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Decomposition of the cartesian product of complete graphs into paths and cycles of length six

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ABSTRACT

Let P_k and C_k respectively denote a path and a cycle on k vertices. In this paper, we give necessary and sufficient conditions for the existence of a *complete* $\{P_7, C_6\}$- *decomposition* of the cartesian product of complete graphs.

Keywords: graph decomposition, path, cycle and product graph

1. Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [5]. Let P_k, C_k, S_k, K_k respectively denote a *path*, *cycle*, *star* and *complete* graph on k vertices, and let $K_{m,n}$ denote the *complete bipartite* graph with m and n vertices in the parts. A graph whose vertex set is partitioned into subsets $V_1, ..., V_m$ such that the edge set is $\cup_{i \neq j \in [m]} V_i \times V_j$ is a *complete m-partite* graph, denoted as $K_{n_1,...,n_m}$, when $|V_i| = n_i$ for all i. For $G = K_{2n}$ or $K_{n,n}$, the graph $G - I$ denotes the graph G with a 1-factor I removed. For any integer $\lambda > 0$, λG denotes λ edge-disjoint copies of G. The complement of the graph G is denoted by $\overline{G}$. For two graphs G and H we define their *Cartesian product*, denoted by $G \square H$, as follows: the vertex set is $V(G) \times V(H)$ and its edge set is

$$E(G \square H) = \{(g, h)(g', h') : g = g',\ hh' \in E(H),\ or\ gg' \in E(G),\ h = h'\}.$$

It is well known that the Cartesian product is commutative and associative. For a graph G, a partition of G into edge-disjoint sub graphs $H_1, \cdots, H_k$ such that $E(G) = E(H_1) \cup \cdots \cup E(H_k)$ is called a *decomposition* of G and we write G as $G = H_1 \oplus \cdots \oplus H_k$. For $1 \leq i \leq k$, if $H_i \cong H$, we say

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that G has a H-*decomposition*. If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a $\{pH_1, qH_2\}$-*decomposition*. If such a decomposition exists for all possible values of p and q satisfying trivial necessary conditions, then we say that G has a $\{H_1, H_2\}_{\{p,q\}}$-*decomposition* or *complete* $\{H_1, H_2\}$-*decomposition*.

The study of $\{H_1, H_2\}_{\{p,q\}}$-decomposition of graphs is not new. Authors in [2, 4] completely determined the values of n for which $K_n(\lambda)$ admits a $\{pH_1, qH_2\}$-decomposition such that $H_1 \cup H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, when $t \in \{4, 5\}$. Abueida and Daven [3] proved that there exists a $\{pK_k, qS_{k+1}\}$-decomposition of K_n, for $k \geq 3$ and $n \equiv 0, 1 (mod\ k)$. Abueida and O'Neil [1] proved that for $k \in \{3, 4, 5\}$, there exists a $\{pC_k, qS_k\}$-decomposition of $K_n(\lambda)$, whenever $n \geq k + 1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1), (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. Farrell and Pike [7] shown that the necessary conditions are sufficient for the existence of C_6-decomposition of $K_m \Box K_n$. Fu et al. [8] established necessary and sufficient condition for the existence of $\{C_3, S_4\}_{\{p,q\}}$-decomposition of K_n. Shyu [12] obtained a necessary and sufficient condition on $\{p, q\}$ for the existence of $\{P_5, C_4\}_{\{p,q\}}$-decomposition of K_n. Priyadharsini and Muthusamy [11] established necessary and sufficient condition for the existence of the $\{pG_n, qH_n\}$-decomposition of $K_n(\lambda)$ when $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$. Jeevadoss and Muthusamy [9] obtained some necessary and sufficient conditions for the existence of $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition of $K_{m,n}$. Jeevadoss and Muthusamy [10] obtained necessary and sufficient conditions for the existence of $\{P_5, C_4\}_{\{p,q\}}$-decomposition of $K_m \times K_n$, $K_m \otimes \overline{K_n}$ and $K_m \Box K_n$. Pauline Ezhilarasi and Muthusamy [6] have obtained necessary and sufficient conditions for the existence of a decomposition of product graphs into paths and stars with three edges.

In this paper, we show that the necessary condition $mn(m + n - 2) \equiv 0 \pmod{12}$ is sufficient for the existence of a $\{P_7, C_6\}_{\{p,q\}}$-decomposition of $K_m \Box K_n$. We abbreviate the $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition as $(k; p, q)$-decomposition.

To prove our results we state the following:

Theorem 1.1 ([9]). *Let p, q be non-negative integers, k be an even integer and $n > 4k$ be an odd integer. If $k(p + q) = \binom{n}{2}$ and $p \neq 1$, then K_n has a $(k; p, q)$-decomposition.*

Theorem 1.2 ([9]). *Let $s, t > 0$ be integers and $k \geq 4$ be an even integer. Then the graph $K_{sk,tk}$ has a $(k; p, q)$-decomposition.*

Remark 1.3. If G and H have a $(6; p, q)$-decomposition, then $G \cup H$ has a such decomposition. In this paper, we denote $G \cup H$ as $G \oplus H$.

Construction 1.4. *Let C_6^1 and C_6^2 be two cycles of length 6, where $C_6^1 = (x_0x_1x_2x_3x_4x_5x_0)$ and $C_6^2 = (y_0y_1y_2y_3y_4y_5y_0)$. If v is a common vertex of C_6^1 and C_6^2 such that at least one neighbour of v from each cycle (say, x_i and y_j) does not belong to the other cycle then we have two edge-disjoint paths of length 6, say P_7^1 and P_7^2 from C_6^1 and C_6^2 as follows:*

$$P_7^1 = (C_6^1 - vx_i) \cup vy_j,$$

and

$$P_7^2 = (C_6^2 - vy_j) \cup vx_i.$$

2. Base Constructions

In this section we prove some basic lemmas which are used to prove our results. Throughout this paper, we denote $V(K_n) = \{x_i : 1 \leq i \leq n\}$.

Lemma 2.1. *There exists a* $(6; p, q)$*-decomposition of* $K_7 \backslash E(K_3)$*,* $p \neq 1$.

Proof. First we decompose $K_7 \backslash E(K_3)$ into $3C_6$ as follows:

$$\{(x_2\mathbf{x_5x_1x_4x_3x_6x_2}),\ (x_2x_4x_6x_1x_3\mathbf{x_7x_2}),\ (x_1x_2x_3x_5x_6x_7x_1)\}.$$

The bold edges (resp., ordinary edges) gives $2P_7$ from first two cycles. Now, the $3P_7$ are

$$\{x_4x_6x_7x_1x_2x_3x_5, x_7x_3x_6x_1x_4x_2x_5, x_7x_2x_6x_5x_1x_3x_4\}.$$

Hence $K_7 \backslash E(K_3)$ has a $(6; p, q)$-decomposition. □

Lemma 2.2. *There exists a* $(6; p, q)$*-decomposition of* $K_8 - I$*,* $p \neq 1$.

Proof. First we decompose $K_8 - I$ into C_6's as follows:

$$\{(x_1\mathbf{x_5x_8x_2x_6x_7x_1}), (\mathbf{x_1x_3}x_2x_7x_4x_8x_1)\}, \{(x_5\mathbf{x_7x_3x_8x_6x_4x_5}), (\mathbf{x_5x_2}x_4x_1x_6x_3x_5)\}.$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_5x_8x_2x_6x_7x_1x_3, x_3x_2x_7x_5x_4x_8x_6, x_6x_4x_7x_3x_8x_1x_5\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 2.3. *There exists a* $(6; p, q)$*-decomposition of* K_9*,* $p \neq 1$.

Proof. First we decompose K_9 into C_6's as follows:

$$\{(x_1\mathbf{x_3x_5x_7x_9x_2x_1}), (\mathbf{x_1x_4}x_6x_8x_2x_5x_1)\}, \{(x_1\mathbf{x_6x_2x_3x_4x_7x_1}),$$
$$(\mathbf{x_1x_8}x_3x_7x_6x_9x_1)\}, \{(x_4\mathbf{x_5x_6x_3x_9x_8x_4}), (\mathbf{x_4x_2}x_7x_8x_5x_9x_4)\}.$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_1x_8x_3x_7x_6x_9x_5, x_5x_8x_7x_2x_4x_9x_3, x_3x_6x_5x_4x_8x_9x_1\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 2.4. *There exists a* $(6; p, q)$*-decomposition of* $K_{6k,4l}$*,* $k, l \in \mathbb{Z}^+$ *and* $p \neq 1$.

Proof. Let $V(K_{6,4}) = \{x_1, \cdots, x_6\} \cup \{y_1, \cdots, y_4\}$. First we decompose $K_{6,4}$ into C_6's as follows:

$$\{(y_2\mathbf{x_2y_3x_3y_1x_1y_2}), (\mathbf{y_2x_5}y_3x_6y_4x_4y_2)\}, \{(\mathbf{y_1x_2}y_4x_1y_3x_4y_1), (y_1\mathbf{x_5y_4x_3y_2x_6y_1})\}.$$

From the first $3C_6$ we can find $3P_7$ as follows:

$$\{y_1x_1y_3x_4y_2x_5y_3, y_3x_6y_4x_4y_1x_2y_4, y_4x_1y_2x_2y_3x_3y_1\}.$$

Now, using Construction 1.4 we get a required number of paths and cycles from the C_6-decomposition given above. Hence $K_{6,4}$ has a $(6; p, q)$-decomposition. Now, we can write $K_{6k,4l} = klK_{6,4}$. Hence by Remark 1.3, $K_{6k,4l}$ has a $(6; p, q)$-decomposition. □

Lemma 2.5. *There exists a $(6; p, q)$-decomposition of*
(i). *$K_{12l+1}\backslash E(2lC_6)$ with $p \neq 1$ and*
(ii). *$K_{12k}\backslash E(2kC_6)$ with $p \geq 6k$, where $2lC_6$ and $2kC_6$ are vertex disjoint cycles and $k, l \in \mathbb{Z}^+$.*

Proof. (i). Let

$$\{\{(x_1\mathbf{x_4x_2x_6x_3x_5x_1}), (\mathbf{x_1x_{10}}x_4x_7x_2x_{12}x_1)\}, \{(x_7\mathbf{x_{13}x_{10}x_{12}x_8x_{11}x_7}), (\mathbf{x_7x_9}x_{12}x_{13}x_8x_{10}x_7)\},$$
$$\{(x_8\mathbf{x_2x_{10}x_3x_7x_1x_8}), (\mathbf{x_8x_5}x_{10}x_6x_{12}x_4x_8)\}, \{(x_9\mathbf{x_1x_3x_{13}x_{11}x_2x_9}), (\mathbf{x_9x_4}x_{13}x_5x_{11}x_6x_9)\},$$
$$\{(x_9x_{11}\mathbf{x_1x_{13}x_2x_5x_9}), (x_9x_3x_{11}x_4\mathbf{x_6x_{13}x_9})\}, (x_3x_8x_6x_7x_5x_{12}x_3)\},$$

be the C_6-decomposition of $K_{13}\backslash E(2C_6)$, where $2C_6$ removed from K_{13} are given by $(x_1x_2x_3x_4x_5x_6x_1)$, $(x_7x_8x_9x_{10}x_{11}x_{12}x_7)$. The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{12}x_3x_9x_{13}x_6x_4x_{11}, x_3x_8x_6x_7x_5x_9x_{11}, x_3x_{11}x_1x_{13}x_2x_5x_{12}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. Hence $K_{13}\backslash E(2C_6)$ has a $(6; p, q)$-decomposition, where the $2C_6$ removed from K_{13} are vertex disjoint cycles. We can write $K_{12l+1} = K_{12(l-1)+1} \oplus K_{13} \oplus K_{12(l-1),12}$. Applying this relation recursively to $K_{12(l-1)+1}$ and using Theorem 1.2, we can have a $(6; p, q)$-decomposition of $K_{12l+1}\backslash E(2lC_6)$, where the $2lC_6$ removed from K_{12l+1} are vertex disjoint cycles.

(ii). Since the degree of each vertex $v \in V(K_{12k}\backslash E(2kC_6))$ is odd, then $p \geq 6k$. Let

$$\{x_1x_{12}x_2x_{11}x_3x_{10}x_4, x_3x_9x_4x_8x_5x_7x_6, x_2x_4x_6x_8x_{10}x_1x_5, x_7x_9x_{12}x_3x_6x_2x_{10}, x_9x_{11}x_1x_3x_5x_{10}x_{12},$$
$$x_{11}x_4x_7x_{10}x_6x_{12}x_8, \{(x_1\mathbf{x_8x_3x_7x_2x_9x_1}), (x_1x_7x_{11}x_5x_{12}\mathbf{x_4x_1})\}, (x_2x_5x_9x_6x_{11}x_8x_2)\},$$

be the $\{6P_7,\ 3C_6\}$-decomposition of $K_{12}\backslash E(2C_6)$, where the $2C_6$ removed from K_{12} are $(x_1x_2x_3x_4x_5x_6x_1)$, $(x_7x_8x_9x_{10}x_{11}x_{12}x_7)$. For $p = 7$, we decompose the last cycle and the first path into $2P_7$ as follows:

$$\{x_1x_{12}x_2x_{11}x_6x_9x_5, x_5x_2x_8x_{11}x_3x_{10}x_4\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the decomposition given above. Hence $K_{12}\backslash E(2C_6)$ has a $(6; p, q)$-decomposition, where the $2C_6$ removed from K_{12} are vertex disjoint cycles. We can write $K_{12k} = K_{12(k-1)} \oplus K_{12} \oplus K_{12(k-1),12}$. Applying this relation recursively to $K_{12(k-1)}$ and using Theorem 1.2, we can have a $(6; p, q)$-decomposition of $K_{12k}\backslash E(2kC_6)$, where $2kC_6$ removed from K_{12k} are vertex disjoint cycles. □

Lemma 2.6. *There exists a* $(6;p,q)$*-decomposition of* $K_m \backslash E(K_3), m \equiv 3,\ 7 \pmod{12}$, $m > 3$ *and* $p \neq 1$.

Proof. Let $m = 12k + i$, where $i = 3, 7$. We prove it in two cases.

Case 1. $m = 12k + 3$. When $m = 15$, $K_{15} \backslash E(K_3) = K_9 \oplus (K_7 \backslash E(K_3)) \oplus K_{8,6}$. By Lemmas 2.1, 2.3 and 2.4 and Remark 1.3, $K_{15} \backslash E(K_3)$ has a $(6;p,q)$-decomposition. For $m > 15$, we can write $K_m \backslash E(K_3) = K_{12(k-1)+1} \oplus (K_{15} \backslash E(K_3)) \oplus K_{12(k-1),14} = K_{12(k-1)+1} \oplus (K_{15} \backslash E(K_3)) \oplus K_{12(k-1),6} \oplus K_{12(k-1),8}$. By Theorems 1.1, 1.2, Lemma 2.4 and Remark 1.3, $K_m \backslash E(K_3)$ has a $(6;p,q)$-decomposition.

