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s} v.$$

□

Theorem 4.4. *Let $I = \langle z_i(x)^\alpha + v z_i(x)^\beta g(x)\rangle$ and define $\Upsilon = \min\{t \mid v z_i(x)^t \in I\}$. Then,*

$$\Upsilon = \begin{cases} \alpha, & \text{if } h(x) = 0, \\ \min\{\alpha, \varepsilon\}, & \text{if } h(x) \neq 0. \end{cases} \tag{6}$$

where

$$\varepsilon = \max\left\{0 \le l \le kp^s \mid z_i(x)^{kp^s+\beta-\alpha} g(x) + k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s} \in \left\langle z_i(x)^l \right\rangle\right\}.$$

Thus, we can write

$$z_i(x)^{kp^s+\beta-\alpha} g(x) + k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s} = z_i(x)^\varepsilon h(x), \tag{7}$$

where $h(x) \in \mathcal{S}_i$ is either 0 or a unit in $\mathcal{S}_i$.

Proof. Consider $v z_i(x)^t \in I$, which is equivalent to

$$v z_i(x)^t = f(x)\left(z_i(x)^\alpha + v z_i(x)^\beta g(x)\right), \tag{8}$$

for some $f(x) \in \mathcal{R}_i$, applying Φ_i yields

$$\Phi_i(f(x)) z_i(x)^\alpha = 0 \quad \text{in } \mathcal{S}_i.$$

Since $\mathcal{S}_i$ is a chain ring with maximal ideal $\langle z_i(x)\rangle$ and nilpotency index kp^s, we obtain

$$\Phi_i(f(x)) = z_i(x)^{kp^s-\alpha} f'(x),$$

with $f'(x) \in \mathcal{S}_i$. Consequently, in $\mathcal{R}_i$,

$$f(x) = z_i(x)^{kp^s-\alpha} f'(x) + v f''(x),$$

where $f''(x) \in \mathcal{R}_i$. Substituting this into Eq. (8), we obtain:

$$vz_i(x)^t = \left(z_i(x)^{kp^s-\alpha} f'(x) + v f''(x)\right)\left(z_i(x)^\alpha + vz_i(x)^\beta g(x)\right).$$

Expanding and using $v^2 = 0$, we get:

$$vz_i(x)^t = z_i(x)^{kp^s} f'(x) + v f''(x) z_i(x)^\alpha + vz_i(x)^{kp^s-\alpha+\beta} f'(x)g(x). \tag{9}$$

Using Lemma 4.3,

$$z_i(x)^{kp^s} = k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s} v.$$

Replacing this in Eq. (9), we get:

$$vz_i(x)^t = v\left(z_i(x)^\alpha f''(x) + \left(z_i(x)^{kp^s-\alpha+\beta} g(x) + k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s}\right) f'(x)\right).$$

Using Eq. (7), we obtain:

$$vz_i(x)^t = v\left(z_i(x)^\alpha f''(x) + z_i(x)^\varepsilon h(x) f'(x)\right).$$

We now consider two cases:

- If $h(x) = 0$, then $\Upsilon \geq \alpha$. Since

  $$vz_i(x)^\alpha = v\left(z_i(x)^\alpha + vz_i(x)^\beta g(x)\right) \in I,$$

  it follows that $\Upsilon = \alpha$.
- If $h(x) \neq 0$, then $\Upsilon \geq \min\{\alpha, \varepsilon\}$. Conversely, we have:

  $$vz_i(x)^\alpha = v\left(z_i(x)^\alpha + vz_i(x)^\beta g(x)\right) \in I,$$

  and

  $$vz_i(x)^\varepsilon = h(x)^{-1} z_i(x)^{kp^s-\alpha}\left(z_i(x)^\alpha + vz_i(x)^\beta g(x)\right) \in I.$$

  Therefore, $\Upsilon = \min\{\alpha, \varepsilon\}$.

