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A note on the threshold numbers of cycles

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ABSTRACT

A graph $G = (V, E)$ is said to be a *k-threshold graph* with *thresholds* $\theta_1 < \theta_2 < \ldots < \theta_k$ if there is a map $r : V \longrightarrow \mathbb{R}$ such that $uv \in E$ if and only if the number of $i \in [k]$ with $\theta_i \le r(u) + r(v)$ is odd. The *threshold number* of G, denoted by $\Theta(G)$, is the smallest positive integer k such that G is a k-threshold graph. In this paper, we determine the exact threshold numbers of cycles by proving

$$\Theta(C_n) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ n = 4, \\ 4 & if\ n \ge 5, \end{cases}$$

where C_n is the cycle with n vertices.

Keywords: multithreshold graph, threshold number, cycle

1. Introduction

In this paper, we only consider finite simple graphs. For $a, b \in \mathbb{Z}$ with $a < b$, we denote $\{i \in \mathbb{Z} : a \le i \le b\}$ by $[a, b]$; for $c \in \mathbb{N}$, we denote $\{j \in \mathbb{N} : 1 \le j \le c\}$ by $[c]$.

A graph $G = (V, E)$ is said to be a *threshold graph* if there is a map $f : V \longrightarrow \mathbb{R}$ such that $uv \in E$ if and only if $f(u) + f(v) \ge 0$. Threshold graphs were introduced by Chvátal and Hammer [2] in 1977, and have been extensively studied since then. For example, see [3, 4, 5, 8].

Jamison and Sprague [6] introduced multithreshold graphs as a generalization of threshold graphs. A graph $G = (V, E)$ is said to be a *k-threshold graph* with *thresholds* $\theta_1 < \theta_2 < \ldots < \theta_k$ if there is a map $r : V \longrightarrow \mathbb{R}$ such that $uv \in E$ if and only if the number of $i \in [k]$ with $\theta_i \le r(u)+r(v)$ is odd. We say $r(v)$ is the *rank* of v. We say such a rank assignment $r : V \longrightarrow \mathbb{R}$ is a $(\theta_1, \theta_2, \ldots, \theta_k)$-*representation* of G.

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If G is a k-threshold graph, then the k thresholds divide $\mathbb{R}$ into $k+1$ parts in the form

$$\text{NO} \mid \text{YES} \mid \text{NO} \mid \text{YES} \mid \ldots,$$

and every edge rank sum lands in a YES part, every nonedge rank sum lands in a NO part.

Jamison and Sprague [6] proved that if a graph has n vertices, then it is a k-threshold graph for some $k \leq \binom{n}{2}$. This means for any finite graph G, there is a smallest positive integer $\Theta(G)$, called the *threshold number* of G, such that G is a $\Theta(G)$-threshold graph.

For specific graphs: Jamison and Sprague [6] proved that the threshold number of any path is at most 2, and the threshold number of any caterpillar (obtained by attaching leaves to vertices in a path) is also at most 2; Chen and Hao [1] determined the exact threshold numbers of the complete m-partite graphs with every part having at least $m+1$ vertices; Kittipassorn and Sumalroj [7] determined the exact threshold numbers of the complete multipartite graphs with every part having 3 vertices or every part having 4 vertices; Wang [9] determined the exact threshold numbers of the complete multipartite graphs with every part having at most 3 vertices, and the exact threshold numbers of some variants of paths.

In this paper, we determine the exact threshold numbers of cycles.

Theorem 1.1. *Let $n \geq 3$ be an integer and let C_n be the cycle with n vertices. Then*

$$\Theta(C_n) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ n = 4, \\ 4 & if\ n \geq 5. \end{cases}$$

2. Proof of Theorem 1.1

For C_n, the cycle with n vertices, we randomly pick a vertex and label it v_1, then counterclockwise label the other vertices $v_2, v_3, \ldots, v_n$. Note that we can continue this labeling process, so each vertex v_i can also be represented as v_{i+kn} for any $k \in \mathbb{N}$, e.g. v_1 and v_{n+1} are the same vertex.

