Generalized domination structure in cubic graphs

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ABSTRACT

The minimum dominating set problem asks for a dominating set with minimum size. First, we determine some vertices contained in the minimum dominating set of a graph. By applying a particular scheme, we ensure that the resulting graph is 2-connected and the length of each formed induced cycle is 0 mod 3. We label every three vertices in the induced cycles of length 0 mod 3. Then there is a way of labeling in which the set of all labeled vertices is the minimum dominating set of the resulting graph, and is contained in the minimum dominating set of the original graph. We also consider the remaining vertices of the minimum dominating set of the original graph and determine all vertices contained in the minimum dominating set of a graph with maximum degree 3. The complexity of the minimum dominating set problem for cubic graphs was shown to be APX-complete in 2000 and this problem is solved by our arguments in polynomial time.

Keywords: minimum dominating set, cubic graph, NP-complete

1. Introduction

This paper takes the minimum dominating set problem and aims to determine the minimum dominating set of a cubic graph. A cubic graph is a type of graph where every vertex has degree 3. A dominating set of a graph G is a set S of vertices of G such that every vertex v of G is either in S or adjacent to a vertex of S. A minimum dominating set is a dominating set with minimum size.

Proposition 1.1. [2] *A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.*

First, we determine some vertices contained in the minimum dominating set of a graph. By applying a particular scheme, we ensure that the resulting graph is 2-connected and the length of

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each formed induced cycle is 0 mod 3. The scheme has been relaxed from the published paper [3]. We label every three vertices in the induced cycles of length 0 mod 3. Then there is a way of labeling in which the set of all labeled vertices is the minimum dominating set of the resulting graph, and is contained in the minimum dominating set of the original graph. We also consider the remaining vertices of the minimum dominating set of the original graph and determine all vertices contained in the minimum dominating set of a graph with maximum degree 3.

The complexity of the minimum dominating set problem for cubic graphs was shown to be APX-complete [1] and this problem is solved by our arguments in polynomial time, which implies that all NP-complete problems are contained in P, but does not claim that P = NP in general.

Figure 1 shows an example of the minimum dominating set of a graph with maximum degree 3 derived by the method in this paper.
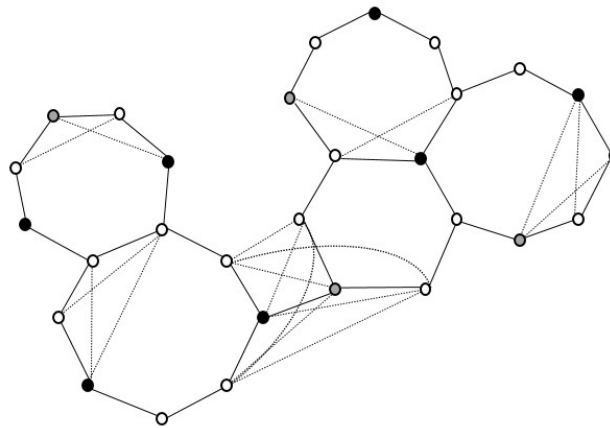
Fig. 1. A graph with maximum degree 3. The added edges are shown with dashed lines. The set of black or grey vertices is the minimum dominating set of the graph

2. Notation

In this paper, a graph G is finite, undirected, and simple with the vertex set V and edge set E. We follow the notations presented in [2]. For a vertex $v \in V(G)$, the open neighborhood, denoted by $N_G(v)$, is $\{u \in V(G): uv \in E(G)\}$, also for a set $W \subseteq V(G)$, let $N_G(W) = \bigcup_{v \in W} N_G(v)$ and $N_G[W] = N_G(W) \cup W$. A *dominating set* $X \subseteq V(G)$ is such that $N_G[X] = V(G)$. A minimum dominating set, called a *d-set*, is a dominating set with the minimum size. *X-3-path* is a path that has *labels* on the vertices so that the distance between the labels is 3 if the length of the path is at least 2, otherwise X-3-path is a path of length 1 that has at most one label on the vertices. Two cycles C_1 and C_2 are said to be *connecting without seams* if X-3-paths can be assigned to C_1 and C_2 as $C_1 \cap C_2$ is one X-3-path.

