

Counting of lattices having nullity up to two

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ABSTRACT

In 2020 Bhavale and Waphare introduced the concept of a nullity of a poset as nullity of its cover graph. According to Bhavale and Waphare, if a dismantlable lattice of nullity k contains r reducible elements then $2 \leq r \leq 2k$. In 2003 Pawar and Waphare counted all non-isomorphic lattices on n elements having nullity one, containing exactly two reducible elements. Recently, Bhavale and Aware counted all non-isomorphic lattices on n elements having nullity two, containing up to three reducible elements. In this paper, we count up to isomorphism the class of all lattices on n elements having nullity two, containing exactly four reducible elements.

Keywords: chain, lattice, poset, counting

1. Introduction

In 1940 Birkhoff [4] raised the open problem, compute for small n all non-isomorphic posets/lattices on a set of n elements. This NP-complete problem was attempted by many authors from all over the world. Enumeration of all non-isomorphic posets with 15 and 16 elements was done by Brinkmann and McKay [5]. The work of enumeration of all non-isomorphic (unlabeled) posets is still in progress for $n \geq 17$. Counting of all non-isomorphic (unlabeled) lattices on up to 18 elements was carried out by Heitzig and Reinhold [7].

The concept of a nullity of a poset as nullity of its cover graph was introduced by Bhavale and Waphare [3]. In the same paper, they introduced the concept of an *RC-lattice* as a lattice in which all the reducible elements are comparable. According to Bhavale and Waphare [3], if a dismantlable lattice of nullity k contains r reducible elements then $2 \leq r \leq 2k$. In particular for $k = 2$, $2 \leq r \leq 4$. In 2003 Pawar and Waphare [9] counted all non-isomorphic lattices with n elements, containing

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exactly two reducible elements, and having nullity one (*that is*, $r = 2$ and $k = 1$). Recently, Bhavale and Aware [2] counted all non-isomorphic lattices on n elements, containing up to three reducible elements, and having nullity two (*that is*, $r \leq 3$ and $k = 2$). In this paper, we count up to isomorphism the class of all non-isomorphic lattices on n elements, containing exactly four (comparable) reducible elements, and having nullity two (*that is*, $r = 4$ and $k = 2$). In fact, this counting covers the counting of all non-isomorphic lattices on n elements having nullity up to two.

Let (P, \leq) be a poset. Elements $a, b \in P$ are said to be *comparable*, if either $a \leq b$ or $b \leq a$, otherwise $a, b \in P$ are said to be *incomparable*, denoted by $a \parallel b$. An element $u \in P$ is *upper bound* (*lower bound*) of $a, b \in P$ if $a \leq u, b \leq u$ ($u \leq a, u \leq b$). A *join* (*meet*) of $a, b \in P$, denoted by $a \vee b$ ($a \wedge b$), is defined as the least upper bound (greatest lower bound) of a and b . A poset L is a *lattice* if $a \vee b$ and $a \wedge b$, exist in $L, \forall a, b \in L$. Lattices L_1 and L_2 are *isomorphic*, denoted by $L_1 \cong L_2$, and the map $\phi : L_1 \rightarrow L_2$ is an *isomorphism* if and only if ϕ is bijective, and $a \leq b$ in L_1 if and only if $\phi(a) \leq \phi(b)$ in L_2 .

An element u in P *covers* an element l in P if $l < u$, and there is no element m in P such that $l < m < u$. We denote this fact by $l \prec u$, and say that pair $\langle l, u \rangle$ is a *covering* or an *edge* (see [12]). An element of a poset P is called *doubly irreducible* if it has at most one upper cover and at most one lower cover in P . Let $Irr(P)$ denotes the set of all doubly irreducible elements in the poset P . Let $Irr^*(P) = \{x \in Irr(P) : x \text{ has exactly one lower cover and exactly one upper cover in } P\}$. The graph on a poset P with edges as covering relations is called the *cover graph* and is denoted by $C(P)$. *Length* of a chain is the number of coverings in a chain.