Let us prove Theorem 1.1.

Proof of Theorem 1.1. It is easy to check that $\Theta(C_3) = 1$.

For C_n with $n \geq 4$, we first show that $\Theta(C_n) \geq 2$.

In C_n with $n \geq 4$, v_1v_2 and v_3v_4 are edges, v_1v_3 and v_2v_4 are nonedges. Assume by contradiction that $\Theta(C_n) = 1$ and $r : V \longrightarrow \mathbb{R}$ is a (θ)-representation of C_n for some θ. Then by the definition of multithreshold graphs, we have $r(v_1) + r(v_2) \geq \theta$, $r(v_3) + r(v_4) \geq \theta$, $r(v_1) + r(v_3) < \theta$, and $r(v_2) + r(v_4) < \theta$. But then we will have

$$(r(v_1) + r(v_2)) + (r(v_3) + r(v_4)) \geq 2\theta > (r(v_1) + r(v_3)) + (r(v_2) + r(v_4)),$$

a contradiction. So $\Theta(C_n) \geq 2$.

For C_4, if we let $\theta_1 = 0.9$ and $\theta_2 = 1.1$, and let $r(v_1) = 0$, $r(v_2) = 1$, $r(v_3) = 0$, and $r(v_4) = 1$, then every edge rank sum is 1, and every nonedge rank sum is either 0 or 2, so we have a $(0.9, 1.1)$-representation of C_4. So $\Theta(C_4) \leq 2$, and hence $\Theta(C_4) = 2$.

Now let us assume $n \geq 5$.

First, we show $\Theta(C_n) \leq 4$ by construction. Let $r(v_i) = (-1)^{i-1}i$ for $i \in [n-1]$, and let $r(v_n) = (-1)^{n-1}(n - 0.5)$.

Case I. n is odd.

In this case $r(v_n) = n - 0.5$. We let $\theta_1 = -1.1$, $\theta_2 = 1.1$, $\theta_3 = r(v_1) + r(v_n) - 0.1 = n + 0.4$, and $\theta_4 = r(v_1) + r(v_n) + 0.1 = n + 0.6$. We have $\theta_1 < \theta_2 < \theta_3 < \theta_4$ because $n \geq 5$. Denote $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ by $\mathscr{S}$.

Then for an edge not involving v_n, its rank sum is -1 or 1, which is between θ_1 and θ_2. For edge $v_{n-1}v_n$, its rank sum is 0.5, which is between θ_1 and θ_2. For edge v_1v_n, its rank sum is $1+(n-0.5) = n + 0.5$, which is between θ_3 and θ_4. We have checked every edge rank sum is greater than or equal to an odd number of elements in $\mathscr{S}$.

For a nonedge not involving v_n, assume its two vertices are v_i and v_j with distinct $i, j \in [n-1]$ and $|i-j| \neq 1$.

- If $r(v_i)+r(v_j) < 0$, then by $|i-j| \neq 1$, we know that $r(v_i)+r(v_j) \leq -3 < \theta_1$, so $|\{s \in \mathscr{S} : s \leq r(v_i) + r(v_j)\}| = 0$, an even number.
- If $r(v_i) + r(v_j) > 0$, then by $|i-j| \neq 1$, we know that $r(v_i) + r(v_j) \geq 3 > \theta_2$. We have that $r(v_i) + r(v_j)$ is an integer, so it is either smaller than θ_3 or greater than θ_4. So $|\{s \in \mathscr{S} : s \leq r(v_i) + r(v_j)\}|$ is 2 or 4, an even number.

For a nonedge involving v_n, assume the other vertex of this nonedge is v_i with $i \in [2, n-2]$.