3. Main results

We consider a connected graph G, otherwise consider each component one by one. We introduce the construction scheme **K** as follows.

K: Input a connected graph G.

(1) Let $G_0 = G$ and $k = 0$.

(2) If there exists a cut vertex in G_k, then proceed as follows. Let v be the cut vertex in G_k. For every pair of components C_1 and C_2 of $G_k - v$ and for every pair of vertices $v_1 \in C_1 \cap N_{G_k}(v)$ and $v_2 \in C_2 \cap N_{G_k}(v)$, add an edge $v_1 v_2$ to G_k. Increase k by 1.

(3) Repeat (2) while they occur.

(4) Find an induced cycle in G_k.

(4-1) If it is an induced cycle of length 0 mod 3, assign X-3-path to it. Increase k by 1.

(4-2) If it is an induced cycle of length 2 mod 3, then proceed as follows. Let D_2 be the induced cycle of length 2 mod 3. Take a 4-path $Q \subseteq D_2$, and set $Q = w_1 w_2 w_3 w_4$. Add three edges $w_1 w_3$, $w_2 w_4$, and $w_1 w_4$ to G_k. Assign X-3-paths to the induced cycles connecting without seams in $D_2 + w_1 w_3 + w_2 w_4 + w_1 w_4$ (say the subgraph (a)). Increase k by 1.

(4-3) If it is an induced cycle of length 1 mod 3, then proceed as follows. Let D_1 be the induced cycle of length 1 mod 3. Take a 4-path $Q \subseteq D_1$, and set $Q = w_1 w_2 w_3 w_4$. Add two edges $w_1 w_3$ and $w_2 w_4$ to G_k. Assign X-3-paths to the induced cycles connecting without seams in $D_1 + w_1 w_3 + w_2 w_4$ (say the subgraph (b)). Increase k by 1.

(5) Find a shortest path P between two vertices v_1 and v_2 on the previously assigned X-3-path in G_k where X-3-path is not assigned to $\mathring{v}_1 P \mathring{v}_2$. Let P' be the previously assigned X-3-path with v_1 and v_2 in G_k.

(5-1) If it forms an induced cycle of length 0 mod 3 with $v_1 P' v_2$ in G_k, then proceed as follows. Let D_0 be the induced cycle of length 0 mod 3. Assign X-3-path to D_0. Increase k by 1.

(5-2) If it forms an induced cycle of length 2 mod 3 with $v_1 P' v_2$ in G_k, then proceed as follows. Let D_2 be the induced cycle of length 2 mod 3. Take a 4-path $Q \subseteq D_2$, and set $Q = w_1 w_2 w_3 w_4$. Add three edges $w_1 w_3$, $w_2 w_4$, and $w_1 w_4$ to G_k. Assign X-3-paths to the induced cycles connecting without seams in $D_2 + w_1 w_3 + w_2 w_4 + w_1 w_4$ (say the subgraph (a)). Increase k by 1.

(5-3) If it forms an induced cycle of length 1 mod 3 with $v_1 P' v_2$ in G_k, then proceed as follows. Let D_1 be the induced cycle of length 1 mod 3. Take a 4-path $Q \subseteq D_1$, and set $Q = w_1 w_2 w_3 w_4$. Add two edges $w_1 w_3$ and $w_2 w_4$ to G_k. Assign X-3-paths to the induced cycles connecting without seams in $D_1 + w_1 w_3 + w_2 w_4$ (say the subgraph (b)). Increase k by 1.

(6) Repeat (5) while they occur.

(7) Return the resulting graph G_k.

Let $\mathbf{K}(G)$ be a graph constructed by applying $\mathbf{K}$ to G. Note that $\mathbf{K}(G)$ is not unique and constructed from G arbitrarily. By the scheme, $\mathbf{K}(G)$ is 2-connected with $V(G) = V(\mathbf{K}(G))$ and $E(G) \subseteq E(\mathbf{K}(G))$, and all edges in $\mathbf{K}(G)$ should be covered by at least one set of X-3-paths but may be covered by other set of X-3-paths with the rotation of labels.