Let G be a graph on l vertices, containing m edges, with n connected components. Then the *nullity of a graph* G is given by $m - l + n$. Bhavale and Waphare [3] defined *nullity of a poset* P , denoted by $\eta(P)$, to be the nullity of its cover graph $C(P)$. For $a < b$, the interval $[a, b) = \{x \in P : a \leq x < b\}$, and $(a, b) = \{x \in P : a < x < b\}$; likewise, we can define $[a, b]$ and $(a, b]$. An element a in a lattice L is *meet-reducible* (*join-reducible*) in L if there exist $b, c \in L$ both distinct from a , such that $b \wedge c = a$ ($b \vee c = a$). *Reducible* element of a lattice L is either meet-reducible or join-reducible element of L . Element of a lattice L is *meet-irreducible* (*join-irreducible*) if it is not meet-reducible (join-reducible); Element of a lattice L is *doubly irreducible* if it is both meet-irreducible and join-irreducible. The set of all doubly irreducible elements in L is denoted by $Irr(L)$, and the set of all reducible elements in L is denoted by $Red(L)$. A *pendant* vertex of a graph is a vertex with degree one.

In 1974 Rival [10] introduced and studied the class of dismantlable lattices.

Definition 1.1. [10] A finite lattice of order n is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \dots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i .

Following result is due to Kelly and Rival [8].

Theorem 1.2. [8] A finite lattice is dismantlable if and only if it contains no crown.

The concept of *adjunct operation of lattices*, is introduced by Thakare, Pawar and Waphare [12]. Suppose L_1 and L_2 are two disjoint lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\leq b$. Define the partial order \leq on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows: $x \leq y$ in L if $x, y \in L_1$ and $x \leq y$ in L_1 , or $x, y \in L_2$ and $x \leq y$ in L_2 , or $x \in L_1, y \in L_2$ and $x \leq a$ in L_1 , or $x \in L_2, y \in L_1$ and $b \leq y$ in L_1 .

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure for obtaining

L in this way is called *adjunct operation (or adjunct sum)* of L_1 with L_2 . We call the pair (a, b) as an *adjunct pair* and L as an *adjunct* of L_1 with L_2 with respect to the adjunct pair (a, b) and write $L = L_1]_a^b L_2$. A diagram of L is obtained by placing a diagram of L_1 and a diagram of L_2 side by side in such a way that the largest element 1 of L_2 is at lower position than b and the least element 0 of L_2 is at the higher position than a and then by adding the coverings $\langle 1, b \rangle$ and $\langle a, 0 \rangle$. This clearly gives $|E(L)| = |E(L_1)| + |E(L_2)| + 2$. A lattice L is called an *adjunct of lattices* L_0, L_1, \dots, L_k , if it is of the form $L = L_0]_{a_1}^{b_1} L_1]_{a_2}^{b_2} L_2 \cdots]_{a_k}^{b_k} L_k$.

Following result is due to Thakare et al. [12].

Theorem 1.3. [12] *A finite lattice is dismantlable if and only if it is an adjunct of chains.*

The *direct sum* (see [11]) of two disjoint posets, namely P_1 and P_2 is denoted by $P_1 \oplus P_2$, and is defined by taking the order relation \leq on $P_1 \cup P_2$ as: $x \leq y$ in $P_1 \cup P_2$ if and only if $x, y \in P_1$ and $x \leq y$ in P_1 , or $x, y \in P_2$ and $x \leq y$ in P_2 or $x \in P_1, y \in P_2$. In general $P_1 \oplus P_2 \neq P_2 \oplus P_1$. Also, if P_1 and P_2 are lattices then $|E(P_1 \oplus P_2)| = |E(P_1)| + |E(P_2)| + 1$.

Thakare et al. [12] defined a *block* as a finite lattice in which the largest element 1 is join-reducible and the smallest element 0 is meet-reducible. Moreover, if L is any lattice which is different from chain, then L contains a unique maximal sublattice which is a block called as *maximal block*. The lattice L is of the form $C \oplus \mathbf{B}$ or $\mathbf{B} \oplus C$ or $C \oplus \mathbf{B} \oplus C'$, where C, C' are chains and \mathbf{B} is the maximal block. Bhavale and Waphare [3] introduced the following concepts namely, retractible element, basic retract, basic block, basic block associated to a poset.

Definition 1.4. [3] Let P be a poset. Let $x \in Irr(P)$. Then x is called a *retractible* element of P if it satisfies either of the following conditions.

- (a) There are no $y, z \in Red(P)$ such that $y \prec x \prec z$.
- (b) There are $y, z \in Red(P)$ such that $y \prec x \prec z$ and there is no other directed path from y to z in P .