Case 2. $m = 12k + 7$. We can write $K_m \backslash E(K_3) = K_{12k+1} \oplus (K_7 \backslash E(K_3)) \oplus K_{12k,6}$. By Theorems 1.1, 1.2, Lemma 2.1 and Remark 1.3, $K_m \backslash E(K_3)$ has a $(6;p,q)$-decomposition.

□

Lemma 2.7. *There exists a* $(6;p,q)$*-decomposition of* $K_m \backslash E(C_4)$ *for* $m \equiv 5 \pmod{12}$ *and* $K_m \backslash E(C_7)$ *for* $m \equiv 11 \pmod{12}$.

Proof.

Case 1. $m \equiv 5 \pmod{12}$. When $m = 17$, $K_{17} \backslash E(C_4) = K_8 \oplus K_9 \oplus (K_{8,9} \backslash E(C_4))$. Let $V_1 = V(K_8) = \{x_i : 1 \leq i \leq 8\}$ and $V_2 = V(K_9) = \{y_i : 1 \leq i \leq 9\}$. So, $V(K_{8,9}) = V_1 \cup V_2$. Let

$$\{(x_1\mathbf{y_2x_2y_4x_3y_5x_1}), (\mathbf{x_1y_1}x_3x_4y_3x_2x_1)\}, \{(x_5\mathbf{y_7x_2y_6x_4y_9x_5}), (\mathbf{x_5x_6}y_8x_7x_8y_6x_5)\},$$
$$\{(x_6x_8x_4x_1x_7\mathbf{x_2x_6}), (x_6\mathbf{x_3x_1x_8x_5x_4x_6})\}, \{(y_8\mathbf{x_2y_9x_6y_4x_1y_8}), (\mathbf{y_8x_5}y_1x_7y_6x_3y_8)\},$$
$$\{(x_8\mathbf{y_7x_6y_5x_2y_1x_8}), (\mathbf{x_8y_8}x_4y_7x_7y_9x_8)\}, \{(y_3\mathbf{x_3y_2x_6y_6x_1y_3}), (\mathbf{y_3x_7}y_4x_4y_5x_5y_3)\},$$
$$\{(x_2x_5x_1\mathbf{x_6x_7}x_3x_2), (\mathbf{x_4x_2x_8x_3x_5x_7}x_4)\}, \{(x_8\mathbf{y_5x_7y_2x_5y_4x_8}), (\mathbf{x_8y_3}x_6y_1x_4y_2x_8)\},$$

be the C_6-decomposition of $K_8 \oplus (K_{8,9} \backslash E(C_4))$. The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_5y_7x_2y_6x_4x_3y_1, y_2x_2y_4x_3y_5x_1y_1, x_5y_9x_4y_3x_2x_1y_2\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition of $K_8 \oplus (K_{8,9} \backslash E(C_4))$ given above. Hence by Lemma 2.3 and Remark 1.3, $K_{17} \backslash E(C_4)$ has a $(6;p,q)$-decomposition. When $m > 17$, we can write $K_m \backslash E(C_4) = K_{12k+5} \backslash E(C_4) = K_{12(k-1)+1} \oplus (K_{17} \backslash E(C_4)) \oplus K_{12(k-1),16}$. By Theorem 1.1 and Lemma 2.4, $K_{12(k-1)+1}$ and $K_{12(k-1),16}$ have a $(6;p,q)$-decomposition. Hence by Remark 1.3, $K_m \backslash E(C_4)$ has a $(6;p,q)$-decomposition.

Case 2. $m \equiv 11 \pmod{12}$. We can write $K_m \backslash E(C_7) = K_{12l+11} \backslash E(C_7) = K_{12l+1} \oplus K_{12l,10} \oplus (K_{11} \backslash E(C_7))$ and $K_{12l,10} = K_{12l,6} \oplus 2lK_{6,4}$. Let

$$\{(x_1\mathbf{x_4x_{10}x_5x_7x_{11}x_1}), (\mathbf{x_1x_8}x_5x_{11}x_{10}x_3x_1)\}, \{(x_2\mathbf{x_7x_3x_9x_6x_8x_2}), (\mathbf{x_2x_{11}}x_3x_5x_1x_{10}x_2)\},$$
$$\{(x_9\mathbf{x_4x_{11}x_6x_{10}x_7x_9}), (\mathbf{x_9x_1}x_6x_4x_8x_{10}x_9)\}, \{(x_9\mathbf{x_8x_3x_6x_2x_5x_9}), (\mathbf{x_9x_{11}}x_8x_7x_4x_2x_9)\},$$

be the C_6-decomposition of $K_{11} \backslash E(C_7)$. The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{(x_1x_{11}x_4x_3x_7x_2x_8, x_8x_1x_4x_{10}x_5x_{11}x_7, x_1x_3x_9x_6x_8x_5x_7\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above for $K_{11}\backslash E(C_7)$. By Theorems 1.1, 1.2, Lemma 2.4 and Remark 1.3, $K_m\backslash E(C_7)$ has a $(6;p,q)$-decomposition. □

Lemma 2.8. *There exists a $(6;p,q)$-decomposition of K_m, where $m \equiv 0,\ 4 \pmod{12}$ and $p \geq m/2$.*

Proof. Since the degree of each vertex $v \in V(K_m)$ is odd, then $p \geq m/2$. We prove the required decomposition in two Cases.

Case 1. $m \equiv 0 \pmod{12}$. Let $m = 12k$ and

$$\{x_3x_9x_4x_8x_5x_7x_6, x_1x_{12}x_2x_{11}x_3x_{10}x_4, x_2x_4x_6x_8x_{10}x_1x_5, x_7x_9x_{12}x_3x_6x_2x_{10}, x_9x_{11}x_1x_3x_5x_{10}x_{12}, x_{11}x_4x_7x_{10}x_6x_{12}x_8, \{(x_1\mathbf{x_8x_3x_7x_2x_9x_1}), (x_1x_7x_{11}x_5x_{12}\mathbf{x_4x_1})\}, \{(\mathbf{x_2x_5x_9x_6x_{11}x_8}x_2), (\mathbf{x_2x_3}x_4x_5x_6x_1x_2)\}, (x_7x_8x_9x_{10}x_{11}x_{12}x_7)\},$$

be a $\{6P_7,\ 5C_6\}$-decomposition of K_{12}. For $p = 7$, we decompose the last cycle and first path into $2P_7$ as follows:

$$\{x_3x_9x_4x_8x_5x_7x_{12}, x_6x_7x_8x_9x_{10}x_{11}x_{12}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6's given above for $p \geq 8$. When $k > 1$, $K_{12k} = K_{12(k-1)} \oplus K_{12} \oplus K_{12(k-1),12}$. Applying this relation recursively to $K_{12(k-1)}$ and using Theorem 1.2, we can prove that K_{12k} has a $(6;p,q)$-decomposition.

Case 2. $m \equiv 4 \pmod{12}$. Let $m = 12k + 4$ and

$$\{x_1x_5x_7x_4x_6x_8x_2, x_3x_6x_7x_2x_5x_8x_4, x_5x_3x_7x_8x_1x_2x_6, x_7x_1x_6x_5x_4x_3x_8, x_{14}x_{13}x_{15}x_{12}x_9x_{11}x_{10}, x_{16}x_9x_{15}x_{10}x_{13}x_{11}x_{12}, x_{13}x_{16}x_{15}x_{11}x_{14}x_{10}x_9, x_{15}x_{14}x_9x_{13}x_{12}x_{16}x_{11}\},$$

$$\begin{gathered}\{(x_3\mathbf{x_9x_5x_{11}x_2x_{10}x_3}), (\mathbf{x_3x_{12}}x_5x_{10}x_4x_{11}x_3)\},\\ \{(x_9\mathbf{x_2x_{12}x_1x_{15}x_8x_9}), (\mathbf{x_9x_4}x_{16}x_6x_{13}x_7x_9)\},\\ \{(x_{13}\mathbf{x_3x_{15}x_5x_{16}x_2x_{13}}), (\mathbf{x_{13}x_4}x_{15}x_7x_{11}x_1x_{13})\},\\ \{(x_8\mathbf{x_{12}x_7x_{14}x_5x_{13}x_8}), (\mathbf{x_8x_{11}}x_6x_9x_1x_{10}x_8)\},\\ \{(x_{14}\mathbf{x_6x_{10}x_7x_{16}x_8x_{14}}), (\mathbf{x_{14}x_2}x_{15}x_6x_{12}x_4x_{14})\},\\ \{(x_1x_3x_{14}x_{12}\mathbf{x_{10}x_{16}x_1}), (\mathbf{x_1x_{14}x_{16}x_3x_2}x_4x_1)\}\end{gathered}$$

be a $\{8P_7,\ 12C_6\}$-decomposition of K_{16}.

For $p = 9$, we decompose the last cycle and last path into $2P_7$ as follows:

$$\{x_{15}x_{14}x_9x_{13}x_{12}x_{16}x_3, x_3x_2x_4x_1x_{14}x_{16}x_{11}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6's given above for $p \geq 10$. When $k > 1$, $K_{12k+4} = K_{12(k-1)} \oplus K_{16} \oplus K_{12(k-1),16}$. By Lemma 2.4 and by applying Case 1, K_{12k+4} has a $(6;p,q)$-decomposition. □

3. $(6;p,q)$-decomposition of $K_m \Box K_n$

In this section we investigate the existence of $(6;p,q)$-decomposition of Cartesian product of complete graphs. Throughout this paper, we denote $V(K_m \Box K_n) = \{x_{i,j} : 1 \le i \le m, 1 \le j \le n\}$.

Lemma 3.1. *There exists a* $(6;p,q)$*-decomposition of* $K_3 \Box K_3$*,* $p \neq 1$.

Proof. First we decompose $K_3 \Box K_3$ into C_6's as follows:

$$\{(x_{1,1}\mathbf{x_{1,2}}\mathbf{x_{3,2}}\mathbf{x_{3,3}}\mathbf{x_{2,3}}\mathbf{x_{1,3}}\mathbf{x_{1,1}}), (\mathbf{x_{1,1}}\mathbf{x_{2,1}}x_{2,3}x_{2,2}x_{3,2}x_{3,1}x_{1,1}), (x_{1,2}x_{1,3}x_{3,3}x_{3,1}x_{2,1}x_{2,2}x_{1,2})\}.$$

The bold edges (resp., ordinary edges) gives $2P_7$ from first two cycles. Also, we can decompose the given graph into $3P_7$ as follows:

$$\{x_{1,2}x_{1,3}x_{3,3}x_{3,2}x_{3,1}x_{1,1}x_{2,1}, x_{2,1}x_{2,2}x_{1,2}x_{1,1}x_{1,3}x_{2,3}x_{3,3}, x_{3,3}x_{3,1}x_{2,1}x_{2,3}x_{2,2}x_{3,2}x_{1,2}\}.$$

Hence $K_3 \Box K_3$ has a $(6;p,q)$-decomposition. □

Lemma 3.2. *There exists a* $(6;p,q)$*-decomposition of* $K_3 \Box K_7$*,* $p \neq 1$.

Proof. First we decompose $K_3 \Box K_7$ into C_6's as follows:

$$\begin{gathered}
\{(x_{1,6}\mathbf{x_{2,6}}\mathbf{x_{3,6}}\mathbf{x_{3,5}}\mathbf{x_{1,5}}\mathbf{x_{1,2}}\mathbf{x_{1,6}}),\ (x_{1,6}x_{3,6}x_{3,4}x_{2,4}x_{1,4}\mathbf{x_{1,3}}\mathbf{x_{1,6}})\},\\
\{(x_{1,7}\mathbf{x_{3,7}}\mathbf{x_{3,3}}\mathbf{x_{3,5}}\mathbf{x_{2,5}}\mathbf{x_{1,5}}\mathbf{x_{1,7}}),\ (\mathbf{x_{1,7}}\mathbf{x_{2,7}}x_{3,7}x_{3,1}x_{3,4}x_{1,4}x_{1,7})\},\\
\{(\mathbf{x_{1,7}}\mathbf{x_{1,1}}\mathbf{x_{1,4}}\mathbf{x_{1,6}}\mathbf{x_{1,5}}\mathbf{x_{1,3}}x_{1,7}),\ (x_{1,7}x_{1,6}x_{1,1}x_{1,5}x_{1,4}\mathbf{x_{1,2}}\mathbf{x_{1,7}})\},\\
\{(\mathbf{x_{2,7}}\mathbf{x_{2,6}}\mathbf{x_{2,1}}\mathbf{x_{2,4}}\mathbf{x_{2,5}}\mathbf{x_{2,3}}x_{2,7}),\ (\mathbf{x_{2,7}}\mathbf{x_{2,2}}x_{2,4}x_{2,6}x_{2,5}x_{2,1}x_{2,7})\},\\
\{(x_{3,7}\mathbf{x_{3,2}}\mathbf{x_{3,6}}\mathbf{x_{3,3}}\mathbf{x_{3,4}}\mathbf{x_{3,5}}\mathbf{x_{3,7}}),\ (\mathbf{x_{3,7}}\mathbf{x_{3,6}}x_{3,1}x_{3,5}x_{3,2}x_{3,4}x_{3,7})\},\\
\{(x_{1,3}\mathbf{x_{3,3}}\mathbf{x_{3,1}}\mathbf{x_{2,1}}\mathbf{x_{2,2}}\mathbf{x_{1,2}}\mathbf{x_{1,3}}),\ (\mathbf{x_{1,3}}\mathbf{x_{2,3}}x_{3,3}x_{3,2}x_{1,2}x_{1,1}x_{1,3})\},\\
\{(x_{2,2}\mathbf{x_{2,6}}\mathbf{x_{2,3}}\mathbf{x_{2,4}}\mathbf{x_{2,7}}\mathbf{x_{2,5}}\mathbf{x_{2,2}}),\ (\mathbf{x_{2,2}}\mathbf{x_{3,2}}x_{3,1}x_{1,1}x_{2,1}x_{2,3}x_{2,2})\}.
\end{gathered}$$

The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,3}x_{1,6}x_{3,6}x_{3,5}x_{2,5}x_{1,5}x_{1,7}, x_{1,7}x_{3,7}x_{3,3}x_{3,5}x_{1,5}x_{1,2}x_{1,6}, x_{1,6}x_{2,6}x_{3,6}x_{3,4}x_{2,4}x_{1,4}x_{1,3}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.3. *There exists a* $(6;p,q)$*-decomposition of* $K_3 \Box K_8$*,* $p \ge 12$.

