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The p-smooth and p-golden partial fields

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ABSTRACT

Given a prime p, a *p-smooth* integer is an integer whose prime factors are all at most p. Let S_p be the multiplicative subgroup of $\mathbb{Q}$ generated by -1 and the p-smooth integers. Define the *p-smooth* partial field as $\mathbb{S}_p = (\mathbb{Q}, S_p)$. Let g be the golden ratio $(1+\sqrt{5})/2$. Let G_p to be the multiplicative subgroup of $\mathbb{R}$ generated by g, -1, and the p-smooth integers. Define the *p-golden* partial field as $\mathbb{G}_p = (\mathbb{R}, G_p)$. The partial field $\mathbb{S}_2$ is actually the well-known dyadic partial field and $\mathbb{S}_3$ has sometimes been called the Gersonides partial field. We calculate the fundamental elements of $\mathbb{S}_5$, $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_5$. Our proofs make use of the SageMath computational package.

Keywords: partial field, golden ratio, p-smooth integer, matroid representation, matroid orientation

1. Introduction

The introduction of this paper assumes that the reader is familiar with matroid theory and linear optimization; however, these subjects are only necessary for motivating our study. The remainder of the paper is readable for any mathematically literate person.

A *partial field* $\mathbb{P}$ is a pair (R, G) in which R is a unitary ring (in this paper R will also be assumed to be commutative) and a subgroup G of the multiplicative group of units of R for which $-1 \in G$. A matrix A is a $\mathbb{P}$-matrix when every non-zero sub-determinant of A is in G. One of the first widely studied examples of such matrices are *totally unimodular* matrices which are $\mathbb{U}_0$-matrices with $\mathbb{U}_0 = (\mathbb{Q}, \{+1, -1\})$. Consider the polyhedron $A\mathbf{x} \leq \mathbf{b}$ in which $\mathbf{b}$ is any integral vector. Hoffman and Kruskal [3] showed that every vertex of this polyhedron is integral for any $\mathbf{b}$ if and only if A is totally unimodular. This implies that linear optimization and integer optimization are equivalent precisely when A is totally unimodular. This is highly significant in that integer optimization is NP-hard in general while linear optimization is quasi-polynomial (and is widely conjectured to be polynomial).

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More recently studies concerning partial fields and $\mathbb{P}$-matrices have proliferated; in particular, within the field of matroid theory. Included among the many highly influential works are those of Tutte [15], Seymour [12], Lee [4, 5], Whittle [17, 18], and Pendavingh and Van Zwam [9, 10].

One thread of investigations concerns the relationship between the matroids which are both orientable and representable over some finite field. The *dyadic* partial field is $\mathbb{D} = (\mathbb{Q}, \{\pm 2^i : i \in \mathbb{Z}\})$ and the *golden-mean* partial field is $\mathbb{G} = \{\mathbb{R}, \{\pm g^i : i \in \mathbb{Z}\})$ in which g is the golden ratio $\frac{1+\sqrt{5}}{2}$. A matroid M is $\mathbb{P}$*-representable* when there is a $\mathbb{P}$-matrix A for which $M = M(A)$.

Theorem 1.1 (Bland and Las Vergnas [1] and Minty [8])**.** *If M is a $GF(2)$-representable (i.e., binary) matroid, then M is orientable if and only if M has a $\mathbb{U}_0$-representation.*

Theorem 1.2 (Lee and Scobee [6])**.** *If M is a $GF(3)$-representable matroid, then M is orientable if and only if M has a $\mathbb{D}$-representation.*

Theorem 1.3 (Robbins and Slilaty [11])**.** *If M is a $GF(4)$-representable matroid, then M has a consistently ordered orientation if and only if M has a $\mathbb{G}$-representation.*

While Theorems 1.1 and 1.2 tell us that the relatively simple partial fields $\mathbb{U}_0$ and $\mathbb{D}$ completely describe all orientations of $GF(2)$- and $GF(3)$-representable matroids, Theorem 1.3 reveals that $GF(4)$-representable matroids have enough complexity so that a simple partial field like $\mathbb{G}$ is not enough to describe all orientations. This leaves us with two obvious questions. One, how might we describe the remaining orientations of $GF(4)$-representable matroids? Two, what about orientations of $GF(q)$-representable matroids for $q > 4$?