- If $r(v_i) > 0$, then $r(v_i)$ is at least 3, so $r(v_i) + r(v_n) \geq 3 + (n-0.5) = n + 2.5 > \theta_4$, so $|\{s \in \mathscr{S} : s \leq r(v_i) + r(v_j)\}| = 4$, an even number.
- If $r(v_i) < 0$, then $-(n-3) \leq r(v_i) \leq -2$. Note that we have $-(n-3) \leq -2$ because $n \geq 5$. We have $\theta_2 < 2.5 \leq r(v_i) + r(v_n) \leq n - 2.5 < \theta_3$, so $|\{s \in \mathscr{S} : s \leq r(v_i) + r(v_j)\}| = 2$, an even number.

We have also checked every nonedge rank sum is greater than or equal to an even number of elements in $\mathscr{S}$. So we have a $(\theta_1, \theta_2, \theta_3, \theta_4)$-representation of C_n, and hence $\Theta(C_n) \leq 4$.

Case II. n is even.

In this case, $r(v_n) = -n+0.5$. We let $\theta_1 = r(v_1)+r(v_n)-0.1 = -n+1.4$, $\theta_2 = r(v_1)+r(v_n)+0.1 = -n + 1.6$, $\theta_3 = -1.1$, and $\theta_4 = 1.1$. We have $\theta_1 < \theta_2 < \theta_3 < \theta_4$ because $n \geq 5$. Denote $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ by $\mathscr{S}'$. The same as in Case I, we can check every edge rank sum is greater than or equal to an odd number of elements in $\mathscr{S}'$, and every nonedge rank sum is greater than or equal to an even number of elements in $\mathscr{S}'$. So we also have $\Theta(C_n) \leq 4$ in this case.

Now we have proved the upper bound $\Theta(C_n) \leq 4$.

We already know $\Theta(C_n) \geq 2$. So by eliminating the possibility of $\Theta(C_n) = 2$ or $\Theta(C_n) = 3$, we can conclude that $\Theta(C_n) = 4$.

First assume $\Theta(C_n) = 2$. So we have two thresholds θ_1, θ_2, and a (θ_1, θ_2)-representation $r : V \longrightarrow \mathbb{R}$. Two thresholds divide $\mathbb{R}$ into three parts of the form

$$\text{NO} \mid \text{YES} \mid \text{NO}.$$

The two NO parts will be called smaller NO and larger NO.

We have three cases. The proofs of the first two follow the same pattern. The proof of the third one is slightly different.

Case i. $2 \nmid n$.

For vertices v_1, v_2, v_3, v_4: We know v_1v_2 and v_3v_4 are edges, so both $r(v_1)+r(v_2)$ and $r(v_3)+r(v_4)$ are in YES. We know v_1v_3 and v_2v_4 are nonedges, so both $r(v_1) + r(v_3)$ and $r(v_2) + r(v_4)$ are in NO. By the fact that $(r(v_1) + r(v_2)) + (r(v_3) + r(v_4)) = (r(v_1) + r(v_3)) + (r(v_2) + r(v_4))$, we know that $r(v_1) + r(v_3)$ and $r(v_2) + r(v_4)$ are in different NO's. Without loss of generality, we assume $r(v_1) + r(v_3)$ is in smaller NO and $r(v_2) + r(v_4)$ is in larger NO.

Then for vertices v_2, v_3, v_4, v_5: Similarly we know that $r(v_2)+r(v_4)$ and $r(v_3)+r(v_5)$ are in different NO's, and we already assumed $r(v_2) + r(v_4)$ is in larger NO, so $r(v_3) + r(v_5)$ is in smaller NO.

Repeating this argument, we have that if i is odd, then $r(v_i)+r(v_{i+2})$ is in smaller NO; if i is even, then $r(v_i) + r(v_{i+2})$ is in larger NO. Now we take $i = n + 1$. We know $n + 1$ is even because $2 \nmid n$, so we have $r(v_{n+1}) + r(v_{n+3})$ is in larger NO, but v_{n+1} and v_1 represent the same vertex, v_{n+3} and v_3 represent the same vertex, so $r(v_1) + r(v_3)$ is in larger NO, which contradicts our assumption that $r(v_1) + r(v_3)$ is in smaller NO.

