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On the theorems of Frobenius, Zorn and Hopf's commutativity

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ABSTRACT

Let A be a real algebra. It is called *locally complex algebra* if every non-zero element generates a subalgebra isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. It is said to satisfy the *uniqueness of the square root except the sign* if the equation $x^2 = y^2$ implies $y = \pm x$. We show the following:
1. Every locally complex algebra is a quadratic algebra.
2. Every alternative locally complex algebra is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.
3. Every commutative locally complex algebra without divisors of zero is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.
4. Every finite-dimensional algebra satisfying the uniqueness of the square root except the sign has dimension ≤ 2 and contains non-zero idempotents.

Keywords: quadratic (alternative, flexible, locally complex) algebra, divisors of zero, Cayley-Dickson process, quaternions, octonions, sedenions, nearly absolute-valued algebra

1. Introduction

In this present paper, all algebras are considered over the field $\mathbb{R}$ of real numbers. An algebra is a vector space A endowed with a bilinear product $A \times A \to A \ \ (x, y) \mapsto xy$. A finite-dimensional algebra A is said to be a (real) *division algebra* if it is $\neq \{0\}$ and contains no divisors of zero. The study of division algebras, began with the discovery of the quaternion algebra $\mathbb{H}$ in 1843 and the octonion algebra $\mathbb{O}$ in 1843 and, independently, 1845.

An algebra A is said to be *algebraic* (resp. *quadratic*) if every element in A generates a finite-

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dimensional subalgebra (resp. if A contains an unit element e and every element $x \in A$ satisfies a quadratic equation $x^2 = \alpha e + \beta x$ with $\alpha, \beta \in \mathbb{R}$).

Frobenius' associative theorem [11, 10] states that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only associative, quadratic algebras with no divisors of zero. Zorn's alternative theorem [22, 10] states that $\mathbb{O}$ is the only alternative, quadratic algebra with no divisors of zero which is not associative. Combining two above results:

Theorem 1.1. (Frobenius-Zorn) *Every alternative quadratic algebra with no divisors of zero is isomorphic to either* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *or* $\mathbb{O}$.

Many researchers gave minimum additional conditions for an alternative algebra to be isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The Frobenius-Zorn theorem will then be either improved or extended.

A *normed algebra* is a space A endowed with a norm $||.||$ such that $||xy|| \leq ||x||\,||y||$ for all x, y. Let $\mathbb{S}(A) = \{x \in A : ||x|| = 1\}$ be the unit-sphere of the normed algebra $(A, ||.||)$. An element $a \in A \setminus \{0\}$ is said to be a *joint topological divisors of zero* if there exists a sequence $(x_n)_{n \geq 0}$ in $\mathbb{S}(A)$ such that

$$\lim_{n\to\infty} a x_n = \lim_{n\to\infty} x_n a = 0.$$

Let A be a normed algebra with no joint topological divisors of zero. Cabrera-Rodríguez showed that A is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ in the associative case [2, Theorem 2], and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ in the alternative case [2, Theorem 3], [3, Theorem 2.5.50].

A nonzero element a of an algebra A is said to be a *joint divisor of zero* if there exists $b \in A \setminus \{0\}$ such that $ab = ba = 0$. Cabrera-Rodríguez showed that every power-associative (resp. alternative) algebraic algebra with no nonzero joint divisor of zero is quadratic [3, Proposition 2.5.10] (resp. isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ [3, Theorem 2.5.29]).

Diouf-Traoré proved that every weakly alternative algebraic algebra with no nonzero joint divisor of zero is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ [9, Corollary 3]

Cuenca proves that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras satisfying the middle Moufang identity [4, Theorem 2.3].

Brešar-Šemrl-Špenko [1] were the first to have introduced the notion of locally complex algebra, this being in the unitary case. They showed that every unital associative (resp. alternative) locally complexe algebra is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (resp. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$) [1, Theorem 4.7].

Hopf's commutativity theorem has been a blessing within the theory of finite-dimensional division algebras. It states that the dimension of every finite-dimensional commutative division algebra is ≤ 2 [12]. The proof uses a topological method. Attracted by this beautiful result, researchers have added additional minimum conditions to an algebra to force its dimension to be ≤ 2.

Using algebraic geometry arguments, Springer [20] showed that $\mathbb{R}$ and $\mathbb{C}$ are the unique finite-dimensional unital commutative division algebra.