□

We now count the number of codewords in each ideal of $\mathcal{R}_i$. For this, we introduce two notions: the residue and the torsion of I.

$$\mathrm{Res}(I) = \Phi_i(I), \quad \mathrm{Tor}(I) = \{c(x) \in \mathcal{S}_i \mid vc(x) \in I\}.$$

Clearly, $\mathrm{Res}(I)$ and $\mathrm{Tor}(I)$ are ideals of $\mathcal{S}_i$. By Proposition 3.3, they can be expressed as $\langle z_i(x)^l\rangle$, where $0 \leq l \leq kp^s$. Now, consider the ring homomorphism

$$\begin{array}{rccc} T: & I & \longrightarrow & \Phi_i(I), \\ & c(x) & \longmapsto & \Phi_i(c(x)). \end{array}$$

Since $\operatorname{Im} T = \operatorname{Res}(I)$ and $\ker T \cong v \operatorname{Tor}(I)$, the first isomorphism theorem yields

$$|I| = |\operatorname{Res}(I)| \cdot |\operatorname{Tor}(I)|. \tag{10}$$

The following lemma, whose proof is straightforward, provides explicit expressions for $\operatorname{Res}(I)$ and $\operatorname{Tor}(I)$ for any ideal I in $\mathcal{R}_i$.

Lemma 4.5. *With the notation of Theorem 4.2 :*

- *If* $I = \langle 0 \rangle$, *then* $\operatorname{Tor}(I) = \operatorname{Res}(I) = \langle 0 \rangle$.
- *If* $I = \langle 1 \rangle$, *then* $\operatorname{Tor}(I) = \operatorname{Res}(I) = \langle 1 \rangle$.
- *If* $I = \langle v z_i(x)^{\alpha} \rangle$ *is of type* 2, *then* $\operatorname{Tor}(I) = \langle z_i(x)^{\alpha} \rangle$ *and* $\operatorname{Res}(I) = \langle 0 \rangle$.
- *If* $I = \langle z_i(x)^{\alpha} + v z_i(x)^{\beta} g(x) \rangle$ *is of type* 3, *then* $\operatorname{Tor}(I) = \langle z_i(x)^{\Upsilon} \rangle$ *and* $\operatorname{Res}(I) = \langle z_i(x)^{\alpha} \rangle$.
- *If* $I = \langle z_i(x)^{\alpha} + v z_i(x)^{\beta} g(x), v z_i(x)^{\delta} \rangle$ *is of type* 4, *then* $\operatorname{Tor}(I) = \langle z_i(x)^{\delta} \rangle$ *and* $\operatorname{Res}(I) = \langle z_i(x)^{\alpha} \rangle$.

By determining $\operatorname{Res}(I)$ and $\operatorname{Tor}(I)$ in each case, we can compute the cardinalities of all ideals in $\mathcal{R}_i$ using Eq. (10) and Proposition 3.3.

5. Dual codes of λ-constacyclic codes over $R_{v,v}$

In this section, we focus on the dual codes of λ-constacyclic codes of length ηp^s over the ring $R_{v,v}$. By Theorem 3.1,

$$C = \bigoplus_{i=1}^{r} e_i(x) I_i,$$

where I_i is an ideal of $\mathcal{R}_i$ for $1 \le i \le r$. By Eq. (1), we have

$$\begin{aligned}
d(x) = \sum_{i=1}^{r} e_i(x) \cdot d(x) \in \mathcal{A}(C) \quad & \Leftrightarrow \quad \forall c(x) \in C, \left(\sum_{i=1}^{r} e_i(x) \cdot d(x) \right) \cdot \left(\sum_{i=1}^{r} e_i(x) \cdot c(x) \right) = 0 \\
& \Leftrightarrow \quad \forall c(x) \in C, \left(\sum_{i=1}^{r} e_i(x) \cdot d(x) \cdot c(x) \right) = 0 \\
& \Leftrightarrow \quad \forall c(x) \in C, \forall i \in \{1, ..., r\}, e_i(x) \cdot d(x) \cdot c(x) = 0 \\
& \Leftrightarrow \quad \forall c(x) \in C, \forall i \in \{1, ..., r\}, d(x) \cdot c(x) = 0 \text{ in } \mathcal{R}_i \\
& \Leftrightarrow \quad \forall i \in \{1, ..., r\}, d(x) \in \mathcal{A}(I_i).
\end{aligned}$$