Proof. Since the degree of each vertex $v \in V(K_3 \Box K_8)$ is odd, then $p \ge \frac{24}{2} = 12$. For $p = 12$, the required number of P_7's and C_6's are constructed as follows:

$$\begin{gathered}
x_{1,1}x_{1,4}x_{2,4}x_{3,4}x_{3,8}x_{3,7}x_{3,5},\ x_{1,2}x_{1,1}x_{1,5}x_{1,8}x_{1,3}x_{3,3}x_{2,3},\\
x_{1,3}x_{2,3}x_{2,4}x_{2,6}x_{2,7}x_{2,1}x_{2,5},\ x_{1,4}x_{1,3}x_{1,6}x_{1,5}x_{1,7}x_{3,7}x_{2,7},\\
x_{1,5}x_{1,3}x_{1,1}x_{1,7}x_{1,8}x_{3,8}x_{2,8},\ x_{1,6}x_{1,7}x_{1,2}x_{1,4}x_{3,4}x_{3,1}x_{3,8},\\
x_{1,7}x_{2,7}x_{2,5}x_{2,3}x_{2,2}x_{2,8}x_{1,8},\ x_{2,1}x_{1,1}x_{3,1}x_{3,3}x_{3,4}x_{3,2}x_{3,7},\\
x_{2,2}x_{1,2}x_{3,2}x_{3,1}x_{3,6}x_{3,5}x_{3,3},\ x_{2,4}x_{2,1}x_{2,2}x_{2,7}x_{2,8}x_{2,6}x_{3,6},
\end{gathered}$$

$$x_{2,6}x_{2,3}x_{2,8}x_{2,5}x_{1,5}x_{3,5}x_{3,4},\ x_{3,1}x_{2,1}x_{2,3}x_{2,7}x_{2,4}x_{2,2}x_{3,2},$$
$$\{(\mathbf{x_{1,2}x_{1,5}}x_{1,4}x_{1,8}x_{1,1}x_{1,6}x_{1,2}),\ (x_{1,2}\mathbf{x_{1,3}x_{1,7}x_{1,4}x_{1,6}x_{1,8}x_{1,2}})\},$$
$$\{(x_{2,6}\mathbf{x_{1,6}x_{3,6}x_{3,2}x_{3,5}x_{2,5}x_{2,6}}),\ (\mathbf{x_{2,6}x_{2,2}}x_{2,5}x_{2,4}x_{2,8}x_{2,1}x_{2,6})\},$$
$$\{(x_{3,3}\mathbf{x_{3,6}x_{3,7}x_{3,1}x_{3,5}x_{3,8}x_{3,3}}),\ (x_{3,3}x_{3,7}x_{3,4}x_{3,6}x_{3,8}\mathbf{x_{3,2}x_{3,3}})\}.$$

For $p = 13$, we decompose the last path and the first cycle into $2P_7$ as follows:

$$\{x_{2,6}x_{2,3}x_{2,8}x_{2,5}x_{1,5}x_{1,4}x_{1,8},\ x_{1,8}x_{1,1}x_{1,6}x_{1,2}x_{1,5}x_{3,5}x_{3,4}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6's given above for $p \geq 14$. □

Lemma 3.4. *There exists a $(6; p, q)$-decomposition of $K_3 \square K_{11}$, $p \neq 1$.*

Proof. First we decompose $K_3 \square K_{11}$ into C_6's as follows:

$$\{(x_{1,2}\mathbf{x_{1,1}x_{1,8}x_{1,3}x_{1,9}x_{1,10}x_{1,2}}),\ (\mathbf{x_{1,2}x_{1,4}}x_{1,6}x_{1,1}x_{1,7}x_{1,9}x_{1,2})\},$$
$$\{(x_{1,7}\mathbf{x_{1,11}x_{1,5}x_{1,3}x_{1,10}x_{1,6}x_{1,7}}),\ (x_{1,7}x_{1,10}x_{1,11}x_{1,3}x_{1,2}\mathbf{x_{1,8}x_{1,7}})\},$$
$$\{(x_{1,11}\mathbf{x_{1,8}x_{1,9}x_{1,5}x_{1,10}x_{1,4}x_{1,11}}),\ (x_{1,11}x_{1,9}x_{1,4}x_{1,8}x_{1,5}\mathbf{x_{1,6}x_{1,11}})\},$$
$$\{(x_{2,1}\mathbf{x_{2,11}x_{2,8}x_{2,10}x_{2,4}x_{2,7}x_{2,1}}),\ (\mathbf{x_{2,1}x_{2,2}}x_{2,4}x_{2,5}x_{2,6}x_{2,9}x_{2,1})\},$$
$$\{(x_{2,6}\mathbf{x_{2,3}x_{2,5}x_{2,9}x_{2,7}x_{2,2}x_{2,6}}),\ (\mathbf{x_{2,6}x_{2,10}}x_{2,9}x_{2,2}x_{2,5}x_{2,1}x_{2,6})\},$$
$$\{(x_{2,11}\mathbf{x_{2,6}x_{2,8}x_{2,3}x_{2,10}x_{2,5}x_{2,11}}),\ (\mathbf{x_{2,11}x_{2,2}}x_{2,10}x_{2,7}x_{2,3}x_{2,4}x_{2,11})\},$$
$$\{(x_{3,1}\mathbf{x_{3,11}x_{3,8}x_{3,7}x_{3,9}x_{3,4}x_{3,1}}),\ (\mathbf{x_{3,1}x_{3,3}}x_{3,10}x_{3,11}x_{3,6}x_{3,9}x_{3,1})\},$$
$$\{(x_{3,2}\mathbf{x_{3,3}x_{3,6}x_{3,5}x_{3,9}x_{3,10}x_{3,2}}),\ (\mathbf{x_{3,2}x_{3,8}}x_{3,5}x_{3,10}x_{3,6}x_{3,4}x_{3,2})\},$$
$$\{(x_{3,1}\mathbf{x_{3,2}x_{3,11}x_{3,3}x_{3,8}x_{3,6}x_{3,1}}),\ (x_{3,1}x_{3,8}x_{3,4}x_{3,11}x_{3,7}\mathbf{x_{3,5}x_{3,1}})\},$$
$$\{(x_{1,1}\mathbf{x_{1,3}x_{1,4}x_{2,4}x_{2,1}x_{3,1}x_{1,1}}),\ (\mathbf{x_{1,1}x_{1,5}}x_{3,5}x_{3,3}x_{2,3}x_{2,1}x_{1,1})\},$$
$$\{(x_{2,11}\mathbf{x_{3,11}x_{3,9}x_{1,9}x_{2,9}x_{2,3}x_{2,11}}),\ (\mathbf{x_{2,11}x_{1,11}}x_{1,1}x_{1,10}x_{3,10}x_{2,10}x_{2,11})\},$$
$$\{(x_{1,5}\mathbf{x_{1,7}x_{2,7}x_{3,7}x_{3,4}x_{1,4}x_{1,5}}),\ (\mathbf{x_{1,5}x_{2,5}x_{3,5}}x_{3,11}x_{1,11}x_{1,2}x_{1,5})\},$$
$$\{(x_{1,3}\mathbf{x_{1,6}x_{2,6}x_{2,4}x_{3,4}x_{3,3}x_{1,3}}),\ (\mathbf{x_{1,3}x_{1,7}}x_{3,7}x_{3,2}x_{2,2}x_{2,3}x_{1,3})\},$$
$$\{(x_{2,8}\mathbf{x_{3,8}x_{1,8}x_{1,10}x_{2,10}x_{2,1}x_{2,8}}),\ (\mathbf{x_{2,8}x_{2,4}}x_{2,9}x_{2,11}x_{2,7}x_{2,5}x_{2,8})\},$$
$$\{(x_{3,10}\mathbf{x_{3,4}x_{3,5}x_{3,2}x_{3,6}x_{3,7}x_{3,10}}),\ (\mathbf{x_{3,10}x_{3,8}}x_{3,9}x_{3,3}x_{3,7}x_{3,1}x_{3,10})\},$$
$$\{(x_{2,8}\mathbf{x_{1,8}x_{1,6}x_{3,6}x_{2,6}x_{2,7}x_{2,8}}),\ (\mathbf{x_{2,8}x_{2,9}}x_{3,9}x_{3,2}x_{1,2}x_{2,2}x_{2,8})\},$$
$$(x_{1,1}x_{1,9}x_{1,6}x_{1,2}x_{1,7}x_{1,4}x_{1,1}).$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,1}x_{1,9}x_{1,6}x_{1,8}x_{2,8}x_{2,9}x_{3,9},\ x_{1,1}x_{1,4}x_{1,7}x_{1,2}x_{1,6}x_{3,6}x_{2,6},\ x_{2,6}x_{2,7}x_{2,8}x_{2,2}x_{1,2}x_{3,2}x_{3,9}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.5. *There exists a $(6; p, q)$-decomposition of $K_3 \square K_{16}$, $p \geq 24$.*

Proof. Since the degree of each vertex $v \in V(K_3 \Box K_{16})$ is odd, $p \geq \frac{48}{2} = 24$. For $p = 24$, the required number of C_6's and P_7's are constructed as follows:

$$\{(x_{1,1}\mathbf{x_{1,14}x_{1,3}x_{1,7}x_{1,4}x_{1,8}x_{1,1}}),\ (\mathbf{x_{1,1}x_{1,16}}x_{1,3}x_{1,2}x_{1,8}x_{1,6}x_{1,1})\},$$
$$\{(x_{1,3}\mathbf{x_{1,9}x_{1,5}x_{1,11}x_{1,2}x_{1,10}x_{1,3}}),\ (\mathbf{x_{1,3}x_{1,12}}x_{1,5}x_{1,10}x_{1,4}x_{1,11}x_{1,3})\},$$
$$\{(x_{1,9}\mathbf{x_{1,2}x_{1,12}x_{1,1}x_{1,15}x_{1,8}x_{1,9}}),\ (\mathbf{x_{1,9}x_{1,4}}x_{1,16}x_{1,6}x_{1,13}x_{1,7}x_{1,9})\},$$
$$\{(x_{1,8}\mathbf{x_{1,12}x_{1,7}x_{1,14}x_{1,5}x_{1,13}x_{1,8}}),\ (\mathbf{x_{1,8}x_{1,11}}x_{1,6}x_{1,9}x_{1,1}x_{1,10}x_{1,8})\},$$
$$\{(x_{1,13}\mathbf{x_{1,3}x_{1,15}x_{1,5}x_{1,16}x_{1,2}x_{1,13}}),\ (\mathbf{x_{1,13}x_{1,4}}x_{1,15}x_{1,7}x_{1,11}x_{1,1}x_{1,13})\},$$
$$\{(x_{1,14}\mathbf{x_{1,6}x_{1,10}x_{1,7}x_{1,16}x_{1,8}x_{1,14}}),\ (\mathbf{x_{1,14}x_{1,2}}x_{1,15}x_{1,6}x_{1,12}x_{1,4}x_{1,14})\},$$
$$\{(x_{2,7}\mathbf{x_{2,9}x_{2,3}x_{2,11}x_{2,2}x_{2,10}x_{2,7}}),\ (\mathbf{x_{2,7}x_{2,12}}x_{2,3}x_{2,10}x_{2,5}x_{2,11}x_{2,7})\},$$
$$\{(x_{2,9}\mathbf{x_{2,2}x_{2,12}x_{2,4}x_{2,15}x_{2,8}x_{2,9}}),\ (\mathbf{x_{2,9}x_{2,5}}x_{2,16}x_{2,6}x_{2,13}x_{2,1}x_{2,9})\},$$
$$\{(x_{2,8}\mathbf{x_{2,12}x_{2,1}x_{2,14}x_{2,3}x_{2,13}x_{2,8}}),\ (\mathbf{x_{2,8}x_{2,11}}x_{2,6}x_{2,9}x_{2,4}x_{2,10}x_{2,8})\},$$
$$\{(x_{2,13}\mathbf{x_{2,7}x_{2,15}x_{2,3}x_{2,16}x_{2,2}x_{2,13}}),\ (\mathbf{x_{2,13}x_{2,5}}x_{2,15}x_{2,1}x_{2,11}x_{2,4}x_{2,13})\},$$
$$\{(x_{2,14}\mathbf{x_{2,6}x_{2,10}x_{2,1}x_{2,16}x_{2,8}x_{2,14}}),\ (\mathbf{x_{2,14}x_{2,2}}x_{2,15}x_{2,6}x_{2,12}x_{2,5}x_{2,14})\},$$
$$\{(x_{2,4}\mathbf{x_{2,14}x_{2,7}x_{2,6}x_{2,3}x_{2,8}x_{2,4}}),\ (\mathbf{x_{2,4}x_{2,16}}x_{2,7}x_{2,1}x_{2,8}x_{2,5}x_{2,4})\},$$
$$\{(x_{3,1}\mathbf{x_{3,9}x_{3,3}x_{3,11}x_{3,5}x_{3,10}x_{3,1}}),\ (\mathbf{x_{3,1}x_{3,12}}x_{3,3}x_{3,10}x_{3,4}x_{3,11}x_{3,1})\},$$
$$\{(x_{3,9}\mathbf{x_{3,5}x_{3,12}x_{3,2}x_{3,15}x_{3,8}x_{3,9}}),\ (\mathbf{x_{3,9}x_{3,4}}x_{3,16}x_{3,6}x_{3,13}x_{3,7}x_{3,9})\},$$
$$\{(x_{3,8}\mathbf{x_{3,12}x_{3,7}x_{3,14}x_{3,3}x_{3,13}x_{3,8}}),\ (\mathbf{x_{3,8}x_{3,11}}x_{3,6}x_{3,9}x_{3,2}x_{3,10}x_{3,8})\},$$
$$\{(x_{3,13}\mathbf{x_{3,1}x_{3,15}x_{3,3}x_{3,16}x_{3,5}x_{3,13}}),\ (\mathbf{x_{3,13}x_{3,4}}x_{3,15}x_{3,7}x_{3,11}x_{3,2}x_{3,13}\},$$
$$\{(x_{3,14}\mathbf{x_{3,6}x_{3,10}x_{3,7}x_{3,16}x_{3,8}x_{3,14}}),\ (\mathbf{x_{3,14}x_{3,5}}x_{3,15}x_{3,6}x_{3,12}x_{3,4}x_{3,14}\},$$
$$\{(x_{3,2}\mathbf{x_{3,14}x_{3,1}x_{3,5}x_{3,8}x_{3,3}x_{3,2}}),\ (\mathbf{x_{3,2}x_{3,16}}x_{3,1}x_{3,7}x_{3,6}x_{3,8}x_{3,2})\},$$
$$\{(x_{1,10}\mathbf{x_{1,11}x_{1,15}x_{1,12}x_{1,14}x_{1,16}x_{1,10}}),\ (\mathbf{x_{1,10}x_{1,13}}x_{1,12}x_{1,16}x_{1,9}x_{1,14}x_{1,10})\},$$
$$\{(x_{2,14}\mathbf{x_{1,14}x_{3,14}x_{3,10}x_{3,13}x_{2,13}x_{2,14}}),\ (\mathbf{x_{2,14}x_{2,10}}x_{2,13}x_{2,12}x_{2,16}x_{2,9}x_{2,14})\},$$
$$\{(x_{3,11}\mathbf{x_{3,14}x_{3,15}x_{3,9}x_{3,13}x_{3,16}x_{3,11}}),\ (x_{3,11}x_{3,15}x_{3,12}x_{3,14}x_{3,16}\mathbf{x_{3,10}x_{3,11}})\},$$
$$\{(x_{1,4}x_{1,5}x_{2,5}x_{2,2}x_{2,6}x_{1,6}x_{1,4}),\ (x_{3,2}x_{3,5}x_{3,7}x_{3,8}x_{3,4}x_{3,6}x_{3,2})\},$$