Cabrera-Rodríguez [3, Corollary 2.5.16] showed that every commutative quadratic algebra with no divisors of zero is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. They used purely algebraic tools.

Yang [21] proved, via a differential geometry method, that every unital division algebra of finite dimension ≥ 2 contains a subalgebra isomorphic to $\mathbb{C}$. The same result was obtained by Petro [16] who used algebraic topology arguments accompanied by the intermediate value theorem.

Recently, Cuenca-Miguel used topological methods to extend the Hopf's commutativity theorem. They showed the following two results:

Every commutative nearly absolute valued division algebra has finite dimension ≤ 2 [6, Theorem

2.4].

For every commutative nearly absolute-valued algebra, the following statements are equivalent [5]:

1. A satisfies the uniqueness of the square roots except the sign.
2. A is a 2-dimensional division algebra.

Motivated by these results, we were interested in the study of minimum additional conditions in an algebra to be finite-dimensional and possibly isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

We show that every locally complex algebra contains an unit-element and is a quadratic algebra (Theorem 3.3). A locally complex algebra is alternative if and only if it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (Theorem 4.3). This refines both Theorem 1.1 of Frobenius-Zorn, [3, Theorem 2.5.29], and [1, Theorem 4.7].

By easy and short algebraic method, we prove that every algebra A satisfying the uniqueness of the square roots except the sign does not contain nilpotent elements of index 2. If, in addition, A has finite-dimension n then $n \leq 2$ and A contains non-zero idempotents (Theorem 5.1). This extends the commutativity theorem of Hopf.

We show that a commutative algebra satisfies the uniqueness of the square roots except the sign if and only if it contains no divisors of zero (Theorem 5.2). We give a description (Theorem 5.3) of all finite-dimensional algebras satisfying the uniqueness of the square roots except the sign. This provides some examples (Examples 5.4) of two-dimensional algebras with the uniqueness of the square root except the sign, which are not commutative nor division algebras.

At the end we show that every commutative locally complex algebra without divisors of zero is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ (Theorem 5.5). This extends [20], [21] and [16].

Note that all algebras obtained from $\mathbb{R}$ by the Cayley-Dickson process: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{S}$ (the sedenions), ... etc are flexible algebras [17, Theorem 1]. The sedenions are examples of finite-dimensional quadratic, flexible locally complex algebras which are not alternative.

2. Notations and preliminary results

An algebra is understood to be a (real) vector space A endowed with a bilinear product $A \times A \to A \quad (x, y) \mapsto xy$. Let a, b, c be in A. We denote by (a, b, c), $[a, b]$, $A(a_1, \ldots, a_m)$, L_a, R_a, respectively, the associator $(ab)c - a(bc)$ of a, b, c; the commutator $ab - ba$ of a, b; the subalgebra of A generated by $a_1, \ldots, a_m \in A$; the operator of multiplication by a on the left $x \mapsto ax$; and the operator of multiplication by a on the right $x \mapsto xa$.

Algebra A is said to be *alternative* (resp *flexible*) if it satisfies $(x, x, y) = (y, x, x) = 0$ (resp. $(x, y, x) = 0$) for all $x, y \in A$.

A is said to be *locally complex algebra* if $A(a)$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ for all $a \in A \setminus \{0\}$.

A is said to be a *normed algebra* if the space A is endowed with a norm $||.||$ such that $||xy|| \leq ||x||\,||y||$ for all $x, y \in A$. This is equivalent to say that the product of algebra A is continuous with respect to the norm $||.||$. Note then $\mathbb{S}(A)$ the unit-sphere $\{x \in A : ||x|| = 1\}$ of the normed algebra $(A, ||.||)$.

A is said to be a *nearly absolute-valued algebra* if it is endowed with a norm $||.||$ of algebra such that $||xy|| \geq \lambda ||x||\,||y||$ for some positive real number λ and all $x, y \in A$ [14].

Let A be an algebra, we denote by A^+ the algebra called *symmetrization of* A having underlying space A and product $x \diamond y = \frac{1}{2}(xy + yx)$.

Algebra A is said to satisfies the *uniqueness of the square roots except the sign* if the equation $x^2 = y^2$ in A implies $y = \pm x$. Such an algebra has no nilpotent elements of index 2. Note that the (real) division algebra $\mathbb{R}$ satisfies the above property.

A is said to be a (real) *division algebra* if it is finite-dimensional with $A \neq \{0\}$ and contains no divisors of zero.