Since $\mathbb{R}$-representable matroids are a proper subset of orientable matroids, partial fields $\mathbb{P} = (R, G)$ in which $R \le \mathbb{R}$ are naturally of importance in analyzing matroids which are both orientable and $GF(q)$-representable. In order to utilize any given partial field $\mathbb{P}$ (for example, to prove results of the above type) it is necessary to know what are the *fundamental elements* of a $\mathbb{P}$. The *elements* of $\mathbb{P}$ are just the elements of $G \cup 0$. An element $f \in \mathbb{P}$ is *fundamental* when $1 - f \in \mathbb{P}$. A fundamental element is *trivial* when $f \in \{0, 1\}$. In this paper we will define and calculate the fundamental elements of one rational partial field and three new real partial fields. Aside from the possible utility of these partial fields, we feel this study is also appealing in that we utilize several rather old and interesting results from number theory.

Many of our proofs rely on computer calculations. We do these calculations in SageMath version 9.3 installed on a personal computer running the Windows-11 operating system. The code and output of our calculations are contained in the unpublished technical report [13].

2. Preliminaries

A *partial field* $\mathbb{P}$ is a pair (R, G) in which R is a unitary ring (in this paper R will also be assumed to be commutative) and a subgroup G of the multiplicative group of units of R for which $-1 \in G$. A matrix A is a $\mathbb{P}$-matrix when every non-zero subdeterminant of A is in G.

The *elements* of a partial field $\mathbb{P} = (R, G)$ are just the elements of $G \cup 0$. A *fundamental* element of $\mathbb{P}$ is an element a for which $1 - a \in G$. A fundamental element a is *trivial* when $a = 0$ or 1. If a

is a non-trivial fundamental element of $\mathbb{P}$, then its *associates* are the elements of the set

$$\mathrm{Assoc}(a) = \left\{ a, 1-a, \frac{1}{a}, \frac{1}{1-a}, \frac{a}{a-1}, \frac{a-1}{a} \right\}.$$

The relation of being associates is an equivalence relation on the set of non-trivial fundamental elements of $\mathbb{P}$. Thus it is customary to describe the set of fundamental elements of $\mathbb{P}$ by a set of representatives from each equivalence class.

Proposition 2.1. *If $a \in \mathbb{C}\backslash\{0,1\}$, then $|Assoc(a)|= 6$ if and only if $a \notin \{-1, 2, \frac{1}{2}, \zeta_6, \zeta_6^{-1}\}$ in which ζ_6 is a primitive 6th root of unity.*

Proof. Let a be an unknown element of $\mathbb{C}\backslash\{0,1\}$ and assume that $|\mathrm{Assoc}(a)|< 6$. It must be the case that some two distinct rational expressions in unknown a from $\mathrm{Assoc}(a)$ are equal. There are 15 possible pairs and if we solve for a in each, we always obtain $a \in \{-1, 2, \frac{1}{2}, \zeta_6, \zeta_6^{-1}\}$, as required. Conversely, if $a \in \{-1, 2, \frac{1}{2}, \zeta_6, \zeta_6^{-1}\}$, then $|\mathrm{Assoc}(a)|= \{-1, 2, \frac{1}{2}\}$ or $\{\zeta_6, \zeta_6^{-1}\}$. □

Some examples of partial fields are the following. The *dyadic* partial field is $\mathbb{D} = (\mathbb{Q}, \{\pm 2^i : i \in \mathbb{Z}\})$. It is easy to show that the non-trivial fundamental elements of $\mathbb{D}$ are $\mathrm{Assoc}(2) = \{-1, \frac{1}{2}, 2\}$. The *Gersonides* partial field is $\mathbb{GE} = (\mathbb{Q}, \{\pm 2^i 3^j : i, j \in \mathbb{Z}\})$. Gersonides' Theorem can be used to show [16, Lemma 2.5.40] that the non-trivial fundamental elements of $\mathbb{GE}$ are $\mathrm{Assoc}(2) \cup \mathrm{Assoc}(3) \cup \mathrm{Assoc}(4) \cup \mathrm{Assoc}(9)$.

