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Multidecomposition of hypercube graphs into paths, cycles and stars

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ABSTRACT

Let P_{n+1}, C_n and S_n represent a path, cycle, and star with n edges, Q_n denote the n-dimensional hypercube graph. The $(\mathscr{H}_1, \mathscr{H}_2)-$multidecomposition of G for graphs $\mathscr{H}_1$, $\mathscr{H}_2$, and G is a decomposition of G into copies of $\mathscr{H}_1$ and $\mathscr{H}_2$, where there is at least one copy of $\mathscr{H}_1$ and at least one copy of $\mathscr{H}_2$. In this paper, we prove that the graph Q_n is $(S_{n-2}, C_4)-$multidecomposable for $n \geq 4$ and $(S_{n-4}, P_5)-$multidecomposable for $n \geq 5$.

Keywords: cycle, hypercube graph, multidecomposition, path, star

1. Introduction

The finite, simple and undirected graphs with no loops are considered in this paper. The readers has to refer [5] for better understanding about the terms of standard graph theory. Let P_{n+1} represent the path with n edges. Let C_n represent the cycle with n edges. S_n represents the complete bipartite graph $K_{1,n}$, which is referred to a star. The $n-$dimensional hypercube, represented by the graph Q_n, is a graph whose vertices are 2^n-tuples of $0's$ and $1's$, and whose edge set $E(Q_n)$ is made up of pairs of vertices that differ in only one coordinate. Whether the tuple contains an odd or even number of non-zero coordinates determines whether it is categorized as odd or even.

An edge-disjoint subgraph of a graph G is called a *decomposition* of G if every edge of G is exactly in one $\mathscr{H}_i$. These subgraphs are $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$. G is *decomposed* or *decomposable* in $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$. The decomposition of G is denoted by $\Psi = \{\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n\}$. It is also said that G admits a $\{\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n\}$-decomposition. If each $\mathscr{H}_i \cong \mathscr{H}$, then it is said that G has a $\mathscr{H}-$*decomposition.*

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A $(\mathscr{H}_1, \mathscr{H}_2)-$*multidecomposition* of G is a partitioning of the edge set of G into multiple copies of $\mathscr{H}_1$ and multiple copies of $\mathscr{H}_2$, where there is at least one copy of $\mathscr{H}_1$ and at least one copy of $\mathscr{H}_2$. G has a $\{p\mathscr{H}_1, q\mathscr{H}_2\}-$*decomposition* if its decomposition consists of p copies and q copies of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. G should be *fully* $\{\mathscr{H}_1, \mathscr{H}_2\}-$*decomposable* or if a similar decomposition occurs for all values of p and q that satisfy the necessary requirements, then it has a $\{\mathscr{H}_1, \mathscr{H}_2\}_{\{p,q\}}-$*decomposition*.

Kringel [19] first demonstrated that the graph Q_n has a Hamiltonian cycle decomposition for every even n if $n \geq 2$ and n is a power of 2. Ringle next asked whether Q_n has a Hamiltonian decomposition. Aubert and Schneider [4] implicitly resolved Ringel's question, and for explicit statement refer Alspach et al. in [2]. Apart from Hamiltonian cycle decomposition, in [7] Horak showed that any graph H of size n whose blocks are C_n, n is even, or Q_n is decomposed by an edge. It has been evidenced that certain $n-$cycle's edge sets of Q_n are fundamental sets in [15] by Ramras. Mollard and Ramras established in [14] that the edge sets of $2n-$cycles in Q_n serve as fundamental sets. It has been shown that Q_n admits a path decomposition of length m for odd n with certain conditions by Anick and Ramras [3]. Wagner and Wild [20] demonstrated that the hypercube Q_n decomposed into isomorphic copies of trees, each of which contains n edges.