An algebra A with unit element e is called a *quadratic algebra* if every element $x \in A$ satisfies a quadratic equation $x^2 = \alpha e + \beta x$ with $\alpha, \beta \in \mathbb{R}$ [15]. The set $Im(A) = \{x \in A : x^2 \in \mathbb{R}e$ and $x \notin \mathbb{R}e \backslash \{0\}\}$ of *purely imaginary* elements of A constitutes a linear subspace of A which is supplementary to $\mathbb{R}e$ ([8], [10]).

Remark 2.1. 1. Clearly, every commutative algebra with no divisors of zero satisfies the uniqueness of the square roots except the sign. Such an algebra can be infinite-dimensional and associative as the algebra $\mathbb{R}[X]$ of polynômials with one indeterminate with coefficients in $\mathbb{R}$.

2. Every finite-dimensional algebra A with no divisors of zero is nearly absolute-valued algebra. Indeed, it is well known that the product of A is continuous with respect to every norm $||.||$ of the space A. The minimum of this product is reached on the compact $\mathbb{S}(A) \times \mathbb{S}(A) := \mathbb{S}$ and there exists $(a, b) \in \mathbb{S}$ such that

$$\inf_{(x,y)\in\mathbb{S}} ||xy|| = ||ab||.$$

As A contains no divisors of zero we have $||ab|| := \lambda > 0$. Now, let x, y be arbitrary in $A \setminus \{0\}$, we have: $\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \in \mathbb{S}$. This implies that $\left|\left|\frac{x}{||x||} \frac{y}{||y||}\right|\right| \geq \lambda$ and then $||xy|| \geq \lambda ||x||\, ||y||$. So A is nearly absolute-valued algebra.

3. Locally complex algebras are quadratic algebras

In order to prove the main result of this section, we need the following two preliminary results where A is assumed to be a locally complex algebra containing two distinct nonzero idempotents e, f :

Lemma 3.1. $A(e-f) \cap A(e+f) = \mathbb{R}(e-f)^2$ *and there exists an idempotent* $g \in A \setminus \{0\}$ *such that*

$$(e-f)^2 \quad = \quad \alpha g, \;\; \alpha \in \mathbb{R} \setminus \{0\}. \tag{1}$$

In particular, $(e-f)^2, e-f, e+f$ *are linearly independent.*

Proof. $(e-f)^2 \in A(e-f) \setminus \{0\}$ because $A(e-f)$ contains no nilpotent elements of index two. Now, $A(e-f) \neq A(e+f)$ otherwise $A(e-f)$ would contain both

$$e = \frac{(e+f)+(e-f)}{2} \quad \text{and} \quad f = \frac{(e+f)-(e-f)}{2},$$

which is absurd. So

$$\dim\left(A(e-f) \cap A(e+f)\right) \leq 1.$$

On the other hand

$$(e-f)^2 = 2(e+f) - (e+f)^2 \in A(e-f) \cap A(e+f).$$

So $A(e-f) \cap A(e+f)$ has dimension one and we have

$$A(e-f) \cap A(e+f) = \mathbb{R}(e-f)^2 = \mathbb{R}g,$$

where g is a nonzero idempotent collinear to $(e-f)^2$. □

Lemma 3.2. *Let $u \in A \setminus \mathbb{R}g$ such that $u^2 \in \mathbb{R}g$ of the forme βg with $\beta \in \mathbb{R}$. Then $\beta < 0$. In particular the scalar α in equality 1 is negative.*

Proof. The multiplication table of algebra $A(u)$ with respect to the basis g, u is given by:

$A(u)$	g	u
g	g	u
u	u	βg

As $A(u)$ is isomorphic to $\mathbb{C}$ there exists $v = \lambda g + \mu u \in A(u)$ such that $v^2 = -g$. We have $v^2 = (\lambda^2 + \mu^2\beta)g + 2\lambda\mu u$. Now

$$v^2 = -g \quad \Leftrightarrow \quad \begin{cases} \lambda^2 + \mu^2\beta = -1, \\ \lambda\mu = 0, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \mu^2\beta = -1, \\ \lambda = 0, \quad \text{because } \mu \text{ cannot be equal to } 0. \end{cases}$$

So $\beta = -\frac{1}{\mu^2}$ is negative. □

Theorem 3.3. *Every locally complex algebra A contains an unit-element and is a quadratic algebra.*