Therefore,

$$\mathcal{A}(C) = \bigoplus_{i=1}^{r} e_i(x) \mathcal{A}(I_i). \tag{11}$$

Thus, by Proposition 2.5,

$$C^{\perp} = \mathcal{A}(C)^{*} = \bigoplus_{i=1}^{r} e_i^{*}(x) \mathcal{A}(I_i)^{*}.$$

We aim to determine $\mathcal{A}(I)^{*}$, where I is an ideal of $\mathcal{R}_i$. To this end, we first establish the following three lemmas.

Lemma 5.1. *Let* I *be an ideal of* $\mathcal{R}_i$. *Then,* $|I| \cdot |\mathcal{A}(I)| = |\mathcal{R}_i|$.

Proof. Let $C = \bigoplus_{j=1}^{r} e_j(x) I_j$ be an ideal of $\mathcal{R}_\lambda$, where

$$I_j = I \quad \text{if } j = i, \quad \text{and} \quad I_j = \langle 1 \rangle \quad \text{otherwise}.$$

Then, by Eq. (11), we obtain

$$\mathcal{A}(C) = \bigoplus_{j=1}^{r} e_j(x) J_j,$$

where

$$J_j = \mathcal{A}(I) \quad \text{if } j = i, \quad \text{and} \quad J_j = \langle 0 \rangle \quad \text{otherwise}.$$

According to Propositions 2.3 and 2.5, we have $|C| \cdot |\mathcal{A}(C)| = |R_{v,v}|^{\eta p^s}$. Therefore,

$$\Big(\prod_{\substack{1 \le j \le r \\ j \ne i}} |\mathcal{R}_j|\Big) \cdot |I| \cdot |\mathcal{A}(I)| = |R_{v,v}|^{\eta p^s}.$$

It follows that

$$|I| \cdot |\mathcal{A}(I)| = \frac{|R_{v,v}|^{\eta p^s}}{\prod_{\substack{1 \le j \le r \\ j \ne i}} |\mathcal{R}_j|} = |\mathcal{R}_i|.$$

□

Lemma 5.2. *Let I be an ideal of $\mathcal{R}_i$ and a, b two integers such that*

$$\mathrm{Res}(I) = \langle z_i(x)^a \rangle \quad \textit{and} \quad \mathrm{Tor}(I) = \langle z_i(x)^b \rangle.$$

Then, we have:

$$\mathrm{Res}(\mathcal{A}(I)) = \langle z_i(x)^{kp^s - b} \rangle, \quad \mathrm{Tor}(\mathcal{A}(I)) = \langle z_i(x)^{kp^s - a} \rangle.$$

Proof. Let

$$\mathrm{Res}(\mathcal{A}(I)) = \langle z_i(x)^{a'} \rangle, \quad \mathrm{Tor}(\mathcal{A}(I)) = \langle z_i(x)^{b'} \rangle,$$

where a' and b' are two integers.

Since $v z_i(x)^b \in I$ and $v z_i(x)^{b'} \in \mathcal{A}(I)$, there exist two polynomials $g(x)$ and $f(x)$ in $\mathcal{R}_i$ such that

$$z_i(x)^a + v g(x) \in I, \quad z_i(x)^{a'} + v f(x) \in \mathcal{A}(I).$$

By the definition of $\mathcal{A}(I)$, we obtain:

$$0 = v z_i(x)^b \left(z_i(x)^{a'} + v f(x) \right) = v z_i(x)^{b+a'}. \tag{12}$$

Similarly, we have:

$$0 = v z_i(x)^{b'} \left(z_i(x)^a + v g(x) \right) = v z_i(x)^{b'+a}. \tag{13}$$