3. The p-smooth partial fields

Let p be a prime. A *p-smooth integer* is an integer whose largest prime factor is at most p. Pairs of consecutive p-smooth integers were characterized by Störmer [14]. Building upon this work Lehmer [7] calculated the complete tables of consecutive p-smooth pairs for every prime $p \leq 41$.

Let G_p be the multiplicative subgroup of $\mathbb{Q}$ generated by -1 along with the p-smooth integers and define the *p-smooth partial field* as $\mathbb{S}_p = (\mathbb{Q}, G_p)$. As stated in Section 2, the fundamental elements are known for $\mathbb{S}_2 = \mathbb{D}$ and $\mathbb{S}_3 = \mathbb{GE}$. Theorem 3.1 gives the complete list of fundamental elements of $\mathbb{S}_5$. Our proof of Theorem 3.1 relies on Propositions 3.2 and 3.3 and so extending both results to p-smooth integers for $p \geq 7$ would be required for determining the fundamental elements of $\mathbb{S}_p$. Indeed, Lehmer [7] accomplished the former task for all $p \leq 41$; however, extensions of Proposition 3.3 to three-term vanishing sums of p-smooth integers seems to be unexplored.

Theorem 3.1. *The there are 99 non-trivial fundamental elements of $\mathbb{S}_5$. They partition into classes of associates of the following rational numbers.*

$$\frac{3}{128}, \frac{2}{27}, \frac{5}{32}, \frac{9}{25}, \frac{3}{8}, \frac{2}{5}, \frac{4}{9}, 2, 3, 4, 5, 6, 9, 10, 16, 25, 81.$$

Proposition 3.2 (Lehmer [7, Table IA]). *The only pairs of consecutive (and positive) 5-smooth integers are the following.*

$$(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (8, 9), (9, 10), (15, 16), (24, 25), (80, 81).$$

Proposition 3.3. *Consider positive-integer triples (a, b, c).*

(1) *The only solutions to* $2^a + 3^b = 5^c$ *are* $(1, 1, 1)$ *and* $(4, 2, 2)$.
(2) *The only solutions to* $2^a + 5^c = 3^b$ *are* $(2, 2, 1)$ *and* $(1, 3, 2)$.
(3) *The only solutions to* $3^b + 5^c = 2^a$ *are* $(3, 1, 1)$, $(5, 3, 1)$, *and* $(7, 1, 3)$.

Proof. (1) This is a well-known exercise in basic number theory. There are many independent proofs available online.

(2) By checking all values for $a, b, c \in \{1, 2, 3\}$ we find that the only two solutions are those listed. So now assume by way of contradiction that $a, b, c \geq 4$ and $2^a + 5^c = 3^b$. Reducing the equation mod 6 we obtain $2^a + (-1)^c \equiv 3 \mod 6$ which requires that $a \not\equiv c \mod 2$. Reduce the equation mod 5 to obtain $2^a \equiv 3^b \mod 5$ which requires that $a \equiv b \mod 2$. In the first case say that a and b are both odd and c is even. Thus

$$2^a = 3^b - 5^c = (1+2)^b - (1+4)^c = \left(1 + 2b + 2^2\binom{b}{2} + \dots\right) - \left(1 + 4c + 4^2\binom{c}{2} + \dots\right).$$

Dividing both sides by 2 now yields

$$2^{a-1} = b + 2\binom{b}{2} + 2^2\binom{b}{3} + \cdots - 2c - 2^3\binom{c}{2} - 2^5\binom{c}{3} - \dots.$$

In this equation, however, the left hand side is even while the right is odd, a contradiction.