Case ii. $2 \mid n$, but $4 \nmid n$.

Because $2 \mid n$, we know $\frac{n}{2}$ is an integer.

For vertices $v_1, v_2, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}$: We know v_1v_2 and $v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}$ are edges, so both $r(v_1) + r(v_2)$ and $r(v_{\frac{n}{2}+1}) + r(v_{\frac{n}{2}+2})$ are in YES. We know $v_1v_{\frac{n}{2}+1}$ and $v_2v_{\frac{n}{2}+2}$ are nonedges, so both $r(v_1) + r(v_{\frac{n}{2}+1})$ and $r(v_2) + r(v_{\frac{n}{2}+2})$ are in NO. By the fact that $(r(v_1) + r(v_2)) + (r(v_{\frac{n}{2}+1}) + r(v_{\frac{n}{2}+2})) = (r(v_1) + r(v_{\frac{n}{2}+1})) + (r(v_2) + r(v_{\frac{n}{2}+2}))$, we know that $r(v_1) + r(v_{\frac{n}{2}+1})$ and $r(v_2) + r(v_{\frac{n}{2}+2})$ are in different NO's. Without loss of generality, we assume $r(v_1)+r(v_{\frac{n}{2}+1})$ is in smaller NO and $r(v_2)+r(v_{\frac{n}{2}+2})$ is in larger NO.

Then for vertices $v_2, v_3, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}$: Similarly we know that $r(v_2) + r(v_{\frac{n}{2}+2})$ and $r(v_3) + r(v_{\frac{n}{2}+3})$ are in different NO's, and we already assumed $r(v_2) + r(v_{\frac{n}{2}+2})$ is in larger NO, so $r(v_3) + r(v_{\frac{n}{2}+3})$ is in smaller NO.

Repeating this argument, we have that if i is odd, then $r(v_i)+r(v_{\frac{n}{2}+i})$ is in smaller NO; if i is even, then $r(v_i) + r(v_{\frac{n}{2}+i})$ is in larger NO. Now we take $i = \frac{n}{2} + 1$. We know $\frac{n}{2} + 1$ is even because $4 \nmid n$, so we have $r(v_{\frac{n}{2}+1}) + r(v_{n+1})$ is in larger NO, but v_{n+1} is the same vertex as v_1, so $r(v_{\frac{n}{2}+1}) + r(v_1)$ is in larger NO, which contradicts our assumption that $r(v_1) + r(v_{\frac{n}{2}+1})$ is in smaller NO.

Case iii. $4 \mid n$.

For vertices $v_1, v_2, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$: We know v_1v_2 and $v_{\frac{n}{2}}v_{\frac{n}{2}+1}$ are edges, so both $r(v_1)+r(v_2)$ and $r(v_{\frac{n}{2}})+r(v_{\frac{n}{2}+1})$ are in YES. We know $v_1v_{\frac{n}{2}}$ and $v_2v_{\frac{n}{2}+1}$ are nonedges, so both $r(v_1)+r(v_{\frac{n}{2}})$ and $r(v_2)+r(v_{\frac{n}{2}+1})$ are in NO. By the fact that $(r(v_1) + r(v_2)) + (r(v_{\frac{n}{2}}) + r(v_{\frac{n}{2}+1})) = (r(v_1) + r(v_{\frac{n}{2}})) + (r(v_2) + r(v_{\frac{n}{2}+1}))$, we know that $r(v_1) + r(v_{\frac{n}{2}})$ and $r(v_2) + r(v_{\frac{n}{2}+1})$ are in different NO's. Without loss of generality, we assume $r(v_1) + r(v_{\frac{n}{2}})$ is in smaller NO and $r(v_2) + r(v_{\frac{n}{2}+1})$ is in larger NO.