$$x_{1,2}x_{1,6}x_{3,6}x_{3,3}x_{3,7}x_{3,4}x_{2,4},\ x_{1,1}x_{1,2}x_{1,5}x_{1,8}x_{1,3}x_{3,3}x_{2,3},\ x_{2,5}x_{2,1}x_{2,2}x_{2,7}x_{2,8}x_{2,6}x_{3,6},\ x_{2,6}x_{2,1}x_{2,4}x_{1,4}x_{1,1}x_{1,5}x_{3,5},$$
$$x_{2,1}x_{1,1}x_{3,1}x_{3,3}x_{3,4}x_{3,2}x_{3,7},\ x_{1,4}x_{1,3}x_{1,6}x_{1,5}x_{1,7}x_{3,7}x_{2,7},\ x_{1,5}x_{1,3}x_{1,1}x_{1,7}x_{1,8}x_{3,8}x_{2,8},\ x_{2,2}x_{1,2}x_{3,2}x_{3,1}x_{3,6}x_{3,5}x_{3,3},$$
$$x_{3,1}x_{2,1}x_{2,3}x_{2,7}x_{2,4}x_{2,2}x_{3,2},\ x_{1,7}x_{2,7}x_{2,5}x_{2,3}x_{2,2}x_{2,8}x_{1,8},\ x_{1,6}x_{1,7}x_{1,2}x_{1,4}x_{3,4}x_{3,1}x_{3,8},\ x_{1,3}x_{2,3}x_{2,4}x_{2,6}x_{2,5}x_{3,5}x_{3,4},$$
$$x_{1,9}x_{1,12}x_{2,12}x_{3,12}x_{3,16}x_{3,15}x_{3,13},\ x_{1,10}x_{1,9}x_{1,13}x_{1,16}x_{1,11}x_{3,11}x_{2,11},\ x_{1,11}x_{2,11}x_{2,12}x_{2,14}x_{2,15}x_{2,9}x_{2,13},$$
$$x_{1,12}x_{1,11}x_{1,14}x_{1,13}x_{1,15}x_{3,15}x_{2,15},\ x_{1,13}x_{1,11}x_{1,9}x_{1,15}x_{1,16}x_{3,16}x_{2,16},\ x_{1,14}x_{1,15}x_{1,10}x_{1,12}x_{3,12}x_{3,9}x_{3,16},$$
$$x_{1,15}x_{2,15}x_{2,13}x_{2,11}x_{2,10}x_{2,16}x_{1,16},\ x_{2,9}x_{1,9}x_{3,9}x_{3,11}x_{3,12}x_{3,10}x_{3,15},\ x_{2,10}x_{1,10}x_{3,10}x_{3,9}x_{3,14}x_{3,13}x_{3,11},$$
$$x_{2,12}x_{2,9}x_{2,10}x_{2,15}x_{2,16}x_{2,14}x_{3,14},\ x_{2,14}x_{2,11}x_{2,16}x_{2,13}x_{1,13}x_{3,13}x_{3,12},\ x_{3,9}x_{2,9}x_{2,11}x_{2,15}x_{2,12}x_{2,10}x_{3,10}.$$

For $p = 25$, we decompose the last cycle and first path into $2P_7$ as follows:

$$x_{1,2}x_{1,6}x_{3,6}x_{3,3}x_{3,7}x_{3,4}x_{3,8},\ x_{2,4}x_{3,4}x_{3,6}x_{3,2}x_{3,5}x_{3,7}x_{3,8}.$$

For $p = 26$, we can decompose

$$\{(x_{1,4}x_{1,5}x_{2,5}x_{2,2}x_{2,6}x_{1,6}x_{1,4}), x_{1,1}x_{1,2}x_{1,5}x_{1,8}x_{1,3}x_{3,3}x_{2,3}\},$$

into $2P_7$ in $p = 25$ as follows:

$$x_{1,1}x_{1,2}x_{1,5}x_{2,5}x_{2,2}x_{2,6}x_{1,6}, x_{1,6}x_{1,4}x_{1,5}x_{1,8}x_{1,3}x_{3,3}x_{2,3}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from paired C_6's given above for $p \geq 27$. So, we have the desired decomposition for $K_3 \square K_{16}$. □

Lemma 3.6. *There exists a $(6; p, q)$-decomposition of $K_6 \square K_2$, $p \neq 1$.*

Proof. First we decompose $K_6 \square K_2$ into C_6's as follows:

$$\{(x_{1,2}\mathbf{x_{5,2}x_{4,2}x_{3,2}x_{2,2}x_{6,2}x_{1,2}}), (\mathbf{x_{2,1}x_{4,1}}x_{4,2}x_{6,2}x_{6,1}x_{5,1}x_{2,1})\},$$
$$\{(x_{3,2}\mathbf{x_{6,2}x_{5,2}x_{2,2}x_{4,2}x_{1,2}x_{3,2}}), (\mathbf{x_{3,2}x_{3,1}}x_{6,1}x_{1,1}x_{5,1}x_{5,2}x_{3,2})\},$$
$$\{(x_{1,1}\mathbf{x_{2,1}x_{6,1}x_{4,1}x_{5,1}x_{3,1}x_{1,1}}), (\mathbf{x_{1,1}x_{1,2}}x_{2,2}x_{2,1}x_{3,1}x_{4,1}x_{1,1})\}.$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{2,1}x_{3,1}x_{5,1}x_{4,1}x_{1,1}x_{1,2}x_{2,2}, x_{2,2}x_{2,1}x_{1,1}x_{6,1}x_{3,1}x_{3,2}x_{5,2}, x_{2,1}x_{6,1}x_{4,1}x_{3,1}x_{1,1}x_{5,1}x_{5,2}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.7. *There exists a $(6; p, q)$-decomposition of $K_6 \square K_4$, $p \neq 1$.*

Proof. First we decompose $K_6 \square K_4$ into C_6's as follows:

$$\{(x_{3,3}\mathbf{x_{1,3}x_{2,3}x_{2,4}x_{3,4}x_{3,2}x_{3,3}}), (\mathbf{x_{3,3}x_{4,3}}x_{4,4}x_{6,4}x_{6,3}x_{5,3}x_{3,3})\},$$
$$\{(x_{1,3}\mathbf{x_{1,1}x_{1,2}x_{2,2}x_{2,4}x_{1,4}x_{1,3}}), (\mathbf{x_{1,3}x_{6,3}}x_{3,3}x_{3,4}x_{5,4}x_{5,3}x_{1,3})\},$$
$$\{(x_{4,2}\mathbf{x_{3,2}x_{5,2}x_{6,2}x_{6,3}x_{4,3}x_{4,2}}), (\mathbf{x_{4,2}x_{4,1}}x_{3,1}x_{6,1}x_{6,4}x_{6,2}x_{4,2})\},$$
$$\{(x_{1,2}\mathbf{x_{3,2}x_{2,2}x_{2,1}x_{1,1}x_{1,4}x_{1,2}}), (\mathbf{x_{1,2}x_{1,3}}x_{4,3}x_{2,3}x_{2,2}x_{4,2}x_{1,2})\},$$
$$\{(x_{6,1}\mathbf{x_{6,3}x_{2,3}x_{5,3}x_{4,3}x_{4,1}x_{6,1}}), (\mathbf{x_{6,1}x_{1,1}}x_{5,1}x_{5,4}x_{2,4}x_{2,1}x_{6,1})\},$$
$$\{(x_{3,1}\mathbf{x_{1,1}x_{4,1}x_{2,1}x_{2,3}x_{3,3}x_{3,1}}), (\mathbf{x_{3,1}x_{5,1}}x_{5,2}x_{1,2}x_{6,2}x_{3,2}x_{3,1})\},$$
$$\{(x_{1,4}\mathbf{x_{6,4}x_{5,4}x_{5,2}x_{4,2}x_{4,4}x_{1,4}}), (x_{1,4}x_{5,4}x_{4,4}x_{2,4}x_{6,4}\mathbf{x_{3,4}x_{1,4}})\},$$
$$\{(x_{5,1}\mathbf{x_{2,1}x_{3,1}x_{3,4}x_{4,4}x_{4,1}x_{5,1}}), (\mathbf{x_{5,1}x_{6,1}}x_{6,2}x_{2,2}x_{5,2}x_{5,3}x_{5,1})\}.$$

The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,1}x_{1,2}x_{2,2}x_{2,4}x_{1,4}x_{1,3}x_{2,3}, x_{2,3}x_{2,4}x_{3,4}x_{3,2}x_{3,3}x_{4,3}x_{4,4}, x_{6,4}x_{6,3}x_{5,3}x_{3,3}x_{1,3}x_{1,1}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.8. *There exists a $(6; p, q)$-decomposition of $K_6 \square K_6$, $p \neq 1$.*

Proof. First we decompose $K_6 \square K_6$ into C_6's as follows:

$$\begin{gathered}
\{(x_{1,1}\mathbf{x_{1,2}}\mathbf{x_{2,2}}\mathbf{x_{3,2}}\mathbf{x_{6,2}}\mathbf{x_{6,1}}\mathbf{x_{1,1}}),\ (\mathbf{x_{1,1}}\mathbf{x_{3,1}}x_{4,1}x_{5,1}x_{6,1}x_{2,1}x_{1,1})\},\\
\{(\mathbf{x_{2,1}}\mathbf{x_{2,2}}x_{4,2}x_{4,3}x_{4,4}x_{4,1}x_{2,1}),\ (x_{2,1}\mathbf{x_{3,1}}\mathbf{x_{6,1}}\mathbf{x_{4,1}}\mathbf{x_{1,1}}\mathbf{x_{5,1}}\mathbf{x_{2,1}})\},\\
\{(x_{1,2}\mathbf{x_{1,4}}\mathbf{x_{1,1}}\mathbf{x_{1,3}}\mathbf{x_{3,3}}\mathbf{x_{3,2}}\mathbf{x_{1,2}}),\ (x_{1,2}x_{1,3}x_{4,3}x_{4,1}x_{4,2}\mathbf{x_{6,2}}\mathbf{x_{1,2}})\}\\
\{(x_{1,3}\mathbf{x_{5,3}}\mathbf{x_{5,1}}\mathbf{x_{5,2}}\mathbf{x_{6,2}}\mathbf{x_{6,3}}\mathbf{x_{1,3}}),\ (\mathbf{x_{1,3}}\mathbf{x_{1,6}}x_{1,4}x_{6,4}x_{6,5}x_{1,5}x_{1,3})\},\\
\{(x_{1,4}\mathbf{x_{1,5}}\mathbf{x_{3,5}}\mathbf{x_{2,5}}\mathbf{x_{5,5}}\mathbf{x_{5,4}}\mathbf{x_{1,4}}),\ (\mathbf{x_{1,4}}\mathbf{x_{4,4}}x_{5,4}x_{2,4}x_{6,4}x_{3,4}x_{1,4})\},\\
\{(x_{1,5}\mathbf{x_{5,5}}\mathbf{x_{5,1}}\mathbf{x_{5,6}}\mathbf{x_{5,2}}\mathbf{x_{1,2}}\mathbf{x_{1,5}}),\ (\mathbf{x_{1,5}}\mathbf{x_{4,5}}x_{4,2}x_{1,2}x_{1,6}x_{1,1}x_{1,5})\}\\
\{(x_{1,6}\mathbf{x_{6,6}}\mathbf{x_{3,6}}\mathbf{x_{4,6}}\mathbf{x_{2,6}}\mathbf{x_{5,6}}\mathbf{x_{1,6}}),\ (x_{1,6}x_{4,6}x_{6,6}x_{2,6}x_{2,5}\mathbf{x_{1,5}}\mathbf{x_{1,6}})\},\\
\{(x_{3,3}\mathbf{x_{2,3}}\mathbf{x_{6,3}}\mathbf{x_{5,3}}\mathbf{x_{5,4}}\mathbf{x_{3,4}}\mathbf{x_{3,3}}),\ (\mathbf{x_{3,3}}\mathbf{x_{4,3}}x_{4,5}x_{6,5}x_{5,5}x_{5,3}x_{3,3})\},\\
\{(x_{5,6}\mathbf{x_{4,6}}\mathbf{x_{4,4}}\mathbf{x_{2,4}}\mathbf{x_{2,6}}\mathbf{x_{3,6}}\mathbf{x_{5,6}}),\ (\mathbf{x_{5,6}}\mathbf{x_{6,6}}x_{6,4}x_{4,4}x_{4,5}x_{5,5}x_{5,6})\}\\
\{(x_{3,4}\mathbf{x_{3,5}}\mathbf{x_{4,5}}\mathbf{x_{4,6}}\mathbf{x_{4,2}}\mathbf{x_{4,4}}\mathbf{x_{3,4}}),\ (\mathbf{x_{3,4}}\mathbf{x_{2,4}}x_{2,3}x_{2,6}x_{1,6}x_{3,6}x_{3,4})\},\\
\{(x_{6,3}\mathbf{x_{6,4}}\mathbf{x_{5,4}}\mathbf{x_{5,6}}\mathbf{x_{5,3}}\mathbf{x_{4,3}}\mathbf{x_{6,3}}),\ (\mathbf{x_{6,3}}\mathbf{x_{3,3}}x_{3,5}x_{6,5}x_{6,1}x_{6,6}x_{6,3})\},\\
\{(x_{2,3}\mathbf{x_{2,5}}\mathbf{x_{4,5}}\mathbf{x_{4,1}}\mathbf{x_{4,6}}\mathbf{x_{4,3}}\mathbf{x_{2,3}}),\ (\mathbf{x_{2,3}}\mathbf{x_{2,2}}x_{2,5}x_{2,4}x_{1,4}x_{1,3}x_{2,3})\},\\
\{(x_{3,1}\mathbf{x_{3,6}}\mathbf{x_{3,2}}\mathbf{x_{5,2}}\mathbf{x_{5,5}}\mathbf{x_{3,5}}\mathbf{x_{3,1}}),\ (\mathbf{x_{3,1}}\mathbf{x_{3,4}}x_{3,2}x_{3,5}x_{3,6}x_{3,3}x_{3,1})\},\\
\{(x_{6,2}\mathbf{x_{6,4}}\mathbf{x_{6,1}}\mathbf{x_{6,3}}\mathbf{x_{6,5}}\mathbf{x_{6,6}}\mathbf{x_{6,2}}),\ (x_{6,2}x_{6,5}x_{2,5}x_{2,1}x_{2,6}\mathbf{x_{2,2}}\mathbf{x_{6,2}})\},\\
\{(x_{5,2}\mathbf{x_{5,4}}\mathbf{x_{5,1}}\mathbf{x_{3,1}}\mathbf{x_{3,2}}\mathbf{x_{4,2}}\mathbf{x_{5,2}}),\ (\mathbf{x_{5,2}}\mathbf{x_{5,3}}x_{2,3}x_{2,1}x_{2,4}x_{2,2}x_{5,2})\}.
\end{gathered}$$

The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{4,4}x_{4,3}x_{4,2}x_{2,2}x_{3,2}x_{6,2}x_{6,1},\ x_{4,4}x_{4,1}x_{5,1}x_{6,1}x_{1,1}x_{2,1}x_{2,2},\ x_{6,1}x_{2,1}x_{4,1}x_{3,1}x_{1,1}x_{1,2}x_{2,2}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.9. *There exists a* $(6;p,q)$*-decomposition of* $K_6\square K_8$, $p\neq 1$.