Proof. Assume that A contains two distinct nonzero idempotents e, f. By Lemma 3.1 $e + f, g$ are linearly independent so $a = e + f - g \in A \setminus \mathbb{R}g$. We have:

$$\begin{aligned} a^2 &= (e+f)^2 - 2(e+f) + g \\ &= -(e-f)^2 + g \\ &= (1-\alpha)g. \end{aligned}$$

According to Lemma 3.2: $1 - \alpha < 0$, that is $\alpha > 1$. Absurd. Thus A contains only a nonzero idempotent which is its unit-element. □

4. Improvements for alternative algebras

Let's start with the following preliminary result in the associative case:

Lemma 4.1. *Let A be an associative locally complex algebra. Then A has no divisors of zero.*

Proof. Let 1 be the unit of the quadratic algebra A, let $a, b \in A \setminus \{0\}$ and let a^{-1} (resp b^{-1}) the inverse of a in algebra $A(a)$ (resp. the inverse of b in algebra $A(b)$). The equality $a^{-1}abb^{-1} = 1$ shows that $ab \neq 0$ and then A has no divisors of zero. □

Corollary 4.2. *Let A be a locally complex algebra. Then A is associative if and only if it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Theorem 4.3. *Let A be a locally complex algebra. Then A is alternative if and only if it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. Let's show the "only if" part. The subalgebra of A generated by any pair $a, b \in A$ is associative by Artin's theorem [18, Theorem 3.1]. Lemma 4.1 shows that A contains no divisors of zero. The "only if" part is then concluded by the Theorem 1.1 of Frobenius-Zorn. □

Corollary 4.4. *Let A be an algebra such that $A(x, y)$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ for all nonzero $x, y \in A$. Then A is an alternative algebra and isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. A is clearly locally complex algebra. It is also alternative algebra by Artin's theorem, and Theorem 4.3 concludes. □

Theorem 4.5. *Let A be a locally complex algebra satisfying any one of the following middle Moufang identities:*

$$(xy)(zx) = x((yz)x), \tag{2}$$
$$(xy)(zx) = (x(yz))x. \tag{3}$$

Then A is alternative and isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Proof. Let 1 be the unit of algebra A. By putting $z = 1$ in any one of equalities 2, 3 we get the flexible law: $(x, y, x) = 0$. Now, taking into account flexibility, the equality

$$((1 + z)y)(z(1 + z)) = (1 + z)(yz)(1 + z)$$

gives $yz^2 = (yz)z$. So A is right-alternative. It is also left-alternative and Theorem 4.3 concludes. □

5. Improvements for commutative algebras

Let's start with improvements via the uniqueness of the square root except the sign:

Theorem 5.1. *Let A be an algebra satisfying the uniqueness of the square root except the sign. Then A does not contain nilpotent elements of index 2. If, in addition, A has finite-dimension n then $n \leq 2$ and A contains non-zero idempotents.*

Proof. The first affirmation follows immediately from the fact that A satisfies the uniqueness of the square root except the sign. Now, we consider the algebra A^+ having product $x \diamond y = \frac{1}{2}(xy + yx)$. Let a, b be in A, we have:

$$\begin{aligned} a \diamond b = 0 \quad &\Rightarrow \quad ab + ba = 0, \\ &\Rightarrow \quad (a + b)^2 - (a - b)^2 = 0, \\ &\Rightarrow \quad a + b = \pm(a - b), \\ &\Rightarrow \quad a = 0 \text{ or } b = 0. \end{aligned}$$

So A^+ does not contain divisors of zero. If, in addition, A has finite-dimension n which is also the dimension of A^+ then $n \leq 2$ by the commutativity theorem of Hopf. Also A^+ contains non-zero idempotents [19], as well as A. □

Theorem 5.2. *Let A be a commutative algebra. Then A satisfies the uniqueness of the square roots except the sign if and only if it contains no divisors of zero.*

Proof. To prove the "only if" part, let x, y be A. Then

$$\begin{aligned} xy = 0 \quad &\Rightarrow \quad (x+y)^2 = (x-y)^2 \text{ because } A \text{ is commutative },\\ &\Rightarrow \quad x + y = \pm(x-y). \end{aligned}$$

So either $x = 0$ or $y = 0$.
The "if" part is given in the beginning of Remarks 2.1. □

Theorem 5.3. *Let A be an algebra of finite dimension n satisfying the uniqueness of the square roots except the sign. Then $n \leq 2$ and is isomorphic to either $\mathbb{R}$ or the algebra $\mathbb{R}^2(\alpha, \lambda, \beta, \mu)$ obtained by endowing the real vector space $\mathbb{R}^2$ with the product:*

$$(x, y) \odot (x', y') = (xx' + \alpha xy' + \lambda yx' - yy',\ \beta xy' + \mu yx'),$$

where $\alpha, \lambda, \beta, \mu$ are real numbers with $\beta + \mu \neq 0$.