Remark 3.1. Let X be a dominating set of G. Every subset $D \subseteq X$ is a d-set of $G[N_G[D]]$ if and only if X is a d-set of G.

Consider labeling vertices by assigning X-3-paths to the induced cycles of length 0 mod 3 connecting without seams in $\mathbf{K}(G)$ (Figure 2). Note that certain induced cycles are not counted for the labeling, and have as few labeled vertices as possible.

Proposition 3.2. (i) For every labeling, the set of all labeled vertices is a minimal dominating set of $\mathbf{K}(G)$. In addition, by selecting the first vertex to label, the number of the remaining labeled vertices

is uniquely determined. (ii) For at least one labeling, the set of all labeled vertices is a d-set of $\mathbf{K}(G)$.
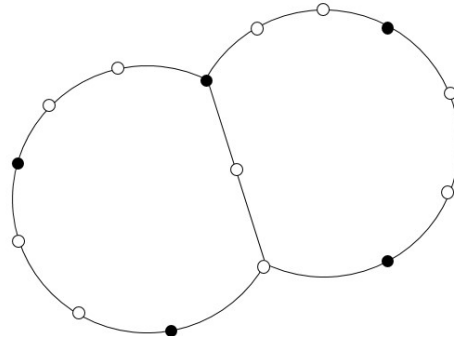

Fig. 2. Labeling vertices on 0 mod 3 cycles connecting without seams

Proof. Since every edge in $\mathbf{K}(G)$ is covered by X-3-path, for every labeling, the set of all labeled vertices is an independent dominating set of $\mathbf{K}(G)$, that is, a minimal dominating set of $\mathbf{K}(G)$. After selecting the vertex to label first, if there are multiple ways to label vertices during the labeling process, those vertices are interchangeable and equivalent. Indeed, those vertices are two adjacent vertices with degree 3 in $K^4 - e \subseteq \mathbf{K}(G)$ for some $e \in E(K^4)$. Thus, the number of the remaining labeled vertices is uniquely determined. Therefore, the statement (i) holds. By Remark 3.1, the statement (ii) obviously holds.   □

Let Y be a d-set of $\mathbf{K}(G)$ that is obtained by labeling vertices. Let $\mathscr{Y}$ be the set of all Y. For $Y_1, Y_2 \in \mathscr{Y}$, if $Y_1 \cap Y_2 \neq \emptyset$ and $Y_1 \setminus Y_2$ has a cut vertex in G, then let $\mathscr{Y}$ be $\mathscr{Y} \setminus \{Y_2\}$. Repeat this operation while they occur and let $\mathscr{A} = \mathscr{Y}$. By the definition of $\mathbf{K}(G)$, a cut vertex in G should be labeled rather than its adjacent vertex as we see in Lemma 3.4. For $A_1, A_2 \in \mathscr{A}$, if $A_1 \cap A_2 \neq \emptyset$, then let A_1 and A_2 be equivalent and denote $A_1 \equiv A_2$.

Lemma 3.3. $\mathscr{A}/\equiv$ *is determined in polynomial time and* $|\mathscr{A}/\equiv| \leq |V(G)|$.

Proof. By the definition of $\mathbf{K}$, $\mathbf{K}(G)$ is constructed from G in polynomial time. By the proof of Proposition 3.2 and the definition of $\mathscr{A}/\equiv$, $\mathscr{A}/\equiv$ is determined in polynomial time and $|\mathscr{A}/\equiv| \leq |V(G)|$.   □

Let X be a d-set of G. Let $\mathscr{X}$ be the set of all X.

Lemma 3.4. *For some* $\mathscr{B} \in \mathscr{A}/\equiv$ *and for each* $Y \in \mathscr{B}$, *there exists* $X \in \mathscr{X}$ *such that* $Y \subseteq X$.