Proof. By Lemma 2.2, $K_8 - I$ has a $(6;p,q)$-decomposition. We decompose $8K_6 \oplus 6I$ into C_6's as follows:

$$\begin{gathered}
\{(x_{3,j}x_{2,j}x_{6,j}\mathbf{x_{4,j}}\mathbf{x_{5,j}}\mathbf{x_{1,j}}\mathbf{x_{3,j}}),\ (\mathbf{x_{3,j}}\mathbf{x_{5,j}}\mathbf{x_{5,j+1}}x_{1,j+1}x_{1,j}x_{6,j}x_{3,j})\},\\
\{(x_{3,j+1}x_{2,j+1}\mathbf{x_{2,j}}\mathbf{x_{1,j}}\mathbf{x_{4,j}}\mathbf{x_{3,j}}\mathbf{x_{3,j+1}}),\ (\mathbf{x_{3,j+1}}\mathbf{x_{4,j+1}}x_{5,j+1}x_{6,j+1}x_{2,j+1}x_{1,j+1}x_{3,j+1})\},\\
\{(x_{4,j+1}\mathbf{x_{1,j+1}}\mathbf{x_{6,j+1}}\mathbf{x_{3,j+1}}\mathbf{x_{5,j+1}}\mathbf{x_{2,j+1}}\mathbf{x_{4,j+1}}),\ ,\ (\mathbf{x_{4,j+1}}\mathbf{x_{4,j}}x_{2,j}x_{5,j}x_{6,j}x_{6,j+1}x_{4,j+1})\},
\end{gathered}$$

where $j = 1, 3, \cdots, 7$. The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{3,j+1}x_{2,j+1}x_{2,j}x_{3,j}x_{6,j}x_{1,j}x_{1,j+1},\ x_{6,j}x_{4,j}x_{1,j}x_{3,j}x_{5,j}x_{5,j+1}x_{1,j+1},\ x_{6,j}x_{2,j}x_{1,j}x_{5,j}x_{4,j}x_{3,j}x_{3,j+1}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.10. *There exists a* $(6;p,q)$*-decomposition of* $K_4\square K_4$, $p\neq 1$.

Proof. First we decompose $K_4\square K_4$ into C_6's as follows:

$$\begin{gathered}
\{(x_{2,1}\mathbf{x_{4,1}}\mathbf{x_{1,1}}\mathbf{x_{1,2}}\mathbf{x_{2,2}}\mathbf{x_{2,3}}\mathbf{x_{2,1}}),\ (\mathbf{x_{2,1}}\mathbf{x_{3,1}}x_{3,4}x_{3,2}x_{2,2}x_{2,4}x_{2,1})\},\\
\{(x_{2,3}\mathbf{x_{1,3}}\mathbf{x_{1,4}}\mathbf{x_{3,4}}\mathbf{x_{4,4}}\mathbf{x_{2,4}}\mathbf{x_{2,3}}),\ (\mathbf{x_{2,3}}\mathbf{x_{3,3}}x_{3,1}x_{3,2}x_{4,2}x_{4,3}x_{2,3})\},\\
\{(x_{1,1}\mathbf{x_{1,3}}\mathbf{x_{3,3}}\mathbf{x_{3,2}}\mathbf{x_{1,2}}\mathbf{x_{1,4}}\mathbf{x_{1,1}}),\ (x_{1,1}x_{1,2}x_{2,2}x_{4,2}x_{4,1}\mathbf{x_{3,1}}\mathbf{x_{1,1}})\},\\
\{(x_{4,4}\mathbf{x_{1,4}}\mathbf{x_{2,4}}\mathbf{x_{3,4}}\mathbf{x_{3,3}}\mathbf{x_{4,3}}\mathbf{x_{4,4}}),\ (\mathbf{x_{4,4}}\mathbf{x_{4,2}}x_{1,2}x_{1,3}x_{4,3}x_{4,1}x_{4,4})\}.
\end{gathered}$$

The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,1}x_{1,2}x_{2,2}x_{2,4}x_{2,1}x_{2,3}x_{1,3}, x_{1,3}x_{1,4}x_{3,4}x_{3,2}x_{2,2}x_{2,3}x_{2,4}, x_{1,1}x_{4,1}x_{2,1}x_{3,1}x_{3,4}x_{4,4}x_{2,4}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.11. *There exists a $(6; p, q)$-decomposition of $K_7 \square K_7$, $p \neq 1$.*

Proof. We can write $K_7 \square K_7 = 7K_7 \backslash E(K_3) \oplus K_3 \oplus 7K_7 \backslash E(K_3) \oplus K_3$. By Lemma 2.1, $K_7 \backslash E(K_3)$ has a $(6; p, q)$-decomposition. Now, we can view $7K_3 \oplus 7K_3$ as $K_3^1 \oplus \cdots \oplus K_3^7 \oplus (K_3^1)' \oplus \cdots \oplus (K_3^7)'$ with $K_3^i = (x_{i,i-2}x_{i,i-1}x_{i,i}x_{i,i-2})$, for $i = 1, \cdots, 7$ and $(K_3^i)' = (x_{i,i}x_{i+1,i}x_{i+2,i}x_{i,i})$, for $i = 1, \cdots, 7$, where the subscripts of x are taken modulo 7 with residues $\{1, \cdots, 7\}$. The C_6-decomposition of $7K_3 \oplus 7K_3$ is given below:

$$\{(x_{3,1}\mathbf{x_{1,1}x_{2,1}x_{2,7}x_{2,2}x_{3,2}x_{3,1}}), (\mathbf{x_{3,1}x_{3,3}}x_{3,2}x_{4,2}x_{2,2}x_{2,1}x_{3,1})\},$$
$$\{(x_{4,4}\mathbf{x_{4,2}x_{4,3}x_{3,3}x_{5,3}x_{5,4}x_{4,4}}), (\mathbf{x_{4,4}x_{6,4}}x_{5,4}x_{5,5}x_{5,3}x_{4,3}x_{4,4})\},$$
$$\{(\mathbf{x_{7,7}x_{1,7}x_{1,6}x_{6,6}x_{6,5}x_{7,5}}x_{7,7}), (\mathbf{x_{7,7}x_{2,7}}x_{1,7}x_{1,1}x_{1,6}x_{7,6}x_{7,7})\},$$
$$(x_{5,5}x_{6,5}x_{6,4}x_{6,6}x_{7,6}x_{7,5}x_{5,5}).$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,1}x_{1,7}x_{2,7}x_{7,7}x_{7,6}x_{7,5}x_{5,5}, x_{5,5}x_{6,5}x_{6,4}x_{6,6}x_{7,6}x_{1,6}x_{1,7}, x_{1,1}x_{1,6}x_{6,6}x_{6,5}x_{7,5}x_{7,7}x_{1,7}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. □

Lemma 3.12. *There exists a $(6; p, q)$-decomposition of $K_3 \square K_{12}$, $p \geq 18$.*

Proof. Since the degree of each vertex $v \in V(K_3 \square K_{12})$ is odd, then $p \geq \frac{36}{2} = 18$. We can write $K_3 \square K_{12} = 12K_3 \oplus 3K_{12} = 12K_3 \oplus 3((K_{12} \backslash E(2C_6)) \oplus 2C_6)$. The graph $12K_3$ along with three rows of $2C_6$ can be viewed as $2G$, where $G = 6K_3 \oplus C_6^1 \oplus C_6^2 \oplus C_6^3$ with $V(G) = \{x_{i,j} : 1 \leq i \leq 3, 1 \leq j \leq 6\}$ and $C_6^1 = (x_{1,1}x_{1,2}x_{1,5}x_{1,4}x_{1,6}x_{1,3}x_{1,1}$, $C_6^2 = (x_{2,1}x_{2,2}x_{2,4}x_{2,5}x_{2,3}x_{2,6}x_{2,1})$, $C_6^3 = (x_{3,1}x_{3,2}x_{3,6}x_{3,4}x_{3,3}x_{3,5}x_{3,1})$ and decompose G into C_6's as follows:

$$\{(x_{1,1}\mathbf{x_{1,2}x_{1,5}x_{2,5}x_{2,3}x_{1,3}x_{1,1}}), (\mathbf{x_{1,1}x_{2,1}}x_{2,2}x_{1,2}x_{3,2}x_{3,1}x_{1,1})\},$$
$$\{(x_{1,6}\mathbf{x_{1,4}x_{2,4}x_{2,2}x_{3,2}x_{3,6}x_{1,6}}), (\mathbf{x_{1,6}x_{2,6}}x_{3,6}x_{3,4}x_{3,3}x_{1,3}x_{1,6})\},$$
$$\{(x_{3,5}\mathbf{x_{2,5}x_{2,4}x_{3,4}x_{1,4}x_{1,5}x_{3,5}}), (\mathbf{x_{3,5}x_{3,1}}x_{2,1}x_{2,6}x_{2,3}x_{3,3}x_{3,5})\}.$$

The first $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{2,1}x_{2,2}x_{1,2}x_{1,1}x_{3,1}x_{3,2}x_{3,6}, x_{2,1}x_{1,1}x_{1,3}x_{2,3}x_{2,5}x_{1,5}x_{1,2}, x_{1,2}x_{3,2}x_{2,2}x_{2,3}x_{1,3}x_{1,6}x_{3,6}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. So, G has a $(6; p, q)$-decomposition. By Lemma 2.5, the remaining edges has a $(6; p, q)$-decomposition. □

Lemma 3.13. *There exists a* $(6;p,q)$*-decomposition of* $K_5 \Box K_{12}$, $p \geq 30$.

Proof. Since the degree of each vertex $v \in V(K_5 \Box K_{12})$ is odd, then $p \geq 30$. We can write $K_5 \Box K_{12} = 12K_5 \oplus 5K_{12} = 12K_5 \oplus 5((K_{12} \backslash E(2C_6)) \oplus 2C_6)$. By Lemma 2.5, $K_{12} \backslash E(2C_6)$ has a $(6;p,q)$-decomposition. Let $12K_5 \oplus 10C_6 = G_1 \oplus G_2$, where $G_1 = (6K_5 \oplus C_6^1 \oplus \cdots \oplus C_6^5) \cong G_2$ with

$$C_6^1 = (x_{1,1}x_{1,2}x_{1,4}x_{1,6}x_{1,5}x_{1,3}x_{1,1}), C_6^2 = (x_{2,1}x_{2,5}x_{2,3}x_{2,6}x_{2,2}x_{2,4}x_{2,1}),$$
$$C_6^3 = (x_{3,1}x_{3,3}x_{3,5}x_{3,2}x_{3,6}x_{3,4}x_{3,1}), C_6^4 = (x_{4,1}x_{4,5}x_{4,2}x_{4,4}x_{4,6}x_{4,3}x_{4,1}),$$
$$C_6^5 = (x_{5,1}x_{5,2}x_{5,4}x_{5,6}x_{5,5}x_{5,3}x_{5,1}).$$

The graph G_1 decomposes into required number of C_6 as follows:

$$\{(x_{1,2}\mathbf{x_{1,4}x_{5,4}x_{4,4}x_{4,2}x_{3,2}x_{1,2}}),\ (\mathbf{x_{1,2}x_{2,2}}x_{2,4}x_{3,4}x_{5,4}x_{5,2}x_{1,2})\},$$
$$\{(x_{1,6}\mathbf{x_{2,6}x_{3,6}x_{3,4}x_{4,4}x_{1,4}x_{1,6}}),\ (x_{1,6}x_{3,6}x_{5,6}x_{5,5}x_{2,5}\mathbf{x_{1,5}x_{1,6}})\},$$
$$\{(x_{1,3}\mathbf{x_{2,3}x_{2,6}x_{5,6}x_{4,6}x_{4,3}x_{1,3}}),\ (\mathbf{x_{1,3}x_{1,5}}x_{5,5}x_{3,5}x_{3,3}x_{5,3}x_{1,3})\},$$
$$\{(x_{2,1}\mathbf{x_{2,5}x_{3,5}x_{4,5}x_{4,1}x_{1,1}x_{2,1}}),\ (\mathbf{x_{2,1}x_{2,4}}x_{1,4}x_{3,4}x_{3,1}x_{4,1}x_{2,1})\},$$
$$\{(x_{4,2}\mathbf{x_{5,2}x_{5,1}x_{3,1}x_{1,1}x_{1,2}x_{4,2}}),\ (\mathbf{x_{4,2}x_{2,2}}x_{3,2}x_{3,5}x_{1,5}x_{4,5}x_{4,2})\},$$
$$\{(x_{4,6}\mathbf{x_{3,6}x_{3,2}x_{5,2}x_{2,2}x_{2,6}x_{4,6}}),\ (\mathbf{x_{4,6}x_{1,6}}x_{5,6}x_{5,4}x_{2,4}x_{4,4}x_{4,6})\},$$
$$\{(x_{3,3}\mathbf{x_{4,3}x_{4,1}x_{5,1}x_{1,1}x_{1,3}x_{3,3}}),\ (\mathbf{x_{3,3}x_{2,3}}x_{5,3}x_{5,1}x_{2,1}x_{3,1}x_{3,3})\},$$
$$(x_{2,3}x_{2,5}x_{4,5}x_{5,5}x_{5,3}x_{4,3}x_{2,3}).$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{5,5}x_{5,3}x_{2,3}x_{4,3}x_{4,1}x_{5,1}x_{2,1}, x_{2,1}x_{3,1}x_{3,3}x_{1,3}x_{1,1}x_{5,1}x_{5,3}, x_{5,3}x_{4,3}x_{3,3}x_{2,3}x_{2,5}x_{4,5}x_{5,5}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. Hence G_1 has a $(6;p,q)$-decomposition and so the graph G_2. □

Lemma 3.14. *There exists a* $(6;p,q)$*-decomposition of* $K_7 \Box K_{12}$, $p \geq 42$.