In the second case assume that $a = 2a'$ and $b = 2b'$ are both even and c is odd. Now

$$5^c = 3^b - 2^a = (3^{b'} - 2^{a'})(3^{b'} + 2^{a'}),$$

which implies that $5^m = 3^{b'} - 2^{a'}$ and $5^n = 3^{b'} + 2^{a'}$ for some $0 \leq m \leq n$. Adding we get that $5^m + 5^n = 2 \cdot 3^{b'}$. Since the right-hand side of this equation is not divisible by 5, we must have that $5^m + 5^n$ is not divisible by 5 which can only happen if $m = 0$. In the case that $m = 0$ we have $1 = 3^{b'} - 2^{a'}$ which is a difference of two consecutive 3-smooth integers. Again in [7, Table IA] we obtain that the only pairs of consecutive 3-smooth integers are $(1, 2)$, $(2, 3)$, and $(3, 4)$. Thus $b' = a' = 1$ which makes $b = a = 2$ which contradicts the assumption that $a, b, c \geq 4$.

(3) This part of the proof uses SageMath calculations. These are contained in Annex A of [13]. For $a \leq 9$ any solution also has $b \leq 5$ and $c \leq 3$. Thus we can easily verify by computer that $(3, 1, 1)$, $(5, 3, 1)$, and $(7, 1, 3)$ are the only solutions with $a \leq 9$. As a consequence we have also determined all possible solutions with b and $c \leq 3$ because any such solution requires $a \leq 8$.

Now assume by way of contradiction that (a, b, c) is a positive-integer solution with $a \geq 10$. Split the remainder of this proof into two cases. In Case 1 say that b or $c \in \{1, 2\}$ and in Case 2 that b and $c \geq 3$.

Case 1. Because $a \geq 10$, if we reduce our equation mod 1024, then we obtain $3^b + 5^c \equiv 0 \mod 1024$. Now $3, 5 \in (\mathbb{Z}/1024\mathbb{Z})^\times$ and both have order 256 in this multiplicative group. Define S_{1024} to be the set of all pairs (b, c) modulo 256 which satisfy $3^b + 5^c \equiv 0 \mod 1024$.

Reduce our integer equation mod 257 to obtain $3^b + 5^c \equiv 2^a \mod 257$. Now $2, 3, 5 \in (\mathbb{Z}/257\mathbb{Z})^\times$ and respectively have orders 16, 256, and 256 in this multiplicative group. Define S_{257} to be the set of all pairs (b, c) modulo 256 for which $3^b + 5^c \equiv 2^a \mod 257$ for some $a \in \{0, \dots, 15\}$.

Thus the positive-integer solution (a, b, c) to the integer equation produces a mod-256 pair $(b, c) \in S_{1024} \cap S_{257}$. We use SageMath to calculate that

$$S_{1024} \cap S_{257} = \{(19, 25), (83, 217), (147, 153), (211, 89)\}.$$

We have now reached a contradiction because our assumption in Case 1 is that b or $c \in \{1, 2\}$.

Case 2. Consider $M = 732{,}375 = 3^3 \cdot 5^3 \cdot 7 \cdot 31$ and reduce our equation to obtain $3^b + 5^c \equiv 2^a \mod M$. In $\mathbb{Z}_M$, $3^b \in \{3, 9\}$ if and only if $b \in \{1, 2\}$ and $5^c \in \{5, 25\}$ if and only if $c \in \{1, 2\}$. Our assumption in this case is, of course, that $b, c \geq 3$ so we let $L_3 = \{3^b \mod M : 3 \leq b \leq M\}$, $L_5 = \{5^c \mod M : 3 \leq c \leq M\}$, and $L_2 = \{2^a \mod M : 0 \leq a \leq M\}$. We calculate these three sets in SageMath and then calculate that $(L_3 + L_5) \cap L_2 = \emptyset$ as sets in $\mathbb{Z}_M$ which implies that (a, b, c) is not a solution to $3^b + 5^c \equiv 2^a \mod M$, a contradiction.