Proof. Let $\mathscr{B} \in \mathscr{A}/\equiv$. Consider labeling vertices in $G \subseteq \mathbf{K}(G)$. By the definition of $\mathbf{K}(G)$ and $\mathscr{B}$, it suffices that considering the subgraph (a) or (b). For any $Y_1, Y_2 \in \mathscr{B}$ such that $Y_1 \neq Y_2$, $N_{\mathbf{K}(G)}[Y_1 \setminus Y_2] = N_{\mathbf{K}(G)}[Y_2 \setminus Y_1]$. If Y_1 is a dominating set of G, then for some $X \in \mathscr{X}$, $X = Y_1$. Now, $N_G[Y_2 \setminus Y_1] \setminus N_G[Y_1 \setminus Y_2] \subseteq N_G[Y_1] = V(G)$. By considering the subgraph (a), we have $N_G[Y_1 \setminus Y_2] \setminus N_G[Y_2 \setminus Y_1] \subseteq N_G[Y_2]$. Hence, $N_G[Y_1] = N_G[Y_2]$. Suppose that for all $Y \in \mathscr{A}$, Y is not a dominating set of G. Now, each Y_1 and Y_2 is not a dominating set of G. By considering the subgraph (b), we have $N_G[Y_2 \setminus Y_1] \setminus N_G[Y_1 \setminus Y_2] \not\subseteq N_G[Y_1]$ and $N_G[Y_1 \setminus Y_2] \setminus N_G[Y_2 \setminus Y_1] \not\subseteq N_G[Y_2]$. Hence, $N_G[Y_1] \setminus N_G[Y_2] \neq \emptyset$ and $N_G[Y_2] \setminus N_G[Y_1] \neq \emptyset$. Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{A}/\equiv$ with $\mathscr{B}_1 \neq \mathscr{B}_2$. For any

$Y_3 \in \mathscr{B}_1$ and any $Y_4 \in \mathscr{B}_2$, since $Y_3 \cap Y_4 = \emptyset$ and each Y_3 and Y_4 is not a dominating set of G, it follows that $N_G[Y_3] \setminus N_G[Y_4] \neq \emptyset$ and $N_G[Y_4] \setminus N_G[Y_3] \neq \emptyset$. Now, Y is a d-set of $G[N_G[Y]]$. Let W be a subset of $V(G) \setminus Y$ with minimum size such that $Y \cup W$ is a dominating set of G. By Remark 3.1, $Y \cup W$ is a d-set of G. $\qquad \square$

Suppose that Y is a d-set of $\mathbf{K}(G)$ that satisfies Lemma 3.4.

Lemma 3.5. *Let G' be a graph obtained by deleting Y from, and adding edges to all vertex pairs of $\bigcup_{y \in Y} N_G(y)$ to G. Let Z_1 be a d-set of G'. Let $G'' = G - N_G[Y]$ and Z_2 be a d-set of G''. If $|Z_1| < |Z_2|$, then $Y \cup Z_1$ is a d-set of G, and $Z_1 \cap \bigcup_{y \in Y} N_G(y) \neq \emptyset$. If $|Z_1| \geq |Z_2|$, then $Y \cup Z_2$ is a d-set of G.*

Proof. Obviously, $Y \cup Z_2$ is a d-set of G if $|Z_1| \geq |Z_2|$. For a set $A \subseteq V(G)$, suppose that $Y \cup A$ is a d-set of G. Let $A' = A \setminus N_G[Y]$. Suppose that $|Z_1| < |Z_2|$. Now, A' is not a dominating set of G'', and $A \cap \bigcup_{y \in Y} N_G(y) \neq \emptyset$. By the definition of G', it suffices that $A = Z_1$. $\qquad \square$

Theorem 3.6. *The d-set of a graph G with maximum degree 3 is determined in polynomial time.*