Proof. Since the degree of each vertex $v \in V(K_7 \Box K_{12})$ is odd, then $p \geq 42$. We can write $K_7 \Box K_{12} = 12K_7 \oplus 7K_{12} = 12(K_7 \backslash E(K_3)) \oplus 4K_{12} \oplus (K_3 \Box K_{12})$. By Lemmas 2.1, 2.8 and 3.12, the given graph has a $(6;p,q)$-decomposition. □

Lemma 3.15. *There exists a* $(6;p,q)$*-decomposition of* $K_{11} \Box K_{12}$, $p \geq 66$.

Proof. Since the degree of each vertex $v \in V(K_{11} \Box K_{12})$ is odd, then $p \geq 66$. We can write $K_{11} \Box K_{12} = 12K_{11} \oplus 11K_{12} = 12(K_{11} \backslash E(C_7)) \oplus 11K_{12}$. Consider K_{12} in rows 1, 3, 4, 7 as $(K_{12} \backslash E(2C_6)) \oplus 2C_6$, where $2C_6$ are vertex disjoint cycles. Now, these $8C_6$ along with $12C_7$ in columns form a graph $G = (4C_6 \oplus 6C_7) \oplus (4C_6 \oplus 6C_7) = G_1 \oplus G_2$, $G_1 \cong G_2$. Let $G_1 = C_6^1 \oplus \cdots \oplus C_6^4 \oplus C_7^1 \oplus \cdots \oplus C_7^6$, where

$$C_6^1 = (x_{1,1}x_{1,2}x_{1,5}x_{1,6}x_{1,4}x_{1,3}x_{1,1}), C_6^2 = (x_{3,1}x_{3,2}x_{3,3}x_{3,6}x_{3,4}x_{3,5}x_{3,1}),$$
$$C_6^3 = (x_{4,1}x_{4,2}x_{4,5}x_{4,6}x_{4,4}x_{4,3}x_{4,1}), C_6^4 = (x_{7,1}x_{7,2}x_{7,3}x_{7,4}x_{7,5}x_{7,6}x_{7,1}),$$

and

$$C_7^1 = (x_{1,1}x_{2,1}x_{4,1}x_{7,1}x_{3,1}x_{5,1}x_{6,1}x_{1,1}), C_7^2 = (x_{1,2}x_{3,2}x_{7,2}x_{6,2}x_{5,2}x_{4,2}x_{2,2}x_{1,2}),$$
$$C_7^3 = (x_{1,3}x_{3,3}x_{5,3}x_{6,3}x_{7,3}x_{4,3}x_{2,3}x_{1,3}), C_7^4 = (x_{1,4}x_{2,4}x_{3,4}x_{7,4}x_{6,4}x_{5,4}x_{4,4}x_{1,4}),$$
$$C_7^5 = (x_{1,5}x_{3,5}x_{5,5}x_{6,5}x_{7,5}x_{4,5}x_{2,5}x_{1,5}), C_7^6 = (x_{1,6}x_{2,6}x_{3,6}x_{4,6}x_{5,6}x_{6,6}x_{7,6}x_{1,6}).$$

This can be decomposed into required number of C_6 as follows:

$$\{(x_{1,2}\mathbf{x_{3,2}x_{3,1}x_{5,1}x_{6,1}x_{1,1}x_{1,2}}), (\mathbf{x_{1,2}x_{1,5}}x_{2,5}x_{4,5}x_{4,2}x_{2,2}x_{1,2})\},$$
$$\{(x_{7,2}\mathbf{x_{7,1}x_{4,1}x_{4,2}x_{5,2}x_{6,2}x_{7,2}}), (\mathbf{x_{7,2}x_{3,2}}x_{3,3}x_{5,3}x_{6,3}x_{7,3}x_{7,2})\},$$
$$\{(x_{7,4}\mathbf{x_{7,3}x_{4,3}x_{4,4}x_{5,4}x_{6,4}x_{7,4}}), (\mathbf{x_{7,4}x_{3,4}}x_{3,5}x_{5,5}x_{6,5}x_{7,5}x_{7,4})\},$$
$$\{(x_{7,6}\mathbf{x_{7,5}x_{4,5}x_{4,6}x_{5,6}x_{6,6}x_{7,6}}), (\mathbf{x_{7,6}x_{7,1}}x_{3,1}x_{3,5}x_{1,5}x_{1,6}x_{7,6})\},$$
$$\{(x_{1,3}\mathbf{x_{1,4}x_{2,4}x_{3,4}x_{3,6}x_{3,3}x_{1,3}}), (\mathbf{x_{1,3}x_{2,3}}x_{4,3}x_{4,1}x_{2,1}x_{1,1}x_{1,3})\},$$
$$(x_{1,4}x_{1,6}x_{2,6}x_{3,6}x_{4,6}x_{4,4}x_{1,4}).$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{4,1}x_{2,1}x_{1,1}x_{1,3}x_{1,4}x_{1,6}x_{2,6}, x_{2,6}x_{3,6}x_{4,6}x_{4,4}x_{1,4}x_{2,4}x_{3,4}, x_{3,4}x_{3,6}x_{3,3}x_{1,3}x_{2,3}x_{4,3}x_{4,1}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition given above. Hence G_1 also has a $(6;p,q)$-decomposition and so the graph G_2. Also by Lemmas 2.5, 2.7, $K_{11}\backslash E(C_7)$ and $K_{12}\backslash E(C_6)$ have a $(6;p,q)$-decomposition. Hence by Remark 1.3, $K_{11}\square K_{12}$ has a $(6;p,q)$-decomposition. □

Lemma 3.16. *There exists a $(6;p,q)$-decomposition of $C_6\square(K_8\backslash E(2C_6))$, where $p = 24$ and the $2C_6$ are $\{(x_{2,i}x_{3,i}x_{7,i}x_{4,i}x_{6,i}x_{8,i}x_{2,i})$, $(x_{2,i}x_{5,i}x_{4,i}x_{8,i}x_{1,i}x_{6,i}x_{2,i})\}$, $1 \le i \le 6$.*

Proof. Let $V(G = C_6\square(K_8\backslash E(2C_6))) = \{x_{i,j} : 1 \le i \le 6, 1 \le j \le 8\}$. Since the degree of each vertex $v \in V(G)$ is odd and $|E(G)| = 144$, then $p = 24$. Now, the $24P_7$ are given below: $\{x_{i,1}x_{i+1,1}x_{i+1,5}x_{i+1,3}x_{i+1,8}x_{i+1,7}x_{i,7}$, $x_{i,2}x_{i+1,2}x_{i+1,1}x_{i+1,3}x_{i+1,6}x_{i+1,5}x_{i,5}$, $x_{i,3}x_{i+1,3}x_{i+1,4}x_{i+1,2}x_{i+1,7}x_{i+1,6}x_{i,6}$, $x_{i,4}x_{i+1,4}x_{i+1,1}x_{i+1,7}x_{i+1,5}x_{i+1,8}x_{i,8}$, where $1 \le i \le 6$ and the first coordinate of subscripts of x are taken modulo 6 with residues $\{1,\cdots,6\}\}$. □

Lemma 3.17. *There exists a $(6;p,q)$-decomposition of $K_8\square K_9$, $p \ge 36$.*

Proof. Since the degree of each vertex $v \in V(K_8\square K_9)$ is odd, then $p \ge \frac{72}{2} = 36$. For $p = 36$, the required number of P_7's and C_6's are constructed as follows:

$$\{x_{1,1}x_{1,3}x_{1,2}x_{6,2}x_{5,2}x_{3,2}x_{4,2}, x_{1,2}x_{1,9}x_{1,7}x_{2,7}x_{3,7}x_{3,2}x_{6,2},$$
$$x_{5,7}x_{6,7}x_{6,1}x_{1,1}x_{7,1}x_{7,6}x_{7,3}, x_{1,5}x_{5,5}x_{6,5}x_{8,5}x_{8,2}x_{1,2}x_{3,2},$$
$$x_{1,6}x_{3,6}x_{3,2}x_{7,2}x_{1,2}x_{2,2}x_{2,4}, x_{1,7}x_{1,3}x_{2,3}x_{2,9}x_{2,5}x_{6,5}x_{6,3},$$
$$x_{1,8}x_{5,8}x_{5,6}x_{8,6}x_{8,9}x_{8,7}x_{7,7}, x_{1,9}x_{1,1}x_{2,1}x_{6,1}x_{8,1}x_{8,2}x_{8,8},$$
$$x_{2,1}x_{5,1}x_{1,1}x_{4,1}x_{4,2}x_{6,2}x_{8,2}, x_{2,2}x_{8,2}x_{8,4}x_{3,4}x_{2,4}x_{2,6}x_{2,7},$$
$$x_{2,3}x_{2,4}x_{6,4}x_{7,4}x_{7,6}x_{1,6}x_{6,6}, x_{2,5}x_{2,2}x_{4,2}x_{4,5}x_{3,5}x_{5,5}x_{8,5},$$

$$
\begin{aligned}
&x_{2,6}x_{2,9}x_{2,4}x_{5,4}x_{5,1}x_{3,1}x_{7,1},\ x_{2,8}x_{2,5}x_{2,7}x_{7,7}x_{5,7}x_{4,7}x_{4,6},\\
&x_{2,9}x_{1,9}x_{7,9}x_{7,3}x_{8,3}x_{2,3}x_{4,3},\ x_{3,1}x_{3,6}x_{7,6}x_{7,5}x_{7,1}x_{8,1}x_{8,9},\\
&x_{3,3}x_{6,3}x_{4,3}x_{4,8}x_{3,8}x_{3,1}x_{4,1},\ x_{3,4}x_{3,5}x_{3,3}x_{1,3}x_{8,3}x_{8,8}x_{7,8},\\
&x_{3,5}x_{3,1}x_{3,7}x_{4,7}x_{4,3}x_{3,3}x_{3,6},\ x_{3,7}x_{3,4}x_{4,4}x_{7,4}x_{5,4}x_{5,8}x_{5,9},\\
&x_{3,8}x_{3,7}x_{8,7}x_{4,7}x_{4,5}x_{1,5}x_{7,5},\ x_{4,4}x_{5,4}x_{5,3}x_{5,9}x_{6,9}x_{6,6}x_{6,1},\\
&x_{4,5}x_{7,5}x_{3,5}x_{3,6}x_{6,6}x_{5,6}x_{5,4},\ x_{4,7}x_{4,9}x_{4,4}x_{4,2}x_{4,8}x_{7,8}x_{7,6},\\
&x_{4,8}x_{1,8}x_{1,2}x_{1,1}x_{1,4}x_{5,4}x_{6,4},\ x_{4,9}x_{4,6}x_{2,6}x_{1,6}x_{8,6}x_{7,6}x_{7,2},\\
&x_{5,3}x_{5,2}x_{5,9}x_{5,5}x_{5,8}x_{2,8}x_{6,8},\ x_{5,5}x_{2,5}x_{1,5}x_{6,5}x_{6,2}x_{6,4}x_{6,7},\\
&x_{5,6}x_{3,6}x_{4,6}x_{6,6}x_{8,6}x_{8,7}x_{8,3},\ x_{5,8}x_{6,8}x_{7,8}x_{7,5}x_{6,5}x_{6,4}x_{6,9},\\
&x_{6,5}x_{6,7}x_{6,6}x_{6,3}x_{5,3}x_{5,1}x_{8,1},\ x_{7,4}x_{8,4}x_{8,9}x_{7,9}x_{7,5}x_{8,5}x_{8,7},\\
&x_{8,4}x_{4,4}x_{1,4}x_{1,8}x_{2,8}x_{2,6}x_{8,6},\ x_{7,9}x_{3,9}x_{4,9}x_{4,1}x_{7,1}x_{6,1}x_{5,1},\\
&x_{1,4}x_{2,4}x_{7,4}x_{7,3}x_{7,2}x_{8,2}x_{5,2},\ x_{1,3}x_{1,8}x_{1,9}x_{6,9}x_{8,9}x_{5,9}x_{3,9}\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\{(\mathbf{x_{1,7}x_{1,2}}x_{1,6}x_{1,5}x_{1,1}x_{1,8}x_{1,7}),\ (x_{1,7}\mathbf{x_{1,4}x_{1,5}x_{1,3}x_{1,6}x_{1,1}x_{1,7}})\},\\
&\{(\mathbf{x_{1,7}x_{3,7}}x_{6,7}x_{4,7}x_{2,7}x_{8,7}x_{1,7}),\ (x_{1,7}\mathbf{x_{1,6}x_{1,9}x_{1,4}x_{1,2}x_{1,5}x_{1,7}})\},\\
&\{(x_{1,3}\mathbf{x_{1,4}x_{1,6}x_{1,8}x_{1,5}x_{1,9}x_{1,3}}),\ (\mathbf{x_{1,3}x_{6,3}}x_{8,3}x_{5,3}x_{3,3}x_{7,3}x_{1,3})\},\\
&\{(\mathbf{x_{2,1}x_{2,2}x_{2,8}x_{2,3}x_{2,7}x_{2,9}}x_{2,1}),\ (x_{2,1}x_{2,7}x_{2,4}x_{2,5}x_{2,3}\mathbf{x_{2,6}x_{2,1}})\},\\
&\{(x_{2,1}\mathbf{x_{2,4}x_{2,8}x_{2,9}x_{2,2}x_{2,3}x_{2,1}}),\ (x_{2,1}x_{2,8}x_{2,7}x_{2,2}x_{2,6}\mathbf{x_{2,5}x_{2,1}})\},\\
&\{(x_{3,2}\mathbf{x_{3,8}x_{3,3}x_{3,7}x_{3,9}x_{3,1}x_{3,2}}),\ (\mathbf{x_{3,2}x_{3,5}}x_{3,7}x_{3,6}x_{3,9}x_{3,4}x_{3,2})\},\\
&\{(\mathbf{x_{3,4}x_{3,8}x_{3,9}x_{3,2}x_{3,3}x_{3,1}}x_{3,4}),\ (\mathbf{x_{3,4}x_{3,6}}x_{3,8}x_{3,5}x_{3,9}x_{3,3}x_{3,4})\},\\
&\{(\mathbf{x_{4,4}x_{4,6}x_{4,8}}x_{4,5}x_{4,9}x_{4,3}x_{4,4}),\ (\mathbf{x_{4,4}x_{4,5}x_{4,3}x_{4,6}x_{4,1}}x_{4,7}x_{4,4})\},\\
&\{(x_{4,9}\mathbf{x_{4,2}x_{4,3}x_{4,1}x_{4,4}x_{4,8}x_{4,9}}),\ (\mathbf{x_{4,9}x_{5,9}}x_{7,9}x_{2,9}x_{3,9}x_{8,9}x_{4,9})\},\\
&\{(\mathbf{x_{4,8}x_{4,7}x_{4,2}x_{4,6}x_{4,5}x_{4,1}}x_{4,8}),\ (\mathbf{x_{4,8}x_{2,8}}x_{8,8}x_{1,8}x_{3,8}x_{6,8}x_{4,8})\},\\
&\{(x_{5,2}\mathbf{x_{5,8}x_{5,3}x_{5,7}x_{5,9}x_{5,1}x_{5,2}}),\ (\mathbf{x_{5,2}x_{5,5}}x_{5,7}x_{5,6}x_{5,9}x_{5,4}x_{5,2})\},\\
&\{(x_{5,5}\mathbf{x_{5,1}x_{5,8}x_{5,7}x_{5,2}x_{5,6}x_{5,5}}),\ (\mathbf{x_{5,5}x_{5,3}}x_{5,6}x_{5,1}x_{5,7}x_{5,4}x_{5,5})\},\\
&\{(\mathbf{x_{6,2}x_{6,8}x_{6,3}x_{6,7}x_{6,9}}x_{6,1}x_{6,2}),\ (\mathbf{x_{6,2}x_{6,6}x_{6,5}}x_{6,1}x_{6,8}x_{6,7}x_{6,2})\},\\
&\{(x_{6,4}\mathbf{x_{6,6}x_{6,8}x_{6,5}x_{6,9}x_{6,3}x_{6,4}}),\ (x_{6,4}x_{6,8}x_{6,9}x_{6,2}x_{6,3}\mathbf{x_{6,1}x_{6,4}})\},\\
&\{(x_{7,2}\mathbf{x_{7,8}x_{7,3}x_{7,7}x_{7,9}x_{7,1}x_{7,2}}),\ (\mathbf{x_{7,2}x_{7,5}}x_{7,7}x_{7,6}x_{7,9}x_{7,4}x_{7,2})\},\\
&\{(\mathbf{x_{7,7}x_{7,1}x_{7,4}x_{7,8}x_{7,9}}x_{7,2}x_{7,7}),\ (\mathbf{x_{7,7}x_{7,4}x_{7,5}}x_{7,3}x_{7,1}x_{7,8}x_{7,7})\},\\
&\{(\mathbf{x_{8,6}x_{8,8}x_{8,5}x_{8,9}x_{8,3}x_{8,4}}x_{8,6}),\ (x_{8,6}x_{8,5}x_{8,1}x_{8,8}x_{8,7}\mathbf{x_{8,2}x_{8,6}})\},\\
&\{(\mathbf{x_{8,3}x_{8,1}x_{8,4}x_{8,8}x_{8,9}x_{8,2}}x_{8,3}),\ (\mathbf{x_{8,3}x_{8,6}}x_{8,1}x_{8,7}x_{8,4}x_{8,5}x_{8,3})\},\\
&\{(\mathbf{x_{3,1}x_{6,1}x_{4,1}x_{2,1}x_{8,1}}x_{1,1}x_{3,1}),\ (x_{3,1}x_{8,1}x_{4,1}x_{5,1}\mathbf{x_{7,1}x_{2,1}x_{3,1}})\},\\
&\{(\mathbf{x_{7,2}x_{6,2}x_{2,2}x_{5,2}x_{1,2}}x_{4,2}x_{7,2}),\ (\mathbf{x_{7,2}x_{2,2}x_{3,2}}x_{8,2}x_{4,2}x_{5,2}x_{7,2})\},\\
&\{(\mathbf{x_{2,3}x_{5,3}x_{1,3}x_{4,3}x_{7,3}x_{6,3}}x_{2,3}),\ (\mathbf{x_{2,3}x_{3,3}}x_{8,3}x_{4,3}x_{5,3}x_{7,3}x_{2,3})\},\\
&\{(x_{1,4}\mathbf{x_{6,4}x_{8,4}x_{5,4}x_{3,4}x_{7,4}}x_{1,4}),\ (x_{1,4}x_{3,4}\mathbf{x_{6,4}x_{4,4}x_{2,4}}x_{8,4}x_{1,4})\},\\
&\{(\mathbf{x_{1,5}x_{3,5}x_{6,5}x_{4,5}x_{2,5}}x_{8,5}x_{1,5}),\ (x_{2,5}x_{3,5}x_{8,5}x_{4,5}\mathbf{x_{5,5}x_{7,5}x_{2,5}})\},\\
&\{(\mathbf{x_{1,6}x_{4,6}x_{7,6}x_{6,6}x_{2,6}}x_{5,6}x_{1,6}),\ (\mathbf{x_{2,6}x_{3,6}x_{8,6}}x_{4,6}x_{5,6}x_{7,6}x_{2,6})\},\\
&\{(\mathbf{x_{5,7}x_{1,7}x_{4,7}x_{7,7}x_{6,7}x_{2,7}}x_{5,7}),\ (\mathbf{x_{5,7}x_{3,7}}x_{7,7}x_{1,7}x_{6,7}x_{8,7}x_{5,7})\},\\
&\{(\mathbf{x_{2,8}x_{3,8}x_{8,8}x_{4,8}x_{5,8}x_{7,8}}x_{2,8}),\ (x_{1,8}x_{6,8}x_{8,8}x_{5,8}x_{3,8}\mathbf{x_{7,8}x_{1,8}})\},\\
&\{(x_{1,9}\mathbf{x_{3,9}x_{6,9}x_{4,9}x_{2,9}x_{8,9}x_{1,9}}),\ (x_{1,9}x_{4,9}x_{7,9}x_{6,9}x_{2,9}\mathbf{x_{5,9}x_{1,9}})\}.
\end{aligned}
$$