As demonstrated by Shyu in [17], the complete graph K_n allows a decomposition into p copies of P_5 and q copies of C_4, given the necessary and sufficient conditions. The necessary and sufficient conditions for the $\{C_4, E_2\}-$ decomposition of the cartesian product and tensor product of paths, cycles, and complete graphs were obtained by Abueida and Devan in [1]. Some necessary and sufficient conditions for the existence of $\{P_{k+1}, C_k\}_{\{p,q\}}-$decomposition of K_n and $K_{m,n}$ were obtained by Jeevadoss and Muthusamy [10]. The above decomposition was extended to complete bipartite multigraphs $K_{m,n}(\lambda)$ in [11]. The necessary and sufficient conditions for the existence of $\{P_5, C_4\}_{\{p,q\}}-$decomposition of tensor product and cartesian product of complete graphs were obtained by Jeevadoss and Muthusamy in [12]. Lee and Chen [13] determined the criteria required for a complete graph to be decomposed into p copies of Hamiltonian paths (cycles) and q copies of S_3. Ilayaraja and Muthusamy [9] explored the conditions required for decomposing complete bipartite graphs into cycles and stars having four-edge, establishing the existence of a $\{pC_4, qS_4\}-$ decomposition of $K_{m,n}$. Recently, Shyu [18] proved the decomposition of K_r and $K_{r,s}$ into $P_5{}'s$, $C_4{}'s$ and $S_4{}'s$. In [16], Saranya and Jeevadoss demonstrated that the graph Q_n is $\{P_5, C_4\}_{\{p,q\}}-$decomposable. In [6], Chaadhanaa and Hemalatha explore the conditions that are both necessary and sufficient for the existence of a $\{P_5, Y_5\}-$ multi-decomposition of K_n and $K_{m,n}$. In [8], Ilayaraja and Muthusamy determined the criteria required for the existence of a $\{pP_5, qS_4\}-$decomposition of K_n, providing both necessary and sufficient conditions.

In this paper, we prove that the graph Q_n is $(S_{n-2}, C_4)-$multidecomposable for $n \geq 4$ and $(S_{n-4}, P_5)-$multidecomposable for $n \geq 5$.

2. Preliminaries

We represent the graph Q_n as a $n-$regular bipartite graph. As illustrated in Figure 1, for instance, we describe the graph Q_4 as a 4-regular bipartite graph. The 4-regular bipartite graph has vertex bipartition $(\mathscr{X}, \mathscr{Y})$ such that $\mathscr{X} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ and $\mathscr{Y} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$.

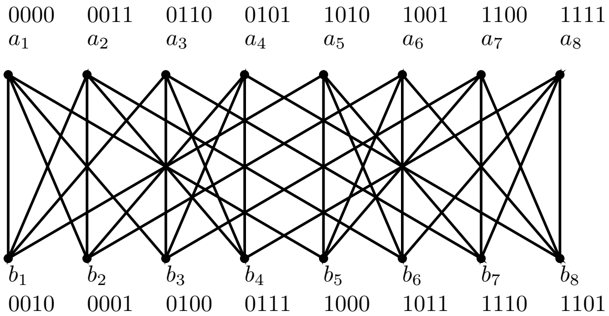

Fig. 1. Q_4 as 4−regular bipartite graph

We represent the graph Q_5 into 4 copies of Q_3 and edges between Q_3 as shown in Figure 2.