Proof. By Lemma 3.4, for some $X \in \mathscr{X}$, $Y \subseteq X$. Let G be a graph with maximum degree 3 and let $G_0 = G$. Let G_1 be constructed by deleting Y, and for every pair $w_1, w_2 \in \bigcup_{y \in Y} N_{G_0}(y)$, adding an edge $w_1 w_2$ to G_0. Let $G_2 = G_0 - N_{G_0}[Y]$. Let W_1 be a d-set of G_1 and W_2 be a d-set of G_2. Since G_0 is a graph with maximum degree 3 and by the definition of G_2, each component of G_2 is a path or cycle, or $G_2 = \emptyset$. Indeed, each component of G_2 does not have a vertex of degree 3 in its subgraph. Thus W_2 is determined in polynomial time. Suppose that $|W_1| < |W_2|$. Let Y_1 be a d-set of $\mathbf{K}(G_1)$ that satisfies Lemma 3.4. By the definition of Y_1, it suffices that $Y_1 \cap \bigcup_{y \in Y} N_{G_0}(y) \neq \emptyset$. Let G_3 be constructed by deleting Y_1, and for every pair $w_1, w_2 \in \bigcup_{y \in Y_1} N_{G_1}(y)$, adding an edge $w_1 w_2$ to G_1. Let $G_4 = G_1 - N_{G_1}[Y_1]$. Let W_3 be a d-set of G_3 and W_4 be a d-set of G_4. By the definition of G_4, each component of G_4 is a path, or $G_4 = \emptyset$. Thus W_4 is determined in polynomial time. Suppose that $|W_3| < |W_4|$. Let Y_2 be a d-set of $\mathbf{K}(G_3)$ that satisfies Lemma 3.4. By the definition of Y_2, it suffices that $Y_2 \cap \bigcup_{y \in Y_1} N_{G_1}(y) \neq \emptyset$. Let G_5 be constructed by deleting Y_2, and for every pair $w_1, w_2 \in \bigcup_{y \in Y_2} N_{G_3}(y)$, adding an edge $w_1 w_2$ to G_3. Let $G_6 = G_3 - N_{G_3}[Y_2]$. Let W_5 be a d-set of G_5 and W_6 be a d-set of G_6. By the definition of G_6, $V(G_6)$ is independent, or $G_6 = \emptyset$. Thus $W_6 = V(G_6)$. Suppose that $|W_5| < |W_6|$. Let Y_3 be a d-set of $\mathbf{K}(G_5)$ that satisfies Lemma 3.4. By the definition of Y_3, it suffices that $Y_3 \cap \bigcup_{y \in Y_2} N_{G_3}(y) \neq \emptyset$. Now, Y_3 is a dominating set of G_5 and so it suffices that $W_5 = Y_3$. By Lemma 3.3, Y_3 is determined in polynomial time. By Lemma 3.5, if $|W_5| < |W_6|$, then it suffices that $W_3 = Y_2 \cup W_5$, otherwise, it suffices that $W_3 = Y_2 \cup W_6$. By Lemma 3.3, Y_2 is determined in polynomial time. By Lemma 3.5, if $|W_3| < |W_4|$, then it suffices that $W_1 = Y_1 \cup W_3$, otherwise, it suffices that $W_1 = Y_1 \cup W_4$. By Lemma 3.3, Y_1 is determined in polynomial time. By Lemma 3.5, if $|W_1| < |W_2|$, then it suffices that $X = Y \cup W_1$, otherwise, it suffices that $X = Y \cup W_2$. By Lemma 3.3, Y is determined in polynomial time. Thus, X is determined in polynomial time. The proof is completed. $\qquad \square$

References

[1]   P. Alimonti and V. Kann. Some apx-completeness results for cubic graphs. *Theoretical Computer Science*, 237:123–134, 2000. https://doi.org/10.1016/S0304-3975(98)00158-3.

[2]   R. Diestel. *Graph Theory Fourth Edition*. Springer, 2010.

[3]   M. Nakanishi. A note on vertices contained in the minimum dominating set of a graph with minimum degree three. *Theoretical Computer Science*, 956:113831, 2023. https://doi.org/10.1016/j.tcs.2023.113831.