For $p = 37$, we decompose the last path and first cycle into $2P_7$ as follows:

$$
\{x_{1,7}x_{1,2}x_{1,6}x_{1,5}x_{1,1}x_{1,8}x_{1,3},\ x_{1,7}x_{1,8}x_{1,9}x_{6,9}x_{8,9}x_{5,9}x_{3,9}\}.
$$

Now, using Construction 1.4 we get the required number of paths and cycles from C_6's for $p > 37$. So, we have the desired decomposition for $K_8 \Box K_9$. $\Box$

Theorem 3.18. *$K_m \Box K_n$ has a $(6; p, q)$-decomposition if and only if* $mn(m+n-2) \equiv 0 \pmod{12}$.

Proof. Necessity. Since $K_m \Box K_n$ is $(m+n-2)$-regular with mn vertices, $K_m \Box K_n$ has $mn(m+n-2)/2$ edges. Now, assume that $K_m \Box K_n$ has a $(6; p, q)$-decomposition. Then the number of edges in the graph must be divisible by 6, i.e., $12 | mn(m+n-2)$ and hence $mn(m+n-2) \equiv 0 \pmod{12}$.
Sufficiency. We construct the required decomposition in ten cases.

Case 1. $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$.

Subcase 1.1. $m, n \equiv 0 \pmod 6$.
Let $m = 6k$ and $n = 6l$, where $k, l > 0$ are integers. We can write $K_m \Box K_n = kl(K_6 \Box K_6) \oplus 3kl(k+l-2)K_{6,6}$. By Theorem 1.2 and Lemma 3.8, $K_{6,6}$ and $K_6 \Box K_6$ have a $(6; p, q)$-decomposition. Hence by Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition.

Subcase 1.2. $m \equiv 0 \pmod 6$, $n \equiv 4 \pmod 6$.
Let $m = 6k$ and $n = 6l + 4$, where k, l are non-negative integers. We can write $K_m \Box K_n = (K_{6k} \Box K_{6l}) \oplus k(K_6 \Box K_4) \oplus 2k(k-1)K_{6,6} \oplus 6kK_{6l,4}$. By Theorem 1.2, Lemmas 3.7, and 2.4, Subcase 1.1 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition.

Subcase 1.3. $m \equiv 0 \pmod 6$, $n \equiv 2 \pmod 6$.
When $m = 6k$ and $n = 2$, $K_m \Box K_n = k(K_6 \Box K_2) \oplus k(k-1)K_{6,6}$. By Theorem 1.2, Lemma 3.6 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition. When $m = 6k$ and $n = 8$, $K_m \Box K_n = k(K_6 \Box K_8) \oplus 4k(k-1)K_{6,6}$. By Theorem 1.2, Lemma 3.9 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition. When $n > 8$, let $m = 6k$, $n = 6l + 8$, where k, l are non-negative integers. We can write $K_m \Box K_n = (K_{6k} \Box K_{6l}) \oplus (K_{6k} \Box K_8) \oplus 6kK_{6l,8}$. By Theorem 1.2, Lemma 2.4, Subcase 1.1 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition.

Case 2. $m, n \equiv 4 \pmod 6$.
Let $m = 6k + 4$ and $n = 6l + 4$, where k, l are non-negative integers. We can write $K_m \Box K_n = kl(K_6 \Box K_6) \oplus (k+l)(K_6 \Box K_4) \oplus (K_4 \Box K_4) \oplus (3kl(k+l-2)+2k(k-1))K_{6,6} \oplus (12kl+4(l+k))K_{6,4}$. By Theorem 1.2, Lemmas 3.7, 3.8, 3.10 and 2.4 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition.

Case 3. $m \equiv 0, 1, 4$ *or* $9 \pmod{12}$, $n \equiv 1$ *or* $9 \pmod{12}$.
When m is even, the degree of each vertex $v \in V(K_m \Box K_n)$ is odd, then $p \geq mn/2$. Now, $K_m \Box K_n = nK_m \oplus mK_n$. By Lemma 2.8 and Theorem 1.1, K_m and K_n have a $(6; p, q)$-decomposition (with $p \geq m/2$ whenever m is even). Hence by Remark 1.3, $K_m \Box K_n$ has the required decomposition.

Case 4. $m, n \equiv 3$ *or* $7 \pmod{12}$.

Subcase 4.1. $m, n \equiv i \pmod{12}$, $i = 3, 7$.
When $m = n$, if $i = 3$, then $K_m \Box K_n = nK_m \oplus mK_n = 2m(K_m \backslash E(K_3)) \oplus \frac{m}{3}(K_3 \Box K_3)$. If $i = 7$ let $m = 12k+7$, then $K_m \Box K_n = 2(m-7)(K_m \backslash E(K_3)) \oplus \frac{(m-7)}{3}(K_3 \Box K_3) \oplus 14(K_{12k+1} \oplus K_{12k,6}) \oplus K_7 \Box K_7$. By Lemmas 2.6, 3.1, 3.11, Theorems 1.1, 1.2 and Remark 1.3, $K_m \Box K_n$ has a $(6; p, q)$-decomposition.
When $m < n$, let $n = m + h$, where $h = 12l, l \in \mathbb{Z}^+$ $m = 12k + i$, $i = 3, 7$. We can write $K_m \Box K_n = (K_m \Box K_m) \oplus hK_m \oplus m(K_n \backslash E(K_m)) = (K_m \Box K_m) \oplus 12l(K_m \backslash E(K_3)) \oplus 12lK_3 \oplus m(K_{12l+1} \oplus$

$K_{12l,m-1}$). Now, the first three rows of $(K_m \square K_n)\backslash(K_m \square K_m)$ can be viewed as $(K_{12l+1}\backslash E(2lC_6)) \oplus K_{12l,m-1} \oplus 2lC_6$. As in the proof of Lemma 3.12, we can prove $12lK_3$ along with three rows of $2lC_6$ has a $(6;p,q)$-decomposition. By Lemmas 2.4 and 2.5 and Theorem 1.2, $K_{12l+1}\backslash E(2lC_6)$ and $K_{12l,m-1}$ have a $(6;p,q)$-decomposition. Hence by Remark 1.3, $K_m \square K_n$ has a $(6;p,q)$-decomposition.

Subcase 4.2. $m \equiv 3 \pmod{12}$, $n \equiv 7 \pmod{12}$.

Let $m = 12k+3, n = 12l+7$. We can write $K_m \square K_n = nK_m \oplus mK_n$.

When $k = l$, every column of $K_m \square K_n$ can be viewed as $(K_m\backslash E(K_3)) \oplus K_3$ and every first $(m-3)$ rows can be viewed as $(K_n\backslash E(K_3)) \oplus K_3$ and last three rows can be viewed as $(K_n\backslash E(K_7)) \oplus K_7$. Now, the K_3's in first $(m-3)$ rows and columns form $\frac{(m-3)}{3}(K_3 \square K_3)$ and $K_n\backslash E(K_7)$ can be viewed as $K_{12l+1} \oplus K_{12l,6}$ and these graphs have a $(6;p,q)$-decomposition, by Theorems 1.1, 1.2. By Lemmas 2.6 and 3.1, $K_m\backslash E(K_3), K_n\backslash E(K_3)$ and $(K_3 \square K_3)$ have a $(6;p,q)$-decomposition. By Lemma 3.2, the remaining graph $K_3 \square K_7$ has a $(6;p,q)$-decomposition.

When $k < l$, every column of $K_m \square K_n$ can be viewed as $(K_m\backslash E(K_3)) \oplus K_3$ and every first $(m-3)$ rows can be viewed as $(K_n\backslash E(K_3)) \oplus K_3$. Now, the K_3's in first $(m-3)$ rows and columns form $\frac{(m-3)}{3}(K_3 \square K_3)$. By Lemmas 2.6 and 3.1, $K_m\backslash E(K_3), K_n\backslash E(K_3)$ and $(K_3 \square K_3)$ have a $(6;p,q)$-decomposition. Finally, we have to find a $(6;p,q)$-decomposition of the last three rows and $12(l-k)+7$ columns of $K_m \square K_n$. Now, every $12(l-k)+7$ columns of $K_m \square K_n$ can be viewed as $(K_m\backslash E(K_3)) \oplus (K_m\backslash V(K_{m-3}))$ and every last three rows of $K_m \square K_n$ can be viewed as $K_{12k+1} \oplus K_7 \oplus K_{12k,6} \oplus K_{12(l-k),6} \oplus K_{12k,12(l-k)} \oplus (K_{12(l-k)+1}\backslash E(2(l-k)C_6)) \oplus 2(l-k)C_6$ and by Theorem 1.2, $K_{12k,6} \oplus K_{12(l-k),6} = K_{12l,6}$ has a $(6;p,q)$-decomposition. By Lemma 2.5, $K_{12(l-k)+1}\backslash E(2(l-k)C_6)$ has a $(6;p,q)$-decomposition. As in the proof of Lemma 3.12, we can prove $12(l-k)K_n\backslash V(K_{n-3})$ along with the three rows of $2(l-k)C_6$ has a $(6;p,q)$-decomposition. By Lemma 3.2, the remaining graph $K_3 \square K_7$ has a $(6;p,q)$-decomposition.

By using similar proof, we can prove for the case $k > l$ also. Hence $K_m \square K_n$ has a $(6;p,q)$-decomposition.