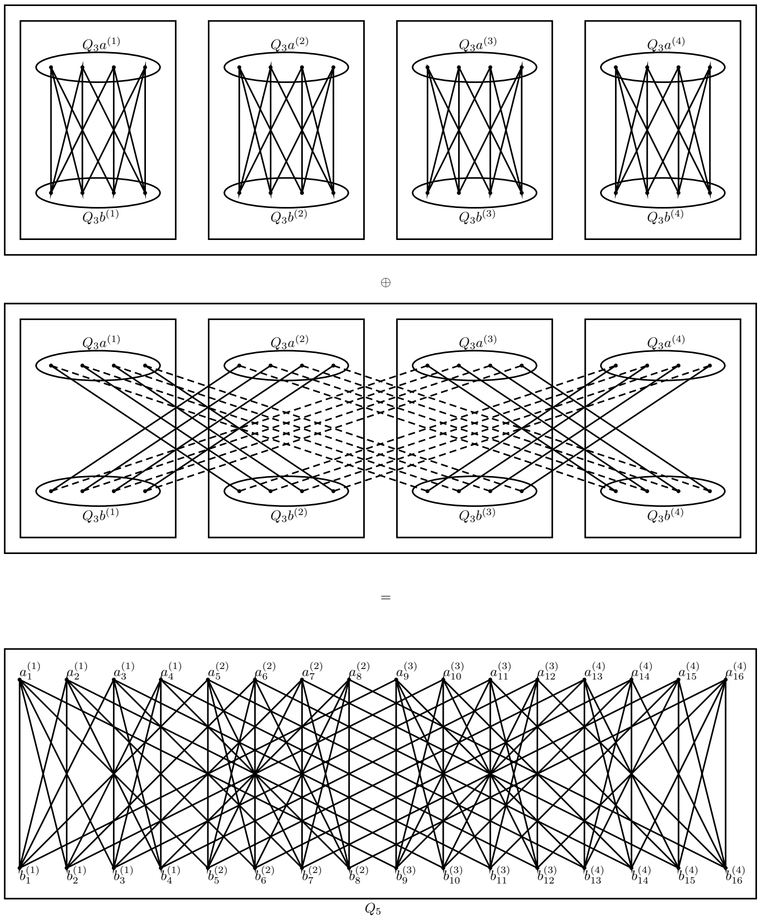

Fig. 2. The graph Q_5

The edges between Q_3 as shown in Figure 3.

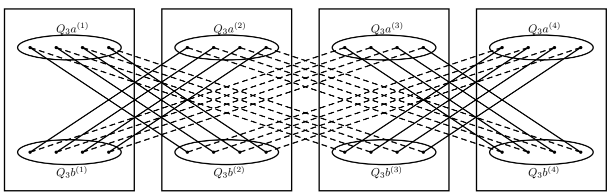

Fig. 3. 2−regular bipartite graph $N_{16,16}$

Similarly the graph Q_n is decomposed into 4 copies of Q_{n-2} and edges between 4 copies of Q_{n-2}.

The edges between 4 copies of Q_{n-2} of Q_n is denoted by a 2−regular bipartite graph $N_{2^{n-1},2^{n-1}}$ as shown in Figure 4. The graph Q_n can be written as $Q_n = 4Q_{n-2} \oplus N_{2^{n-1},2^{n-1}}$.