Definition 1.5. [3] A poset P is a *basic retract* if no element of $Irr^*(P)$ is retractible in the poset P .

Definition 1.6. [3] A poset P is a *basic block* if it is one element or $Irr(P) = \phi$ or removal of any doubly irreducible element reduces nullity by one.

Definition 1.7. [3] B is a *basic block associated to a poset* P if B is obtained from the basic retract associated to P by successive removal of all the pendant vertices.

Following result is due to Bhavale [1].

Theorem 1.8. [1] *Let B be an RC-lattice on n elements. Then B is a basic block having nullity k if and only if $B = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots]_{a_k}^{b_k} C_k$ with $a_i, b_i \in C_0$, satisfying*

- (i) $|C_i| = 1$ for all $i, 1 \leq i \leq k$,
- (ii) $n = |C_0| + k$,
- (iii) $Irr(B) = k + m$, where m is the number of distinct adjunct pairs (a_i, b_i) such that the interval $(a_i, b_i) \subseteq Irr(B)$, and
- (iv) $|C_0| = |Red(B)| + m$.

Following result is due to Bhavale and Waphare [3].

Theorem 1.9. [3] *Let B be the basic block associated to a poset P . Then $Red(B) = Red(P)$.*

For the other definitions, notation, and terminology see [2, 6, 13].

2. Counting of lattices of nullity up to two, containing up to three reducible elements

In this section, we discuss counting of all non-isomorphic lattices on n elements containing up to three comparable reducible elements, and having nullity up to two. Note that chain is the only lattice containing no reducible element and having nullity 0. Obviously, there is no lattice on one reducible element. In 2003 Pawar and Waphare [9] enumerated all non-isomorphic lattices on n elements, containing exactly two reducible elements, and having nullity one. Let $\mathcal{L}(n; r, k)$ be the class of all non-isomorphic lattices on n elements such that every member of it contains r comparable reducible elements, and has nullity k . Let P_n^k denote the number of partitions of n into k non-decreasing positive integer parts. Pawar and Waphare [9] obtained the following result.

Theorem 2.1. [9] $n \geq 4$,

$$|\mathcal{L}(n; 2, 1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6}; & \text{if } n = 2m + 1, \\ \frac{m(m-1)(4m-5)}{6}; & \text{if } n = 2m. \end{cases}$$

Recently, Bhavale and Aware [2] counted all non-isomorphic lattices on n elements, containing three comparable reducible elements and having nullity $k \geq 2$. In particular for $k = 2$, Bhavale and Aware [2] obtained the following result.

Theorem 2.2. [2] *For an integer $n \geq 6$,*

$$|\mathcal{L}(n; 3, 2)| = \sum_{j=0}^{n-6} \sum_{l=1}^{n-j-5} \sum_{i=1}^{n-j-l-4} 2(j+1)P_{n-j-l-i-2}^2 + \sum_{j=0}^{n-7} \sum_{l=4}^{n-j-3} (j+1)P_{l-2}^2 P_{n-j-l-1}^2.$$

3. Counting of lattices of nullity two, containing four reducible elements

In this section, we count all non-isomorphic lattices on n elements containing four comparable reducible elements, and having nullity two.

Let $\mathcal{L}(n; r, k, h)$ be the subclass of $\mathcal{L}(n; r, k)$ such that the basic block associated to a member of it is of height h . Let $\mathcal{B}(j; r, k)$ be the class of all non-isomorphic maximal blocks on j elements such that every member of it contains r comparable reducible elements, and has nullity k . Let $\mathcal{B}(j; r, k, h)$ be the subclass of $\mathcal{B}(j; r, k)$ such that the basic block associated to a member of it is of height h . Let $\mathcal{L}'(n; 2, k)$ be the subclass of $\mathcal{L}(n; 2, k)$ such that any $L \in \mathcal{L}'(n; 2, k)$ is of the form $L = C \oplus \mathbf{B} \oplus C'$ where \mathbf{B} is the maximal block, and C, C' are chains with $|C| \geq 1, |C'| \geq 1$. According to Bhavale [1] (Proposition 3.2.5) there are exactly three basic blocks (see Figure 1) containing four (comparable) reducible elements and having nullity two. Therefore, we have the following result.