Therefore, we necessarily have:

$$a' \ge kp^s - b, \quad b' \ge kp^s - a.$$

By Lemma 5.1, $|I| \cdot |\mathcal{A}(I)| = |\mathcal{R}_i|$, and by Proposition 3.3, we obtain

$$p^{m2k\deg z_i} = |\mathcal{R}_i| = p^{\deg z_i m(kp^s-(a+b+a'+b'))} \leq p^{m2k\deg z_i}$$

It follows that:

$$b' = kp^s - a, \quad a' = kp^s - b.$$

□

Lemma 5.3. *With the previous notation, we have:*

$$\mathcal{A}(I) = \langle z_i(x)^{kp^s-b} + vf(x), vz_i(x)^{kp^s-a} \rangle.$$

Proof. Firstly, it is clear that $vz_i(x)^{kp^s-a} \in \mathcal{A}(I)$ and $z_i(x)^{kp^s-b} + vf(x) \in \mathcal{A}(I)$.

Now, let $c(x) \in \mathcal{A}(I)$. Then, we have $\Phi_i(c(x)) \in \text{Res}(\mathcal{A}(I))$, which implies that there exists $c_0(x) \in \mathcal{S}_i$ such that:

$$\Phi_i(c(x)) = c_0(x)z_i(x)^{kp^s-b} = \Phi_i\left(c_0(x)\left(z_i(x)^{kp^s-b} + vf(x)\right)\right).$$

Therefore, we obtain:

$$c(x) - c_0(x)\left(z_i(x)^{kp^s-b} + vf(x)\right) \in \ker \Phi_i.$$

This implies that:

$$\mathcal{A}(I) \ni c(x) - c_0(x)\left(z_i(x)^{kp^s-a} + vf(x)\right) = vc_1(x),$$

where $c_1(x) \in \mathcal{S}_i$. Since $c_1(x) \in \text{Tor}(\mathcal{A}(I)) = \langle z_i(x)^{kp^s-a} \rangle$, we deduce that:

$$c(x) \in \langle z_i(x)^{kp^s-b} + vf(x), vz_i(x)^{kp^s-a} \rangle.$$

This shows that:

$$\mathcal{A}(I) = \langle z_i(x)^{kp^s-b} + vf(x), vz_i(x)^{kp^s-a} \rangle.$$

□

We now proceed to determine the annihilator of each type of ideal. For type 1 ideals, this is straightforward: if $I = \langle 0 \rangle$, then $\mathcal{A}(I) = \langle 1 \rangle$; if $I = \langle 1 \rangle$, then $\mathcal{A}(I) = \langle 0 \rangle$. For other types, we first identify two integers a and b such that $\text{Res}(I) = \langle z_i(x)^a \rangle$ and $\text{Tor}(I) = \langle z_i(x)^b \rangle$, as given in Lemma 4.3. Once these values are determined, we find a polynomial $f(x)$ satisfying $z_i(x)^{kp^s-b} + vf(x) \in \mathcal{A}(I)$. Finally, applying Lemma 5.3, we obtain $\mathcal{A}(I)$.

Proposition 5.4. *If $I = \langle vz_i(x)^\alpha \rangle$ is an ideal of type 2, then $\mathcal{A}(I) = \langle z_i(x)^{kp^s-\alpha}, v \rangle$.*

Proof. It is clear that $z_i(x)^{kp^s-\alpha} \in \mathcal{A}(I)$. Therefore, we have $\mathcal{A}(I) = \langle z_i(x)^{kp^s-\alpha}, v \rangle$.