□

Proof of Theorem 3.1. The final calculation in Annex A of [13] is to check that the union of the sets of associates of the numbers listed contains $99 = 3 + 6(16)$ distinct numbers. Thus these 17 classes of associates are mutually disjoint.

If f is a fundamental element of $\mathbb{S}_5$, then either $\text{Assoc}(f)$ contains an integer or does not contain an integer. In Case 1 we determine all integer fundamental elements of $\mathbb{S}_5$ and in Case 2 we determine the fundamental elements f for which $\text{Assoc}(f)$ does not contain an integer.

Case 1. Consider a non-negative integer a. If a a non-trivial fundamental element of $\mathbb{S}_5$, then its associate $1 - a$ is a negative integer that is a fundamental element of $\mathbb{S}_5$. Thus $(a - 1, a)$ is a pair of positive and consecutive 5-smooth integers. If $-a$ is a non-trivial fundamental element of $\mathbb{S}_5$, then its associate $1 + a$ is a positive integer that is a fundamental element of $\mathbb{S}_5$. Again, $(a, a + 1)$ is a pair of positive and consecutive 5-smooth integers. The only such pairs are given by Proposition 3.2. Thus the associates of $2, 3, 4, 5, 6, 9, 10, 16, 25, 81$ are all of the fundamental elements of $\mathbb{S}_5$ which are either integers or associates of integers.

Case 2. Let $\{p_1, p_2, p_3\} = \{2, 3, 5\}$ and $\pm p_1^{e_1} p_2^{e_2} p_3^{e_3}$ be a fundamental element which is neither an integer nor the associate of an integer. We now have

$$1 = \pm p_1^{e_1} p_2^{e_2} p_3^{e_3} \pm p_1^{f_1} p_2^{f_2} p_3^{f_3}. \tag{1}$$

Since $p_1^{e_1} p_2^{e_2} p_3^{e_3}$ is not an integer, neither is its associate $p_1^{f_1} p_2^{f_2} p_3^{f_3}$. Thus some $e_i < 0$ and some $f_j < 0$.

We claim that some $e_i > 0$ and some $f_j > 0$ as well. By way of contradiction, if each $e_i \leq 0$, then $\pm p_1^{e_1} p_2^{e_2} p_3^{e_3}$ is the reciprocal of a 5-smooth integer which means that both $\pm p_1^{e_1} p_2^{e_2} p_3^{e_3}$ and $\pm p_1^{f_1} p_2^{f_2} p_3^{f_3}$ are associates of a 5-smooth integer. This is contrary to our assumption.

We also claim that for a fixed i, one of e_i and f_i is non-positive. Suppose by way of contradiction that $h_i = \min\{e_i, f_i\} > 0$. We now have

$$\frac{1}{p_i^{h_i}} = a + b,$$

in which a and b are rational numbers which when written in lowest terms do not have a factor of p_i in their denominator, a contradiction.

Suppose without loss of generality that $e_1, f_2 > 0$ which now implies that $f_1, e_2 \leq 0$. Again without loss of generality assume that $e_3 \leq f_3$ which implies that $e_3 \leq 0$. Eq. (1) now yields the integer equation;

$$p_1^{-f_1} p_2^{-e_2} p_3^{-e_3} = \pm p_1^{e_1 - f_1} \pm p_2^{f_2 - e_2} p_3^{f_3 - e_3}, \tag{2}$$

in which $p_1^{-f_1} p_2^{-e_2} p_3^{-e_3} \neq 1$ (because otherwise, Eq. (1) becomes a difference of two p-smooth integers). Since a prime factor is shared by no more than two of the three terms in Eq. (2), each prime factor is contained in at most one of the three terms. Hence Eq. (2) is actually one of $2^a+3^b=5^c$, $2^a+5^c=3^b$, and $3^b+5^c=2^a$. The seven possible solutions to these equations are given in Proposition 3.3. Each of the seven solutions produces six equations of the form $1-a=b$ and so describes six fundamental elements of $\mathbb{S}_5$ which form a complete class of six associates. These seven classes of associates are exactly the associates of the seven fractions listed in Theorem 3.1. □