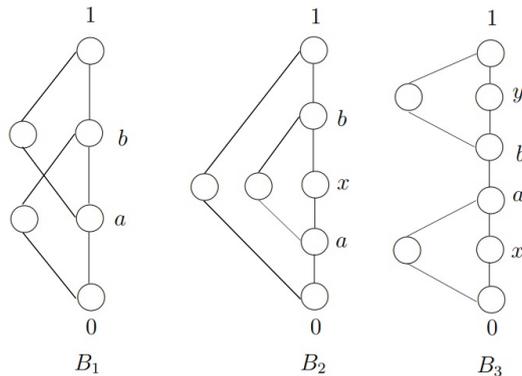


Fig. 1. Basic blocks containing four comparable reducible elements and having nullity two

Proposition 3.1. [1] *If B is the basic block associated to $\mathbf{B} \in \mathcal{B}(j; 4, 2)$ then $B \in \{B_1, B_2, B_3\}$ (see Figure 1).*

Recall that, if L is a lattice of nullity $k \geq 1$ then $2 \leq |Red(L)| \leq 2k$. Therefore, if L is any lattice of nullity 2 then $2 \leq |Red(L)| = r \leq 4$. Also, if $L \in \mathcal{L}(n; 4, 2, h)$ then $3 \leq h \leq 5$. Now firstly, we count the classes of all non-isomorphic maximal blocks $\mathcal{B}(j; 4, 2, h)$ for $3 \leq h \leq 5$ and secondly, we count the classes $\mathcal{L}(n; 4, 2, h)$ for $3 \leq h \leq 5$, and thereby we count the class $\mathcal{L}(n; 4, 2)$. Note that $\mathcal{B}(j; 4, 2) = \mathcal{B}(j; 4, 2, 3) \dot{\cup} \mathcal{B}(j; 4, 2, 4) \dot{\cup} \mathcal{B}(j; 4, 2, 5)$.

Let $p(n, k)$ denote the number of weak compositions of integer n into k non-negative parts. In fact $p(n, k)$ is the number of all non-negative unordered solutions of the equation $x_1 + x_2 + \dots + x_k = n$. $p(n, k)$ is also the number of ways of distributing n (identical) objects into k boxes. Further $p(n, k) = \binom{n+k-1}{k}$.

Now in the following result, we count the class $\mathcal{B}(j; 4, 2, 3)$.

Proposition 3.2. *For $j \geq 6$,*

$$|\mathcal{B}(j; 4, 2, 3)| = \binom{j-2}{4} = \sum_{l=1}^{j-5} \sum_{s=1}^{j-l-4} \sum_{r=1}^{j-s-l-3} (j-s-l-r-2).$$

Proof. Let $\mathbf{B} \in \mathcal{B}(j; 4, 2, 3)$. Let $0 < a < b < 1$ be the reducible elements of \mathbf{B} . Let B be the basic block associated to \mathbf{B} . Then $B = B_1$ (see Figure 1), and hence by Theorem 1.8, an adjunct representation of B is given by $B = C]_0^b\{c_1\}]_a^1\{c_2\}$, where $C : 0 \prec a \prec b \prec 1$ is a 4-chain. Also by Theorem 1.9, $Red(B) = Red(\mathbf{B})$ and $\eta(B) = \eta(\mathbf{B}) = 2$. Therefore an adjunct representation of \mathbf{B} is given by $\mathbf{B} = C_0]_0^b C_1]_a^1 C_2$, where C_0 is a maximal chain containing all the reducible elements of \mathbf{B} , and C_1, C_2 are chains. Let $s = |\mathbf{B} \cap [0, a)| = |C_0 \cap [0, a)|$, $m = |\mathbf{B} \cap [a, b)| = |C_0 \cap [a, b)|$, $t = |\mathbf{B} \cap [b, 1]| = |C_0 \cap [b, 1]|$, $l = |\mathbf{B} \cap C_1|$, and $r = |\mathbf{B} \cap C_2|$. Let $S = \{(s, m, t, l, r) | s, m, t, l, r \in \mathbb{N}, s + m + t + l + r = j, t \geq 2\}$. Now given any $\mathbf{B} \in \mathcal{B}(j; 4, 2, 3)$ there exists unique $(s, m, t, l, r) \in S$ and vice-versa. Also for $\mathbf{B}' \in \mathcal{B}(j; 4, 2, 3)$, suppose there exists $(s', m', t', l', r') \in S$. Then $\mathbf{B} \cong \mathbf{B}'$ if and only if $(s, m, t, l, r) = (s', m', t', l', r')$. Therefore the set S is equivalent to the class $\mathcal{B}(j; 4, 2, 3)$, and hence $|S| = |\mathcal{B}(j; 4, 2, 3)|$. Now let $S' = \{(x_1, x_2, x_3, x_4, x_5) | x_i \in \mathbb{N} \cup \{0\}, 1 \leq i \leq 5, x_1 + x_2 + x_3 + x_4 + x_5 = j - 6\}$. Then $|S| = |S'|$. But $|S'|$ is the number of all unordered non-negative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = j - 6$, which is given by $p(j - 6, 5) = \binom{(j-6)+5-1}{5} = \binom{j-2}{4}$.