□

Proposition 5.5. *If $I = \langle z_i(x)^\alpha + vz_i(x)^\beta g(x) \rangle$ is an ideal of type 3, then:*

$$\mathcal{A}(I) = \begin{cases} \left\langle z_i(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) = 0, \\ \langle z_i(x)^{kp^s-\Upsilon} - vz_i(x)^{\varepsilon-\Upsilon}h(x), vz_i(x)^{kp^s-\alpha} \rangle, & \text{if } h(x) \neq 0. \end{cases}$$

Proof. Since $\mathrm{Tor}(I) = \langle z_i(x)^{\Upsilon} \rangle$, it suffices to determine a polynomial $f(x) \in \mathcal{S}_i$ satisfying

$$z_i(x)^{kp^s-\Upsilon} + vf(x) \in \mathcal{A}(I).$$

By Theorem 4.4, we have

$$\begin{aligned} 0 &= \left(z_i(x)^{kp^s-\Upsilon} + vf(x)\right)\left(z_i(x)^{\alpha} + vz_i(x)^{\beta}g(x)\right) \\ &= z_i(x)^{kp^s-\Upsilon+\alpha} + v\left(z_i(x)^{kp^s-\Upsilon+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= z_i(x)^{\alpha-\Upsilon}z_i(x)^{kp^s} + v\left(z_i(x)^{kp^s-\Upsilon+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= v\left(k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s+\alpha-\Upsilon} + z_i(x)^{kp^s-\Upsilon+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= v\left(z_i(x)^{\alpha-\Upsilon+\varepsilon}h(x) + z_i(x)^{\alpha}f(x)\right). \end{aligned} \tag{14}$$

We now distinguish two cases:

- If $h(x) = 0$, then $\Upsilon = \alpha$, and Eq. (14) simplifies to

$$0 = vz_i(x)^{\alpha}f(x).$$

  In this case, choosing $f(x) = 0$ leads to

$$\mathcal{A}(I) = \langle z_i(x)^{kp^s-\alpha}, vz_i(x)^{kp^s-\alpha} \rangle = \langle z_i(x)^{kp^s-\alpha} \rangle.$$

- If $h(x) \neq 0$, we take $f(x) = -z_i(x)^{\varepsilon-\Upsilon}h(x)$, yielding

$$\mathcal{A}(I) = \langle z_i(x)^{kp^s-\Upsilon} - vz_i(x)^{\varepsilon-\Upsilon}h(x), vz_i(x)^{kp^s-\alpha} \rangle.$$

□

Proposition 5.6. *If $I = \langle z_i(x)^{\alpha} + vz_i(x)^{\beta}g(x), vz_i(x)^{\delta} \rangle$, an ideal of type 4, then:*

$$\mathcal{A}(I) = \begin{cases} \left\langle z_i(x)^{kp^s-\delta}, vz_i(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) = 0, \\ \left\langle z_i(x)^{kp^s-\delta} - vz_i(x)^{\varepsilon-\delta}h(x), vz_i(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) \neq 0. \end{cases}$$

Proof. Since $\mathrm{Tor}(I) = \langle z_i(x)^{\delta} \rangle$, it suffices to determine a polynomial $f(x) \in \mathcal{S}_i$ satisfying

$$z_i(x)^{kp^s-\delta} + vf(x) \in \mathcal{A}(I).$$

Given that

$$\left(z_i(x)^{kp^s-\delta} + vf(x)\right)\left(vz_i(x)^{kp^s-\alpha}\right) = 0,$$

it suffices that

$$\left(z_i(x)^{kp^s-\delta} + vf(x)\right)\left(z_i(x)^{\alpha} + vz_i(x)^{\beta}g(x)\right) = 0.$$

By Theorem 4.4, this is equivalent to:

$$\begin{aligned} 0 &= z_i(x)^{kp^s-\delta+\alpha} + v\left(z_i(x)^{kp^s-\delta+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= z_i(x)^{\alpha-\delta}z_i(x)^{kp^s} + v\left(z_i(x)^{kp^s-\delta+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= v\left(k\mu\left(\widehat{z}(x)^{p^s}\right)^{-1} z_i(x)^{(k-1)p^s+\alpha-\delta} + z_i(x)^{kp^s-\delta+\beta}g(x) + z_i(x)^{\alpha}f(x)\right) \\ &= v\left(z_i(x)^{\alpha-\delta+\varepsilon}h(x) + z_i(x)^{\alpha}f(x)\right). \end{aligned} \tag{15}$$

Then, we choose $f(x) = -z_i(x)^{\varepsilon-\delta}h(x)$ if $h(x) \neq 0$ and $f(x) = 0$ if $h(x) = 0$, resulting in:

- If $h(x) = 0$,

$$\mathcal{A}(I) = \left\langle z_i(x)^{kp^s-\delta}, vz_i(x)^{kp^s-\alpha} \right\rangle.$$

- If $h(x) \neq 0$,

$$\mathcal{A}(I) = \left\langle z_i(x)^{kp^s-\delta} - vz_i(x)^{\varepsilon-\delta}h(x), vz_i(x)^{kp^s-\alpha} \right\rangle.$$

□

We now determine $\mathcal{A}(I)^*$ for any ideal in $\mathcal{R}_i$. If I is of type 1, then $\mathcal{A}(I)^* = \langle 0 \rangle$ when $I = \langle 0 \rangle$, and $\mathcal{A}(I)^* = \langle 1 \rangle$ when $I = \langle 1 \rangle$. For other types, we introduce the following notation: Let

$$\vartheta = \max \left\{ 0 \leq l \leq kp^s \mid x^{(kp^s-\varepsilon)\deg z_i} h(x^{-1}) \in \left\langle z_i(x)^l \right\rangle \right\}.$$

Then, we can write

$$x^{(kp^s-\varepsilon)\deg z_i} h(x^{-1}) = z_i(x)^{\vartheta} h'(x),$$

where $h'(x)$ is either 0 or a unit in $\mathcal{S}_i$.

Corollary 5.7. *If $I = \langle vz_i(x)^{\alpha} \rangle$ is an ideal of type 2, then $\mathcal{A}(I)^* = \langle z_i^*(x)^{kp^s-\alpha}, v \rangle$.*

Corollary 5.8. *If $I = \langle z_i(x)^{\alpha} + vz_i(x)^{\beta} g(x) \rangle$ is an ideal of type 3, then:*

$$\mathcal{A}(I)^* = \begin{cases} \left\langle z_i^*(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) = 0, \\ \left\langle z_i^*(x)^{kp^s-\Upsilon} - vz_i(x)^{\varepsilon-\Upsilon+\vartheta} h'(x), vz_i^*(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) \neq 0. \end{cases}$$

Proof. The result is immediate when $h(x) = 0$. If $h(x) \neq 0$, we have

$$\begin{aligned} \mathcal{A}(I)^* &= \left\langle z_i^*(x)^{kp^s-\Upsilon} - vx^{(kp^s-\Upsilon)\deg z_i} z(x^{-1})^{\varepsilon-\Upsilon} h(x^{-1}), vz_i^*(x)^{kp^s-\alpha} \right\rangle \\ &= \left\langle z_i^*(x)^{kp^s-\Upsilon} - vx^{(kp^s-\varepsilon)\deg z_i} z_i^*(x)^{\varepsilon-\Upsilon} h(x^{-1}), vz_i^*(x)^{kp^s-\alpha} \right\rangle \\ &= \left\langle z_i^*(x)^{kp^s-\Upsilon} - vz_i(x)^{\varepsilon-\Upsilon+\vartheta} h'(x), vz_i^*(x)^{kp^s-\alpha} \right\rangle. \end{aligned}$$

□

Corollary 5.9. *If $I = \langle z_i(x)^{\alpha} + vz_i(x)^{\beta} g(x), vz_i(x)^{\delta} \rangle$, an ideal of type 4, then:*

$$\mathcal{A}(I)^* = \begin{cases} \left\langle z_i^*(x)^{kp^s-\delta}, vz_i^*(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) = 0, \\ \left\langle z_i^*(x)^{kp^s-\delta} - vz_i(x)^{\varepsilon-\delta+\vartheta} h'(x), vz_i^*(x)^{kp^s-\alpha} \right\rangle, & \text{if } h(x) \neq 0. \end{cases}$$

Proof. It is similar to Corollary 5.8; it suffices to substitute Υ with δ. □

6. Hamming distance of λ-constacyclic codes of length p^s over $R_{u,v}$

The Hamming weight of a codeword c, denoted by $wt_H(c)$, represents the number of nonzero components in the vector c. The Hamming distance between two vectors c and c', denoted by $d_H(c, c')$, is defined as $wt_H(c - c')$.