Then for vertices $v_2, v_3, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}$: Similarly we know that $r(v_2) + r(v_{\frac{n}{2}+1})$ and $r(v_3) + r(v_{\frac{n}{2}+2})$ are in different NO's, and we already assumed $r(v_2) + r(v_{\frac{n}{2}+1})$ is in larger NO, so $r(v_3) + r(v_{\frac{n}{2}+2})$ is in smaller NO.

Repeating this argument, we have that if i is odd, then $r(v_i) + r(v_{\frac{n}{2}+i-1})$ is in smaller NO; if i is even, then $r(v_i) + r(v_{\frac{n}{2}+i-1})$ is in larger NO. Now we take $i = \frac{n}{2} + 1$. We know $\frac{n}{2} + 1$ is odd because $4 \mid n$, so $r(v_{\frac{n}{2}+1}) + r(v_n)$ is in smaller NO. But then for vertices $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, v_n, v_1$: We know $v_{\frac{n}{2}}v_{\frac{n}{2}+1}$ and v_nv_1 are edges, so both $r(v_{\frac{n}{2}}) + r(v_{\frac{n}{2}+1})$ and $r(v_n) + r(v_1)$ are in YES. We know $v_1v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}v_n$ are nonedges, and both $r(v_1) + r(v_{\frac{n}{2}})$ and $r(v_{\frac{n}{2}+1}) + r(v_n)$ are in smaller NO. But then we will have $(r(v_{\frac{n}{2}}) + r(v_{\frac{n}{2}+1})) + (r(v_n) + r(v_1)) > (r(v_1) + r(v_{\frac{n}{2}})) + (r(v_{\frac{n}{2}+1}) + r(v_n))$, a contradiction.

In each of the three cases, we get a contradiction, so $\Theta(C_n)$ cannot be 2.

Then assume $\Theta(C_n) = 3$. So we have three thresholds $\theta_1, \theta_2, \theta_3$, and a $(\theta_1, \theta_2, \theta_3)$-representation $r : V \longrightarrow \mathbb{R}$. Three thresholds divide $\mathbb{R}$ into four parts in the form

$$\text{NO} \mid \text{YES} \mid \text{NO} \mid \text{YES}.$$

The two NO parts will be called smaller NO and larger NO; the two YES parts will be called smaller YES and larger YES.

First we note that there must be an edge rank sum in larger YES, as otherwise two thresholds would be enough, and $\Theta(C_n) = 3$ would not hold. We may assume $r(v_1) + r(v_2)$ is in larger YES.

We also make the following claim.

Claim 1. Assume v_iv_{i+1} and v_jv_{j+1} are disjoint edges in C_n. Then $r(v_i)+r(v_{i+1})$ and $r(v_j)+r(v_{j+1})$ cannot both be in larger YES.

Proof of Claim. Because v_iv_{i+1} and v_jv_{j+1} are disjoint edges, we can see that both v_iv_j and $v_{i+1}v_{j+1}$ are nonedges. Assume by contradiction that both $r(v_i) + r(v_{i+1})$ and $r(v_j) + r(v_{j+1})$ are in larger YES. Then by the equation $(r(v_i)+r(v_{i+1}))+(r(v_j)+r(v_{j+1})) = (r(v_i)+r(v_j))+(r(v_{i+1})+r(v_{j+1}))$, we know that $\max\{r(v_i) + r(v_j), r(v_{i+1}) + r(v_{j+1})\} \geq \min\{r(v_i) + r(v_{i+1}), r(v_j) + r(v_{j+1})\}$, so at least one of $r(v_i) + r(v_j)$ and $r(v_{i+1}) + r(v_{j+1})$ is in larger YES. However, both v_iv_j and $v_{i+1}v_{j+1}$ are nonedges, a contradiction. □

By this claim, we know that edge v_2v_3 and edge v_nv_1 cannot both have rank sums in larger YES, because they are disjoint edges. Moreover, any edge v_iv_{i+1} with $i \in [3, n-1]$ cannot have rank sum in larger YES, because v_1v_2 and v_iv_{i+1} are disjoint edges.