If we take p as $m + t$ then in the following, we obtain an alternate formula for $|\mathcal{B}(j; 4, 2, 3)|$.

For fixed s, l, r , there are $p - 2 = j - s - l - r - 2$ choices for b in \mathbf{B} up to isomorphism. That is, for

fixed s, l, r , there are $p - 2 = j - s - l - r - 2$ non-isomorphic maximal blocks in $\mathcal{B}(j; 4, 2, 3)$. Now for fixed $s, l, 1 \leq r = j - s - p - l \leq j - s - l - 3$, since $p \geq 3$. Therefore there are $\sum_{r=1}^{j-s-l-3} (j - s - l - r - 2)$ maximal blocks in $\mathcal{B}(j; 4, 2, 3)$ up to isomorphism. Again for fixed $l, 1 \leq s = j - p - l - r \leq j - l - 4$, since $p \geq 3, r \geq 1$, and there are $\sum_{s=1}^{j-l-4} \sum_{r=1}^{j-s-l-3} (j - s - l - r - 2)$ maximal blocks in $\mathcal{B}(j; 4, 2, 3)$ up to isomorphism. Further $1 \leq l = j - s - p - r \leq j - 5$, since $s \geq 1, p \geq 3, r \geq 1$, and there are $\sum_{l=1}^{j-5} \sum_{s=1}^{j-l-4} \sum_{r=1}^{j-s-l-3} (j - s - l - r - 2)$ maximal blocks in $\mathcal{B}(j; 4, 2, 3)$ up to isomorphism. \square

Following result is due to Bhavale and aware [2].

Lemma 3.3. [2] For the integers $m \geq 4$ and $0 \leq k \leq m - 4$,

$$|\mathcal{B}(m; 2, k + 1)| = P_{m-2}^{k+2}.$$

Using Lemma 3.3, we have the following result.

Proposition 3.4. For $n \geq 6$,

$$|\mathcal{L}'(n; 2, 1)| = \sum_{l=2}^{n-4} (l - 1) P_{n-l-2}^2.$$

Proof. Let $L \in \mathcal{L}'(n; 2, 1)$. Then $L = C \oplus \mathbf{B} \oplus C'$ where C, C' are chains with $|C| \geq 1, |C'| \geq 1, |C| + |C'| = l \geq 2$, and $\mathbf{B} \in \mathcal{B}(m; 2, 1)$ with $m = n - l \geq 4$. For fixed l , by Lemma 3.3, there are P_{n-l-2}^2 maximal blocks of type \mathbf{B} in $\mathcal{B}(n - l; 2, 1)$ up to isomorphism. Also l elements out of n elements can be distributed over the chains C', C'' in $(l - 2) + 1 = l - 1$ ways up to isomorphism. Therefore for fixed l there are $(l - 1) \times P_{n-l-2}^2$ lattices of type L in $\mathcal{L}'(n; 2, 1)$ up to isomorphism. Further, $2 \leq l = n - m \leq n - 4$, since $m \geq 4$. Therefore there are $\sum_{l=2}^{n-4} (l - 1) P_{n-l-2}^2$ lattices of type L in $\mathcal{L}'(k; 2, 1)$ up to isomorphism. \square

Now we count the class $\mathcal{B}(j; 4, 2, 4)$ using Proposition 3.4 in the following.