For a linear code C, the Hamming distance $d_H(C)$ is given by the minimum weight among all nonzero codewords in C.

In this section, we compute the Hamming distance of λ-constacyclic codes of length p^s over the ring $R_{u,v}$. We begin by establishing the structure of these codes, which can be derived from Theorem 4.2.

Corollary 6.1. *λ-constacyclic codes of length p^s over the ring $R_{u,v}$, i.e., ideals of*

$$\frac{R_{u,v}[x]}{\langle x^{p^s}-\lambda\rangle},$$

can be classified as follows:

- *Type 1: The trivial ideals $\langle 0\rangle$ and $\langle 1\rangle$.*
- *Type 2: Ideals of the form $\langle v(x-\theta)^{\alpha}\rangle$, where $0\le\alpha\le kp^s-1$.*
- *Type 3: Ideals of the form $\langle (x-\theta)^{\alpha}+v(x-\theta)^{\beta}g(x)\rangle$, where $1\le\alpha\le kp^s-1$, $0\le\beta<\Upsilon$, and $g(x)$ is either 0 or a unit in $\frac{T_u[x]}{\langle x^{p^s}-\lambda_0-\gamma u\rangle}$,.*
- *Type 4: Ideals of the form $\langle (x-\theta)^{\alpha}+v(x-\theta)^{\beta}g(x), v(x-\theta)^{\delta}\rangle$, where $0\le\beta<\delta<\Upsilon$, and $g(x)$ is either 0 or a unit in $\frac{T_u[x]}{\langle x^{p^s}-\lambda_0-\gamma u\rangle}$.*

Moreover,

$$\Upsilon=\begin{cases}\alpha, & \text{if } h(x)=0,\\ \min\{\alpha,\varepsilon\}, & \text{if } h(x)\neq 0.\end{cases}$$

where

$$\varepsilon=\max\left\{0\le l\le kp^s \mid (x-\theta)^{kp^s+\beta-\alpha}g(x)+k\mu(x-\theta)^{(k-1)p^s}\in\left\langle (x-\theta)^{l}\right\rangle\right\}.$$

The following lemma establishes a relationship between the Hamming distance of a λ-constacyclic code and its torsion $\mathrm{Tor}(C)$.

Lemma 6.2. *For any λ-constacyclic code C of length p^s over $R_{u,v}$, we have*

$$d_H(C)=d_H(\mathrm{Tor}(C)).$$

Proof. We first prove that $d_H(C)\le d_H(\mathrm{Tor}(C))$. Let $a(x)$ be a nonzero polynomial in $\mathrm{Tor}(C)$, so that $va(x)\in C$. Since $a(x)$ does not involve v, both $a(x)$ and $va(x)$ share the same nonzero coefficients, which implies

$$wt_H(va(x))=wt_H(a(x))\neq 0.$$

Hence, we obtain

$$d_H(C)\le d_H(\mathrm{Tor}(C)).$$

To prove the reverse inequality, let $f(x)\in C$ be a nonzero polynomial, and decompose it as

$$f(x)=a(x)+vb(x),$$

where $a(x), b(x)$ are polynomials that do not involve v.

- If $a(x)=0$, then $b(x)\in\mathrm{Tor}(C)$, leading to

$$d_H(\mathrm{Tor}(C))\le wt_H(f(x)).$$

- If $a(x) \neq 0$, then $va(x) = vf(x)$ is a nonzero element of C. Hence, $a(x) \in \mathrm{Tor}(C)$. Moreover, $a(x)$ and $va(x)$ share the same nonzero components, while $vf(x)$ has more zero components than $f(x)$. It follows that

$$d_H(\mathrm{Tor}(C)) \leq wt_H(a(x)) = wt_H(va(x)) \leq wt_H(f(x)).$$

Thus, we conclude that $d_H(\mathrm{Tor}(C)) \leq d_H(C)$, completing the proof. □

The previous lemma reduces the computation of the Hamming distance of a λ-constacyclic code C of length p^s over $R_{u,v}$ to that of its torsion $\mathrm{Tor}(C)$, which corresponds to a $(\lambda_0 + \gamma v)$-constacyclic code of the same length over T_u.