4. The p-golden partial fields

Let g be the *golden ratio* $\frac{1+\sqrt{5}}{2}$; that is, the positive root of the polynomial x^2-x-1. The *golden-mean* partial field is $\mathbb{G}=(\mathbb{R},\{\pm g^i : i\in\mathbb{Z}\})$. The non-trivial fundamental elements of $\mathbb{G}$ are $\text{Assoc}(g)=\{g,-g,g^{-1},-g^{-1},g^2,g^{-2}\}$. For a prime p, define the *p-golden* partial field $\mathbb{G}_p=(\mathbb{R},\{\pm g^i s : i\in\mathbb{Z}$ and $s\in\mathbb{S}_p\})$.

An interesting geometric property of $\mathbb{G}_2$ is that $\cos(\frac{n\pi}{5})\in\{\pm1,\pm\frac{g}{2},\pm\frac{1}{2g}\}\subseteq\mathbb{G}_2$ for every $n\in\mathbb{Z}$ and, furthermore, $\cos(\frac{\pi}{5})=\frac{g}{2}$ and $\cos(\frac{3\pi}{5})=\frac{-1}{2g}$ are both fundamental elements of $\mathbb{G}_2$.

4.1. Fundamental elements of $\mathbb{G}_2$

Theorem 4.1. *The there are 27 non-trivial fundamental elements of $\mathbb{G}_2$. They partition into classes of associates of the following numbers.*

$$2,g,2g^2,g^3,g^6.$$

In order to prove Theorem 4.1 we use the relationship between the golden ratio and the Fibonacci numbers. For $n\geq 0$, define the n^{th} Fibonacci number f_n using $f_0=0$, $f_1=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$. Proposition 4.2 is well known and can easily be proven by induction.

Proposition 4.2. *For $n\geq 0$, $g^n=f_ng+f_{n-1}$ and $g^{-n}=(-1)^n(-f_ng+f_{n+1})$.*

Theorem 4.3 (Carmichael [2])**.** *If $n\notin\{1,2,6,12\}$, then f_n has a prime factor which is not a factor of f_m for any $m<n$.*

Lemma 4.4. *If $\pm 2^i g^j$ is a non-trivial fundamental element of $\mathbb{G}_2$ with $j\neq 0$, then $-6\leq i\leq 3$ and $|j|\in\{1,2,3,6\}$.*

Proof. Fundamental element $\pm 2^i g^j$ yields the equation

$$1\pm 2^i g^j=\pm 2^a g^b, \tag{3}$$

in which $\pm 2^a g^b$ is also a fundamental element. Proposition 4.2 now implies that $2^i f_{|j|}g=2^a f_{|b|}g$ and so we have that $f_{|j|}=2^{a-i}f_{|b|}$. Using Theorem 4.3 and inspecting the values $f_0,\dots,f_{12}$ we find that $|j|,|b|\in\{1,2,3,6\}$ and $|a-i|\leq 3$. Going back to (3) and again using Proposition 4.2 we get that $1\pm 2^i f_x=\pm 2^a f_y$ where $x\in\{|j|-1,|j|+1\}$ and $y\in\{|b|-1,|b|+1\}$. Thus $1=|2^i f_x\pm 2^a f_y|=2^i|f_x\pm 2^{a-i}f_y|$ where $|f_x\pm 2^{a-i}f_y|$ is nonzero and is at most $(f_7+8f_7)=117$ and at least $\frac{1}{8}$. This along with the fact that $2^6<117<2^7$ imply that $-6\leq i\leq 3$. □