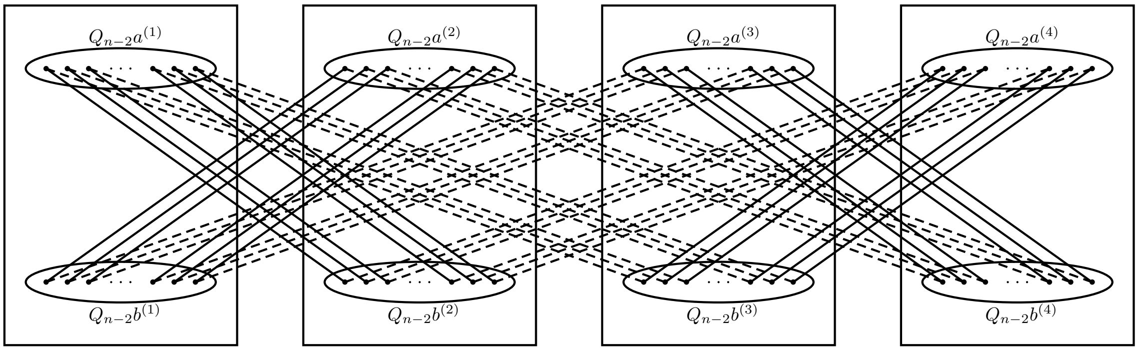

Fig. 4. 2−regular bipartite graph $N_{2^{n-1},2^{n-1}}$

The partite sets $\mathscr{X}$ and $\mathscr{Y}$ are each divided into four equal subsets, labeled as $\mathscr{X}a^{(1)}$, $\mathscr{X}a^{(2)}$, $\mathscr{X}a^{(3)}$, $\mathscr{X}a^{(4)}$ and $\mathscr{Y}b^{(1)}$, $\mathscr{Y}b^{(2)}$, $\mathscr{Y}b^{(3)}$, $\mathscr{Y}b^{(4)}$. The adjacency relations between these subsets are as follow: The vertices in $\mathscr{X}a^{(1)}$ are adjacent to those in $\mathscr{Y}b^{(2)}$ and $\mathscr{Y}b^{(3)}$. The vertices in $\mathscr{X}a^{(2)}$ are adjacent to those in $\mathscr{Y}b^{(1)}$ and $\mathscr{Y}b^{(4)}$. The vertices in $\mathscr{X}a^{(3)}$ are adjacent to those in $\mathscr{Y}b^{(1)}$ and $\mathscr{Y}b^{(4)}$. The vertices in $\mathscr{X}a^{(4)}$ are adjacent to those in $\mathscr{Y}b^{(2)}$ and $\mathscr{Y}b^{(3)}$.

Remark 2.1. $\mathscr{G} \oplus \mathscr{H}$ has a similar decomposition if $\mathscr{G}$ and $\mathscr{H}$ have $\{S_{n-2}, C_4\}$− and $\{S_{n-4}, P_5\}$−multidecomposition.

Here we use the following construction.

Construction 1. Let ${C_4}^{(1)} = (a_1a_2a_3a_4)$ and ${C_4}^{(2)} = (b_1b_2b_3b_4)$ be two 4-length cycles that share a single common vertex x, such that $a_1 = b_1 = x$. It is assumed that each cycle has at least one neighbor of x that is not shared with the other cycle (*e.g.*, $a_4 \notin {C_4}^{(2)}$, $b_2 \notin {C_4}^{(1)}$). This setup allows to construct two edge-disjoint paths of length 4. Let ${P_5}^{(1)}$ and ${P_5}^{(2)}$ be created from ${C_4}^{(1)}$ and ${C_4}^{(2)}$, as shown in Figure 5, where ${P_5}^{(1)} = {C_4}^{(1)} - xa_4 \cup xb_2$, ${P_5}^{(2)} = {C_4}^{(2)} - xb_2 \cup xa_4$.

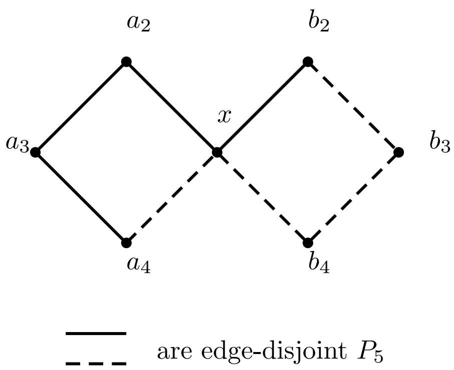

Fig. 5. Decomposition of $2C_4$ into $2P_5$

3. Main theorem

The existence of (S_{n-2}, C_4)−multidecomposition and (S_{n-4}, P_5)−multidecomposition of the graph Q_n is represented in this section. The following lemmas are useful for our main theorem.

Lemma 3.1. *Let n be a positive integer and $n \geq 3$. Then the graph $N_{2^{n-1},2^{n-1}}$ is C_4 decomposable.*

Proof. The graph $N_{2^{n-1},2^{n-1}}$ has two partite sets $\mathscr{X}$ and $\mathscr{Y}$, the corresponding labels are $\mathscr{X} = (a_1, a_2, a_3, \cdots, a_{2^{n-1}})$ and $\mathscr{Y} = (b_1, b_2, b_3, \cdots, b_{2^{n-1}})$. The following construction decomposes the bipartite graph $N_{2^{n-1},2^{n-1}}$ into 2^{n-2} copies of C_4.

$$(a_i\ b_{2^{n-3}+i}\ a_{3*2^{n-3}+i}\ b_{2^{n-2}+i}\ a_i),$$

$$(b_i\ a_{2^{n-3}+i}\ b_{3*2^{n-3}+i}\ a_{2^{n-2}+i}\ b_i),$$

for $i = 1, 2, \cdots, 2^{n-3}$. □

Lemma 3.2. *Let m be a positive integer and $m \geq 2$. Then the graph Q_m is S_m decomposable.*

Proof. The graph Q_m is represented as $m-$regular bipartite graph. There are 2^{m-1} vertices in each partite sets X and Y. Each vertices in X partite set is adjacent to exactly m distinct vertices in Y partite set. Hence the $m-$dimensional hypercube Q_m is decomposed into 2^{m-1} copies of S_m. □