Let $C_\tau = \langle (x-\theta)^\tau \rangle$ be a nonzero $(\lambda_0+\gamma v)$-constacyclic code of length p^s over T_u, where $0 \leq \tau \leq p^s k$. We distinguish two cases:

- If $0 \leq \tau \leq p^s(k-1)$, then the chain of inclusions

$$C_{p^s(k-1)} \subset \cdots \subset C_\tau \subset \cdots \subset C_0 = \langle 1 \rangle$$

  implies

$$d_H(C_{p^s(k-1)}) \geq \cdots \geq d_H(C_\tau) \geq 1.$$

  By Proposition 3.3, $C_{p^s(k-1)} = \langle u^{k-1} \rangle$. Then, $d_H(C_{p^s(k-1)}) = 1$, which implies that $d_H(C_\tau) = 1$ for all $0 \leq \tau \leq p^s(k-1)$.

- If $p^s(k-1)+1 \leq \tau \leq p^s k - 1$, writing $\tau = p^s(k-1) + \varsigma$ with $1 \leq \varsigma \leq p^s - 1$, we obtain

$$C_\tau = \langle v^{k-1}(x-\theta)^\varsigma \rangle.$$

  Thus, each C_τ corresponds to the λ_0-constacyclic code $\langle (x-\theta)^\varsigma \rangle$ over $\mathbb{F}_{p^m}$, multiplied by v^{k-1}, leading to

$$d_H(C_\tau) = d_H(\langle (x-\theta)^\varsigma \rangle).$$

The Hamming distance of constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ is determined by the following proposition.

Proposition 6.3. [9] *Let $C_\varsigma = \langle (x-\theta)^\varsigma \rangle$ be a λ_0-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $\varsigma \in \{0, 1, \ldots, p^s\}$. The Hamming distance $d_H(C_\varsigma)$ is given by*

$$d_H(C_\varsigma) = \begin{cases} 1, & \text{if } \varsigma = 0, \\ \varpi + 2, & \text{if } \varpi p^{s-1} + 1 \leq \varsigma \leq (\varpi+1)p^{s-1}, \quad 0 \leq \varpi \leq p-2, \\ (q+1)p^j, & \text{if } p^s - p^{s-j} + (q-1)p^{s-j-1} + 1 \leq \varsigma \leq p^s - p^{s-j} + qp^{s-j-1}, \\ & \quad 1 \leq q \leq p-1, \quad 1 \leq j \leq s-1, \\ 0, & \text{if } \varsigma = p^s. \end{cases}$$

Thus, we establish the following theorem.

Theorem 6.4. *Let C be a λ-constacyclic code of length p^s over $R_{u,v}$. Then, its Hamming distance is given by*

$$d_H(C) = \begin{cases} 1, & \text{if } 0 \leq \tau \leq p^s(k-1) \text{ or } \varsigma = 0, \\ \varpi + 2, & \text{if } \varpi p^{s-1} + 1 \leq \varsigma \leq (\varpi+1)p^{s-1}, \quad 0 \leq \varpi \leq p-2, \\ (q+1)p^j, & \text{if } p^s - p^{s-j} + (q-1)p^{s-j-1} + 1 \leq \varsigma \leq p^s - p^{s-j} + qp^{s-j-1}, \\ & \quad 1 \leq q \leq p-1, \quad 1 \leq j \leq s-1, \\ 0, & \text{if } \varsigma = p^s, \end{cases}$$

where $0 \leq \tau \leq kp^s$ satisfies $\mathrm{Tor}(C) = \langle (x-\theta)^\tau \rangle$, and if $\tau \geq p^s(k-1)$, then $\varsigma = \tau - p^s(k-1)$.

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