Proof of Theorem 4.1. A rational fundamental element of $\mathbb{G}_2$ must be a fundamental element of $\mathbb{D}=\mathbb{S}_2$. Thus $\text{Assoc}(2)=\{-1,2,\frac{1}{2}\}$ are the rational fundamental elements of $\mathbb{G}_2$. Any other fundamental element $\pm 2^i g^j$ must have $j \neq 0$. By Lemma 4.4, the set $P = \{\pm 2^i g^j : -6 \leq i \leq 3 \text{ and } |j| \in \{1, 2, 3, 6\}\}$ contains the remaining fundamental elements of $\mathbb{G}_2$. From here we use the SageMath software package to check if $1 - a \in P$ for each $a \in P$. The calculations are in Annex B of [13]. We also confirm that the classes of associates of the elements listed in our theorem are mutually disjoint and so contain 27 distinct elements in total. □

4.2. Fundamental elements of $\mathbb{G}_3$

Theorem 4.5. *The there are 81 non-trivial fundamental elements of $\mathbb{G}_3$. They partition into classes of associates of the following numbers.*

$$2, 3, 4, 9, g, 2g^2, \frac{8}{3}g^2, 3g^2, \frac{1}{3}g^3, g^3, 2g^4, 8g^4, g^6, 18g^6.$$

Lemma 4.6. *If $\pm 2^i 3^j g^k$ is a non-trivial fundamental element of $\mathbb{G}_3$ with $k \neq 0$, then $-16 \leq i \leq 8$, $-10 \leq j \leq 5$, and $|k| \in \{1, 2, 3, 4, 6, 12\}$.*

Proof. Fundamental element $\pm 2^i 3^j g^k$ yields an equation

$$1 \pm 2^i 3^j g^k = \pm 2^a 3^b g^c, \tag{4}$$

in which $\pm 2^a 3^b g^c$ is also a fundamental element. Proposition 4.2 now implies that

$$2^{i-a} 3^{j-b} f_{|k|} = f_{|c|}.$$

Using Theorem 4.3 and inspecting the values $f_0, \ldots, f_{12}$ we find that $|k|, |c| \in \{1, 2, 3, 4, 6, 12\}$, $|i - a| \leq 4$ and $|j - b| \leq 2$. Going back to (5) and again using Proposition 4.2 we get that $1 \pm 2^i 3^j f_x = \pm 2^a 3^b f_y$ where $x \in \{|k| - 1, |k| + 1\}$ and $y \in \{|c| - 1, |c| + 1\}$. Thus $1 = |2^i 3^j f_x \pm 2^a 3^b f_y| = 2^i 3^j |f_x \pm 2^{a-i} 3^{b-j} f_y|$ where $|f_x \pm 2^{a-i} 3^{b-j} f_y|$ is nonzero and is at most $(f_{13} + 144 f_{13}) = 33,785$ and at least $\frac{1}{144}$. This along with the fact that neither 144 nor 33,785 is a power of 2 or a power of 3 imply that $-16 \leq i \leq 8$ and $-10 \leq j \leq 5$. □

Proof of Theorem 4.5. The rational fundamental elements of $\mathbb{G}_3$ must be the fundamental elements of $\mathbb{GE} = \mathbb{S}_3$. These are the associates of 2,3,4,9. Any other fundamental element $\pm 2^i 3^j g^k$ must have $k \neq 0$. By Lemma 4.6 the set $P = \{\pm 2^i 3^j g^k : -16 \leq i \leq 8 \text{ and } -10 \leq j \leq 5 \text{ and } |k| \in \{1, 2, 3, 4, 6, 12\}\}$ contains all of the remaining fundamental elements of $\mathbb{G}_3$. From here we use the SageMath software package to check if $1 - a \in P$ for each $a \in P$. The calculations are in Annex C of [13]. We also confirm that the classes of associates of the elements listed in our theorem are mutually disjoint and so contain 81 distinct elements in total. □