Now we prove the main result of this section

Theorem 3.3. *Let n be a positive integer and $n \geq 3$. Then the graph Q_n has a $(S_{n-2}, C_4)-$ multidecomposition.*

Proof. The graph Q_n can be viewed as $4Q_{n-2} \oplus N_{2^{n-1},2^{n-1}}$. By Lemma 3.2, the graph Q_{n-2} is S_{n-2} decomposable and by Lemma 3.1, the graph $N_{2^{n-1},2^{n-1}}$ is C_4 decomposable. Hence the graph Q_n for $n \geq 3$ is $\{2^{n-1}S_{n-2}, 2^{n-2}C_4\}-$multidecomposable.

□

Theorem 3.4. *Let n be a positive integer and $n \geq 5$. Then the graph Q_n is $(S_{n-4}, P_5)-$ multidecomposable.*

Proof.

Represent the graph Q_n into 4 copies of Q_{n-2} and $2-$regular bipartite graph $N_{2^{n-1},2^{n-1}}$ as shown in Figure 3. Represent the graph Q_{n-2} into 4 copies of Q_{n-4} and $2-$regular bipartite graph $N_{2^{n-3},2^{n-3}}$. The graph Q_n can be written as $Q_n = N_{2^{n-1},2^{n-1}} \oplus 4(N_{2^{n-3},2^{n-3}}) \oplus 16Q_{n-4}$. By Lemma 3.2, Q_{n-4} is S_{n-4} decomposable. The $4-$regular bipartite graph $N_{2^{n-1},2^{n-1}} \oplus 4N_{2^{n-3},2^{n-3}}$ is decomposed into 2^{n-1} copies of C_4, since the $2-$regular bipartite graph $N_{2^{n-1},2^{n-1}}$ is C_4 decomposable and the $2-$regular bipartite graph $N_{2^{n-3},2^{n-3}}$ is C_4 decomposable.

Each vertex in the $\mathscr{X}$ partite set is a common vertex of exactly $2C_4$s as shown in Figure 6, since the $4-$regular bipartite graph $N_{2^{n-1},2^{n-1}} \oplus 4(N_{2^{n-3},2^{n-3}})$ is decomposed into 2^{n-1} copies of C_4 and these cycles are edge-disjoint cycles. The common vertex has four neighbors, with two belonging to one 4-cycle and the other two to another 4-cycle. Since at least one neighbor of a common vertex is not shared with the other cycle, by Construction 1 these cycles are decomposed into same copies of P_5. Hence the graph Q_n for $n \geq 5$ is $\{2^{n-1}S_{n-4}, 2^{n-1}P_5\}-$multidecomposable.

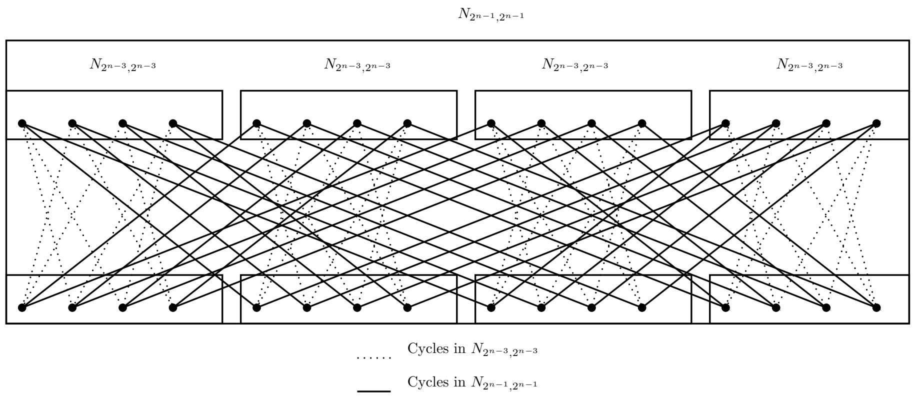

Fig. 6. $N_{2^{n-1},2^{n-1}} \oplus 4(N_{2^{n-3},2^{n-3}})$

□

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