Proposition 3.5. For $j \geq 7$,

$$|\mathcal{B}(j; 4, 2, 4)| = \sum_{i=1}^{j-6} \sum_{l=2}^{j-i-4} (l - 1) P_{j-i-l-2}^2.$$

Proof. Let $\mathbf{B} \in \mathcal{B}(j; 4, 2, 4)$. Let $0 < a < b < 1$ be the reducible elements of \mathbf{B} . Let B be the basic block associated to \mathbf{B} . Then $B = B_2$ (see Figure 1), and hence by Theorem 1.8, an adjunct representation of B is given by $B = C]_a^b \{c_1\}]_0^1 \{c_2\}$, where $C : 0 < a < x < b < 1$ is a 5-chain with $x \in \text{Irr}(\mathbf{B} \cap C)$. Also by Theorem 1.9, $\text{Red}(B) = \text{Red}(\mathbf{B})$ and $\eta(B) = \eta(\mathbf{B}) = 2$. Therefore an adjunct representation of \mathbf{B} is given by $\mathbf{B} = C_0]_a^b C_1]_0^1 C_2$, where C_0 is a maximal chain containing all the reducible elements of \mathbf{B} , and C_1, C_2 are chains. Moreover $\mathbf{B} = L]_0^1 C_2$, where $L \in \mathcal{L}'(k; 2, 1)$ with $k \geq 6$ and $|C_2| = i \geq 1$ with $j = k + i$. As $L \in \mathcal{L}'(k; 2, 1)$, $L = C' \oplus \mathbf{B}' \oplus C''$, where C', C'' are chains with $|C'| \geq 1, |C''| \geq 1, |C'| + |C''| = l \geq 2$, and $\mathbf{B}' \in \mathcal{B}(m; 2, 1)$ where $m = k - l = j - i - l \geq 4$.

For fixed i , there are $|\mathcal{L}'(j-i; 2, 1)|$ maximal blocks of type \mathbf{B} in $\mathcal{B}(j; 4, 2, 4)$ up to isomorphism. Further $1 \leq i = j - k \leq j - 6$, since $k = l + m \geq 6$. Therefore there are $\sum_{i=1}^{j-6} |\mathcal{L}'(j-i; 2, 1)|$ maximal blocks of type \mathbf{B} in $\mathcal{B}(j; 4, 2, 4)$ up to isomorphism. Thus proof follows from Proposition 3.4. \square

In the following result, we count the class $\mathcal{B}(j; 4, 2, 5)$.

Proposition 3.6. For $j \geq 8$,

$$|\mathcal{B}(j; 4, 2, 5)| = \sum_{m=0}^{j-8} \sum_{s=4}^{j-m-4} P_{s-2}^2 P_{j-m-s-2}^2.$$

Proof. Let $\mathbf{B} \in \mathcal{B}(j; 4, 2, 5)$. Let $0 < a < b < 1$ be the reducible elements of \mathbf{B} . Let B be the basic block associated to \mathbf{B} . Then $B = B_3$ (see Figure 1), and hence by Theorem 1.8, an adjunct representation of B is given by $B = C|_0^a \{c_1\}|_b^1 \{c_2\}$, where $C : 0 \prec a \prec x \prec b \prec y \prec 1$ is a 6-chain. Also by Theorem 1.9, $Red(B) = Red(\mathbf{B})$ and $\eta(B) = \eta(\mathbf{B}) = 2$. Therefore an adjunct representation of \mathbf{B} is given by $\mathbf{B} = C_0|_0^a C_1|_b^1 C_2$, where C_0 is a maximal chain containing all the reducible elements of \mathbf{B} , and C_1, C_2 are chains. Let $s = |\mathbf{B} \cap [0, a]| = |C_0 \cap [0, a]|$, $m = |\mathbf{B} \cap (a, b)| = |C_0 \cap (a, b)|$, $t = |\mathbf{B} \cap [b, 1]| = |C_0 \cap [b, 1]|$. Observe that $\mathbf{B} = \mathbf{B}' \oplus C \oplus \mathbf{B}''$, where $\mathbf{B}' \in \mathcal{B}(s; 2, 1)$, $s \geq 4$, $\mathbf{B}'' \in \mathcal{B}(t; 2, 1)$, $t \geq 4$, C is a chain with $|C| = m \geq 0$, and $s + m + t = j \geq 8$.