4.3. Fundamental elements of $\mathbb{G}_5$

Theorem 4.7. *The there are 195 non-trivial fundamental elements of $\mathbb{G}_5$. They partition into classes of associates of the following numbers.*

$$\frac{3}{128}, \frac{2}{27}, \frac{5}{32}, \frac{9}{25}, \frac{3}{8}, \frac{2}{5}, \frac{4}{9}, 2, 3, 4, 5, 6, 9, 10, 16, 25, 81,$$

$$\frac{5}{8}g, g, 2g^2, \frac{5}{2}g^2, \frac{8}{3}g^2, 3g^2, \frac{1}{3}g^3, g^3, 2g^4, 5g^4, 8g^4, \frac{1}{8}g^5, \frac{1}{3}g^5, 2g^5, g^6, 18g^6.$$

Lemma 4.8. *If* $\pm 2^i 3^j 5^k g^\ell$ *is a non-trivial fundamental element of* $\mathbb{G}_5$ *with* $\ell \neq 0$*, then* $-17 \leq i \leq 9$*,* $-10 \leq j \leq 5$*, and* $-7 \leq k \leq 4$*, and* $|\ell| \in \{1, 2, 3, 4, 5, 6, 12\}$*.*

Proof. Fundamental element $\pm 2^i 3^j 5^k g^\ell$ yields an equation

$$1 \pm 2^i 3^j 5^k g^\ell = \pm 2^a 3^b 5^c g^d, \tag{5}$$

in which $\pm 2^a 3^b 5^c g^d$ is also a fundamental element. Proposition 4.2 now implies that

$$2^i 3^j 5^k f_{|\ell|} g = 2^a 3^b 5^c f_{|d|} g,$$

which yields

$$2^{i-a} 3^{j-b} 5^{k-c} f_{|\ell|} = f_{|d|}.$$

Using Theorem 4.3 and inspecting the values $f_0, \dots, f_{12}$ we find that $|\ell|, |d| \in \{1, 2, 3, 4, 5, 6, 12\}$, $|a - i| \leq 4$, $|b - j| \leq 2$, and $|c - k| \leq 1$. Going back to (5) and again using Proposition 4.2 we get that $1 \pm 2^i 3^j 5^k f_x = \pm 2^a 3^b 5^c f_y$ where $x \in \{|\ell| - 1, |\ell| + 1\}$ and $y \in \{|d| - 1, |d| + 1\}$. Thus $1 = 2^i 3^j 5^k |f_x \pm 2^{a-i} 3^{b-j} 5^{c-k} f_y|$. The expression $|f_x \pm 2^{a-i} 3^{b-j} 5^{c-k} f_y|$ is nonzero and is at most $(f_{13} + 720 f_{13}) = 167{,}993$ and at least $\frac{1}{720}$. This implies that $-17 \leq i \leq 9$, $-10 \leq j \leq 5$, and $-7 \leq k \leq 4$. □

Proof of Theorem 4.7. The rational fundamental elements of $\mathbb{G}_5$ must be the fundamental elements of $\mathbb{S}_5$. These are listed in Theorem 3.1. Any other fundamental element $\pm 2^i 3^j 5^k g^\ell$ must have $\ell \neq 0$. By Lemma 4.6 the set $P = \{\pm 2^i 3^j 5^k g^\ell : -17 \leq i \leq 9 \text{ and } -10 \leq j \leq 5 \text{ and } -7 \leq k \leq 4 \text{ and } |\ell| \in \{1, 2, 3, 4, 5, 6, 12\}\}$ contains all of the remaining fundamental elements of $\mathbb{G}_3$. From here we use the SageMath software package to check if $1 - a \in P$ for each $a \in P$. The calculations are in Annex D of [13]. We also confirm that the classes of associates of the elements listed in our theorem are mutually disjoint and so contain 195 distinct elements in total. □

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