We have two cases:

Case 1. Neither v_2v_3 nor v_nv_1 has rank sum in larger YES.

We have that both $r(v_2) + r(v_3)$ and $r(v_n) + (v_1)$ are in smaller YES, so by a similar argument as in the proof of $\Theta(C_n) \neq 2$, we know $r(v_1) + r(v_3)$ and $r(v_2) + r(v_n)$ are in different NO's. Without loss of generality, we may assume $r(v_1) + r(v_3)$ is in smaller NO.

Because v_2v_4 is a nonedge and $r(v_1)+r(v_2)$ is in larger YES, we have $r(v_2)+r(v_4) < r(v_1)+r(v_2)$, so $r(v_4) < r(v_1)$, which implies $r(v_3) + r(v_4) < r(v_3) + r(v_1)$, and we assumed $r(v_3) + r(v_1)$ is in smaller NO, so $r(v_3) + r(v_4)$ is also in smaller NO, which contradicts the fact that v_3v_4 is an edge.

Case 2. Exactly one of v_2v_3 and v_nv_1 has rank sum in larger YES.

By symmetry, we may assume $r(v_2) + r(v_3)$ is in larger YES. Now, $r(v_1) + r(v_2)$ and $r(v_2) + r(v_3)$ are in larger YES, and by the claim we just made, all other edge rank sums are in smaller YES. In particular, both $r(v_3) + r(v_4)$ and $r(v_n) + r(v_1)$ are in smaller YES, so by a similar argument as in the proof of $\Theta(C_n) \neq 2$, we know $r(v_3) + r(v_n)$ and $r(v_4) + r(v_1)$ are in different NO's. Without loss of generality, we may assume $r(v_3) + r(v_n)$ is in smaller NO.

Now we look at vertices v_2, v_3, v_{n-1}, v_n: We know v_2v_3 is an edge with $r(v_2) + r(v_3)$ in larger YES, $v_{n-1}v_n$ is an edge with $r(v_{n-1}) + r(v_n)$ in smaller YES. Now, because $r(v_3) + r(v_n)$ is in smaller NO, we have $r(v_3)+r(v_n) < r(v_{n-1})+r(v_n)$. Because v_2v_{n-1} is a nonedge, we have $r(v_2)+r(v_{n-1})$ is in NO (larger or smaller), and we know $r(v_2)+r(v_3)$ is in larger YES, so $r(v_2)+r(v_{n-1}) < r(v_2)+r(v_3)$. But then we have $(r(v_3)+r(v_n))+(r(v_2)+r(v_{n-1})) < (r(v_{n-1})+r(v_n))+(r(v_2)+r(v_3))$, a contradiction.

Note that if $n = 5$, then v_4 and v_{n-1} are the same vertex, but this does not affect our argument.

In either case, we get a contradiction, so $\Theta(C_n)$ cannot be 3.

In summary, for $n \geq 5$, We have showed $\Theta(C_n) \geq 2$, $\Theta(C_n) \leq 4$, $\Theta(C_n) \neq 2$, and $\Theta(C_n) \neq 3$, thus $\Theta(C_n) = 4$. □

3. Remarks

Actually there is another way to see that $\Theta(C_n) \leq 4$, which is given by combining the following two propositions and the fact that we can get C_n by adding an edge to P_n.

Proposition 3.1 (Jamison and Sprague [6]). *Let n be a positive integer, let P_n be the path with n vertices, then $\Theta(P_n) \leq 2$.*

Proposition 3.2 (Jamison and Sprague [6]). *Let $G = (V, E)$ be a graph with $\Theta(G) = k$, and let $u, v \in V$ be distinct vertices.*

(a) *If uv is an edge, then the graph $G - uv$ obtained by deleting edge uv from E has threshold number at most $k + 2$.*

(b) *If uv is a nonedge, then the graph $G + uv$ obtained by adding edge uv to E has threshold number at most $k + 2$.*

We have given a more constructive proof, as it helps readers understand the concept of multi-threshold graphs.

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