Now for fixed s, m there are $|\mathcal{B}(s; 2, 1)| \times |\mathcal{B}(t; 2, 1)|$, where $t = j - m - s$ maximal blocks of type \mathbf{B} in $\mathcal{B}(j; 4, 2, 5)$ up to isomorphism. Now for fixed m , $4 \leq s = j - m - t \leq j - m - 4$, since $s \geq 4$. Therefore there are $\sum_{s=4}^{j-m-4} (|\mathcal{B}(s; 2, 1)| \times |\mathcal{B}(j-m-s; 2, 1)|)$ maximal blocks of type \mathbf{B} in $\mathcal{B}(j; 4, 2, 5)$ up to isomorphism. Further $0 \leq m = j - s - t \leq j - 8$, since $s \geq 4, t \geq 4$. Thus there are $\sum_{m=0}^{j-8} \left(\sum_{s=4}^{j-m-4} (|\mathcal{B}(s; 2, 1)| \times |\mathcal{B}(j-m-s; 2, 1)|) \right)$ non-isomorphic maximal blocks of type \mathbf{B} in $\mathcal{B}(j; 4, 2, 5)$ to isomorphism. Thus proof follows from Lemma 3.3. \square

Using Proposition 3.2, we have the following result.

Corollary 3.7. For $n \geq 6$,

$$|\mathcal{L}(n; 4, 2, 3)| = \sum_{i=0}^{n-6} (i+1) \binom{n-i-2}{4}.$$

Proof. Let $L \in \mathcal{L}(n; 4, 2, 3)$. Then $L = C \oplus \mathbf{B} \oplus C'$, where $\mathbf{B} \in \mathcal{B}(j; 4, 2, 3)$ and C, C' are chains with $|C| + |C'| = i = n - j \geq 0$. Note that $0 \leq i = n - j \leq n - 6$, since $j \geq 6$. Now remaining $n - j = i$ elements can be distributed over the chains C and C' in $i + 1$ ways up to isomorphism. Therefore by Proposition 3.2,

$$|\mathcal{L}(n; 4, 2, 3)| = \sum_{i=0}^{n-6} (i+1) \times |\mathcal{B}(n-i; 4, 2, 3)| = \sum_{i=0}^{n-6} (i+1) \binom{n-i-2}{4}.$$

\square

Similarly, using Proposition 3.5, we have the following result.

Corollary 3.8. For $n \geq 7$,

$$|\mathcal{L}(n; 4, 2, 4)| = \sum_{i=0}^{n-7} \sum_{p=1}^{n-i-6} \sum_{l=2}^{n-i-p-4} (i+1)(l-1)P_{n-i-p-l-2}^2.$$

Also, using Proposition 3.6, we have the following result.

Corollary 3.9. For $n \geq 8$,

$$|\mathcal{L}(n; 4, 2, 5)| = \sum_{i=0}^{n-8} \sum_{m=0}^{n-i-8} \sum_{s=4}^{n-i-m-4} (i+1)P_{s-2}^2 P_{n-i-m-s-2}^2.$$

Using Corollary 3.7, Corollary 3.8, and Corollary 3.9, we have for $n \geq 6$,

$$|\mathcal{L}(n; 4, 2)| = \sum_{h=3}^5 |\mathcal{L}(n; 4, 2, h)|,$$

and hence we have the following main result.

Theorem 3.10. For $n \geq 6$,

$$\begin{aligned} |\mathcal{L}(n; 4, 2)| &= \sum_{i=0}^{n-6} (i+1) \binom{n-i-2}{4} + \sum_{i=0}^{n-7} \sum_{p=1}^{n-i-6} \sum_{l=2}^{n-i-p-4} (i+1)(l-1)P_{n-i-p-l-2}^2 \\ &\quad + \sum_{i=0}^{n-8} \sum_{m=0}^{n-i-8} \sum_{s=4}^{n-i-m-4} (i+1)P_{s-2}^2 P_{n-i-m-s-2}^2. \end{aligned}$$

Conclusion

Bhavale and Aware [2] proved that, if a lattice contains $r \leq 3$ reducible elements then they are all comparable, but the result is not true for $r \geq 4$. However, there is no lattice containing four reducible elements and having nullity two such that not all the reducible elements are comparable. Therefore the counting of all non-isomorphic lattices containing four (comparable) reducible elements and having nullity two accomplishes the counting of all non-isomorphic lattices having nullity up to two.

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