Case 5. $m \equiv 3 \pmod{12}$, $n \equiv 11 \pmod{12}$.

Let $m = 12k+3, n = 12l+11$. We can write $K_m \square K_n = nK_m \oplus mK_n$. Consider all columns as $(K_m\backslash E(K_3)) \oplus K_3$ except the columns 1, 3, 4 and 7 and consider these columns as $(K_m\backslash (E(K_3)) \oplus E(2kC_6)) \oplus K_3 \oplus 2kC_6$ and all rows can be viewed as $(K_n\backslash E(C_7)) \oplus C_7$ except the last three rows. The last three rows can be viewed as $(K_{12l+1}\backslash E(2lC_6)) \oplus K_{12l,10} \oplus 2lC_6 \oplus K_{11}$. In each column $K_m\backslash E(K_3)$ has a $(6;p,q)$-decomposition and in columns 1, 3, 4 and 7 the graph $(K_m\backslash (E(K_3)) \oplus E(2kC_6))$ has a $(6;p,q)$-decomposition, by Lemma 2.6. So the remaining edges in columns 1, 3, 4 and 7 form $K_3 \oplus 2kC_6$ and in other columns form K_3. By Lemma 2.7 and Theorem 1.1, the graphs $K_n\backslash E(C_7)$ and K_{12l+1} have a $(6;p,q)$-decomposition and $K_{12l,10} = 2l(K_{6,6} \oplus K_{6,4})$ has a $(6;p,q)$-decomposition, by Theorem 1.2 and Lemma 2.4. So the remaining edges in the first $(m-3)$ rows form $12kC_7$ and in the last three rows form $K_{11} \oplus 2lC_6$. The graph $12lK_3$ in the first $12l$ columns along with $2lC_6$ in the last three rows have a $(6;p,q)$-decomposition as in Lemma 3.12. Also, the edges of $12kC_7$ along with four columns of $2kC_6$ can have a $(6;p,q)$-decomposition as in Lemma 3.15.

Now, the remaining edges (K_3's) in the last 11 columns and (K_{11}'s) in the last 3 rows will form $K_3 \square K_{11}$ which has a $(6;p,q)$-decomposition, by Lemma 3.4. Hence by Remark 1.3 $K_m \square K_n$ has a $(6;p,q)$-decomposition.

Case 6. $m \equiv 5 \pmod{12}$, $n \equiv 9 \pmod{12}$.

Let $m = 12k+5,\ n = 12l+9$. We can write $K_m \square K_n = nK_m \oplus mK_n = n((K_m \backslash E(C_4)) \oplus C_4) \oplus mK_n$. Consider the first 5 rows and the last 2 rows as $K_{12l+1} \oplus K_9 \oplus K_{12,8}$ and $(K_{12l+1} \backslash E(2lC_6)) \oplus 2lC_6 \oplus K_9 \oplus K_{12,8}$ respectively. The graph $(n-9)C_4$ in the first $n-9$ columns along with the last 2 rows of $2lC_6$ can be viewed as $2lG$, where $G = 6C_4 \oplus 2C_6 = C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^6 \oplus C_6^3 \oplus C_6^4$ with $V(G) = \{x_{i,j} | 1 \le i \le 4, \le j \le 2\}$ and $C_4^i = (x_{1,i}x_{2,i}x_{3,i}x_{4,i}x_{1,i}),\ 1 \le i \le 6,\ C_6^j = (x_{j,1}x_{j,2}x_{j,3}x_{j,4}x_{j,5}x_{j,6}x_{j,1}),\ j = 3,4$ and G can be decomposed into C_6's as follows:

$$\left\{(x_{3,2i}\mathbf{x_{3,(2i-1)}x_{2,(2i-1)}x_{1,(2i-1)}x_{4,(2i-1)}x_{4,2i}x_{3,2i}}),\ (\mathbf{x_{3,2i}x_{2,2i}}x_{1,2i}x_{4,2i}x_{4,(2i+1)}x_{3,(2i+1)}x_{3,2i})\right\},$$

where $1 \le i \le 6$ and the subscripts of x are taken modulo 6 with residues $\{1, \cdots, 6\}$. The first three cycles can be decomposed into $3P_7$ as follows: $\{x_{1,2}x_{4,2}x_{4,1}x_{1,1}x_{2,1}x_{3,1}x_{3,2},\ x_{3,2}x_{4,2}x_{4,3}x_{1,3}x_{2,3}x_{3,3}x_{3,4},\ x_{3,4}x_{4,4}x_{4,3}x_{3,3}x_{3,2}x_{2,2}x_{1,2}\}$. Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition of G given above. By Theorem 1.1, Lemmas 2.3, 2.4 and 2.5, K_n, $K_{12l+1} \backslash E(2kC_6)$, K_9 and $K_{12,8}$ have a $(6;p,q)$-decomposition. Now, consider the remaining $9C_4$ with $5C_6$ from the first 5 rows in 5×9 block with vertex and edge set as follows:

$$V(G) = \{x_{i,j} | 1 \le i \le 5, \le j \le 6\}$$

and

$$C_4^i = (x_{1,i}x_{2,i}x_{3,i}x_{4,i}x_{1,i}), i = 3,4,8$$

and

$$\begin{aligned}
C_4^1 &= (x_{1,1}x_{2,1}x_{4,1}x_{3,1}x_{1,1}),\\
C_4^2 &= (x_{2,2}x_{3,2}x_{4,2}x_{5,2}x_{2,2}),\\
C_4^5 &= (x_{1,5}x_{3,5}x_{5,5}x_{4,5}x_{1,5}),\\
C_4^6 &= (x_{1,6}x_{4,6}x_{2,6}x_{3,6}x_{1,6}),\\
C_4^7 &= (x_{1,7}x_{3,7}x_{5,7}x_{2,7}x_{1,7}),\\
C_4^9 &= (x_{1,9}x_{4,9}x_{3,9}x_{5,9}x_{1,9}),\\
C_6^1 &= (x_{1,1}x_{1,3}x_{1,5}x_{1,7}x_{1,9}x_{1,8}x_{1,1}),\\
C_6^2 &= (x_{2,1}x_{2,2}x_{2,3}x_{2,6}x_{2,7}x_{2,8}x_{2,1}),\\
C_6^3 &= (x_{3,1}x_{3,3}x_{3,4}x_{3,6}x_{3,5}x_{3,2}x_{3,1}),\\
C_6^4 &= (x_{4,2}x_{4,3}x_{4,4}x_{4,7}x_{4,6}x_{4,9}x_{4,2}),\\
C_6^5 &= (x_{5,2}x_{5,3}x_{5,5}x_{5,7}x_{5,9}x_{5,4}x_{5,2}).
\end{aligned}$$

Now, this G can be decomposed into C_6's as follows:

$$\begin{aligned}
&\left\{(x_{1,8}\mathbf{x_{1,9}x_{5,9}x_{5,7}x_{2,7}x_{2,8}x_{1,8}}),\ (\mathbf{x_{1,8}x_{4,8}}x_{3,8}x_{2,8}x_{2,1}x_{1,1}x_{1,8})\right\},\\
&\left\{(x_{5,5}\mathbf{x_{4,5}x_{1,5}x_{1,7}x_{3,7}x_{5,7}x_{5,5}}),\ (\mathbf{x_{5,5}x_{5,3}}x_{5,2}x_{2,2}x_{3,2}x_{3,5}x_{5,5})\right\},\\
&\left\{(x_{3,3}\mathbf{x_{2,3}x_{2,2}x_{2,1}x_{4,1}x_{3,1}x_{3,3}}),\ (\mathbf{x_{3,3}x_{3,4}}x_{2,4}x_{1,4}x_{4,4}x_{4,3}x_{3,3})\right\},\\
&\left\{(x_{4,2}\mathbf{x_{3,2}x_{3,1}x_{1,1}x_{1,3}x_{4,3}x_{4,2}}),\ (\mathbf{x_{4,2}x_{5,2}}x_{5,4}x_{5,9}x_{3,9}x_{4,9}x_{4,2})\right\},\\
&\left\{(x_{4,6}\mathbf{x_{4,7}x_{4,4}x_{3,4}x_{3,6}x_{1,6}x_{4,6}}),\ (\mathbf{x_{4,6}x_{2,6}}x_{2,7}x_{1,7}x_{1,9}x_{4,9}x_{4,6})\right\},\\
&(x_{1,3}x_{2,3}x_{2,6}x_{3,6}x_{3,5}x_{1,5}x_{1,3}).
\end{aligned}$$

The last $3C_6$ can be decomposed into $3P_7$ as follows:

$$\{x_{1,3}x_{2,3}x_{2,6}x_{4,6}x_{4,9}x_{1,9}x_{1,7}, x_{1,7}x_{2,7}x_{2,6}x_{3,6}x_{1,6}x_{4,6}x_{4,7}, x_{4,7}x_{4,4}x_{3,4}x_{3,6}x_{3,5}x_{1,5}x_{1,3}\}.$$

Now, using Construction 1.4 we get the required number of paths and cycles from the C_6-decomposition of G given above. Hence we have the desired decomposition of $K_m \square K_n$.

Note 3.19. From Case 7 to Case 10 the degree of each vertex $v \in V(K_m \square K_n)$ is odd and so $p \geq mn/2$.

Case 7. $m \equiv 0 \pmod{12}$, $n \equiv i \pmod{12}$, $i = 3, 5, 7, 11$.

Let $m = 12k$ and $n = 12l + i$, $l, k \in Z^+$ and $i \in \{3, 5, 7, 11\}$. We can write $K_m \square K_n = nK_m \oplus mK_n = (n - i)K_m \oplus k(K_i \square K_{12}) \oplus i\frac{k(k-1)}{2}K_{12,12} \oplus m(K_n \backslash E(K_i))$, $i \in \{3, 5, 7, 11\}$. By Lemma 2.6, $K_n \backslash E(K_i)$ has a $(6; p, q)$-decomposition for $i = 3$. For $i \in \{5, 7, 11\}$, $K_n \backslash E(K_i)$ can be viewed as $K_{12l+1} \oplus K_{12l,i-1}$ and these graphs have a $(6; p, q)$-decomposition, by Theorems 1.1, 1.2 and Lemma 2.4. Also by Theorem 1.2 and Lemmas 3.12 to 3.15, $K_{12,12}$ and $K_i \square K_{12}$, $i \in \{3, 5, 7, 11\}$ have a $(6; p, q)$-decomposition. Hence by Remark 1.3, $K_m \square K_n$ has a $(6; p, q)$-decomposition.

Case 8. $m \equiv 4 \pmod{12}$, $n \equiv 3$ *or* $7 \pmod{12}$.

Let $m = 12k + 4$. Then $K_m \square K_n = nK_m \oplus mK_n = nK_m \oplus m((K_n \backslash E(K_3)) \oplus K_3)$. By Lemmas 2.6 and 2.8, K_m has a $(6; p, q)$-decomposition with $p \geq m/2$ and $K_n \backslash E(K_3)$ has a $(6; p, q)$-decomposition. Now, the last three columns can be viewed as $(K_{12(k-1)} \backslash E(2(k-1)C_6)) \oplus 2(k-1)C_6 \oplus K_{16} \oplus K_{12(k-1),16}$. By Lemmas 2.5 and 2.4, $K_{12(k-1)} \backslash E(2(k-1)C_6)$ and $K_{12(k-1),16}$ have a $(6; p, q)$-decomposition. The graph $12(k-1)K_3$ in the first $12(k-1)$ rows along with the last 3 columns of $2(k-1)C_6$ can be viewed as $2(k-1)(6K_3 \oplus 3C_6)$. We can prove this has a $(6; p, q)$-decomposition as in Lemma 3.12. Now, K_{16}'s of last 3 columns and K_3's of last 16 rows form $K_3 \square K_{16}$ and this has a $(6; p, q)$-decomposition, by Lemma 3.5.

Case 9. $m \equiv 8 \pmod{12}$, $n \equiv 3 \pmod{12}$.

Let $m = 12k + 8$ and $n = 12l + 3$. We can write $K_m \square K_n = nK_m \oplus mK_n = n((K_{12k} \oplus 2C_6) \oplus (K_8 \backslash E(2C_6)) \oplus K_{12k,8}) \oplus m((K_n \backslash E(K_3)) \oplus K_3)$, where $2C_6$ are $(x_{2,i}x_{3,i}x_{7,i}x_{4,i}x_{6,i}x_{8,i}x_{2,i})$, $(x_{2,i}x_{5,i}x_{4,i}x_{8,i}x_{1,i}x_{6,i}x_{2,i})$, $1 \leq i \leq n$. Last three columns can be viewed as $(K_{12k} \backslash E(2kC_6)) \oplus 2kC_6 \oplus K_8 \oplus K_{12k,8}$ and first three rows can be viewed as $(K_m \backslash E(2lC_6 \oplus K_3)) \oplus 2lC_6 \oplus K_3$. The graph $12kK_3$ in the last $12k$ rows along with the last 3 columns of $2kC_6$ can be viewed as $2k(6K_3 \oplus 3C_6)$. We can prove this has a $(6; p, q)$-decomposition as in Lemma 3.12. Now, K_8's in last three columns and K_3's in the first 8 rows forms $K_8 \square K_3$ and by Lemma 3.3, which has a $(6; p, q)$-decomposition. Also by Lemma 2.8, $K_{12k} \oplus 2C_6$ in the first $(n - 3)$ columns has a $(6; p, q)$-decomposition. The remaining edges $K_8 \backslash E(2C_6)$ in the first $12l$ columns and $2lC_6$ in first 3 rows form $(K_8 \backslash E(2C_6)) \square C_6$ which has a $(6; p, q)$-decomposition, by Lemma 3.16.

Case 10. $m \equiv 8 \pmod{12}$, $n \equiv 9 \pmod{12}$.

Let $m = 12k + 8$ and $n = 12l + 9$. We can write $K_m \square K_n = nK_m \oplus mK_n = (n - 9)((K_{12k} \oplus 2C_6) \oplus (K_8 \backslash E(2C_6)) \oplus K_{12k,8}) \oplus 9(K_{12k} \oplus K_8 \oplus K_{12k,8}) \oplus mK_n$. The last 8 rows can be viewed as $K_{12l+1} \backslash E(2lC_6) \oplus 2lC_6 \oplus K_9 \oplus K_{12l,8}$. Now, the graph $K_8 \backslash E(2C_6)$ in each $(n - 9)$ columns along with $2lC_6$ in last 8 rows forms $2l((K_8 \backslash E(2C_6)) \square C_6)$ which has a $(6; p, q)$-decomposition, by Lemma 3.16 and the graph K_8's in last 9 columns and K_9's in last 8 rows will form $K_8 \square K_9$ which has a $(6; p, q)$-decomposition, by Lemma 3.17. By Lemmas 2.4, 2.5 and 2.8, the remaining edges have a $(6; p, q)$decomposition.

□

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