Proof. Algebra A has no nilpotent elements of index 2, and contains a nonzero idempotent e [19] which is an idempotent for the commutative division algebra $A^+ := (A, \diamond)$. We have $n \leq 2$ by the commutativity theorem of Hopf, and we can assume that $n = 2$. There exists $u \in A$ such that $u^2 = u \diamond u = -e$ [7, Lemma 2.3], [13, Remark 1]. We get a basis $\{e, u\}$ of algebra A for which the multiplication is given by the table:

	e	u
e	e	$\alpha e + \beta u$
u	$\lambda e + \mu u$	$-e$

where $\alpha, \lambda, \beta, \mu$ are real numbers. We then define on the vector space $\mathbb{R}^2$ the product:

$$(x, y) \odot (x', y') = (xx' + \alpha xy' + \lambda yx' - yy',\ \beta xy' + \mu yx').$$

The mapping $\Phi : A \to (\mathbb{R}^2, \odot)$ $\ ae + bu \mapsto (a, b)$ is then an isomorphism of algebras. Indeed, for every $(x, y), (x', y') \in \mathbb{R}^2$:

$$\begin{aligned} (xe + yu)(x'e + y'u) &= xx'e + xy'(\alpha e + \beta u) + yx'(\lambda e + \mu u) - yy'e \\ &= (xx' + \alpha xy' + \lambda yx' - yy')e + (\beta xy' + \mu yx')u, \\ \Phi((xe + yu)(x'e + y'u)) &= (xx' + \alpha xy' + \lambda yx' - yy',\ \beta xy' + \mu yx') \\ &= (x, y) \odot (x', y') \\ &= \Phi(xe + yu) \odot \Phi(x'e + y'u). \end{aligned}$$

Assume now that $\beta+\mu=0$. We have: $0\neq(e+u)^2=\omega e$ where $\omega=\alpha+\lambda\neq 0$.

1. If $\omega>0$ then $(e+u)^2=(\sqrt{\omega}e)^2$. As A satisfies the uniqueness of the square roots except the sign, we have: $e+u=\pm\sqrt{\omega}e$, absurd.
2. If $\omega<0$ then $(e+u)^2=(\sqrt{-\omega}u)^2$. This gives $e+u=\pm\sqrt{-\omega}u$, absurd.

So $\beta+\mu\neq 0$. □

There are two-dimensional not commutative algebras with divisors of zero satisfying the uniqueness of the square roots except de sign:

Example 5.4. Theorem 4.3 shows that algebras $\mathbb{R}^2(0,0,1,0):=(\mathbb{R}^2,*)$, $\mathbb{R}^2(0,0,1,2):=(\mathbb{R}^2,\odot)$ satisfy the uniqueness of the square roots except the sign. However, they are not commutative. In addition,

1. $(\mathbb{R}^2,*)$ contains divisors of zero.
2. $(\mathbb{R}^2,\odot)$ is a division algebra.

Now let's move on to improvements via the locally complex property:

Theorem 5.5. *Let A be a commutative locally complex algebra without divisors of zero. Then A is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.*

Proof. A is quadratic algebra by Theorem 3.3, with unit e, and we can assume that $Im(A)\neq\{0\}$. Let u,v be in $Im(A)\setminus\{0\}$ which can be taken as $u^2=v^2=-e$. We have: $0=u^2-v^2=(u-v)(u+v)$. So $v=\pm u$ and then A has underlying vector space $\mathbb{R}e+\mathbb{R}u$ and is isomorphic to $\mathbb{C}$. □

Remark 5.6. A commutative locally complex algebra is not necessarily isomorphic to $\mathbb{R}$ or $\mathbb{C}$ as shown by [1, Example 4.3]. So the hypothesis of absence of divisors of zero is necessary in Theorem 5.5. Also the fact that algebras $\mathbb{H}$, $\mathbb{O}$ are locally complex algebra with no divisors of zero shows that the hypothesis of commutativity is necessary in Theorem 5.5.

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