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On multiset dimension of cylindrical graphs

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ABSTRACT

Let $G = (V, E)$ be a simple connected graph and $W \subseteq V$. For $v \in V$, the representation multiset or m-code of v is the multiset $r_m(v) = \{d(v, w) \mid w \in W\}$. If no two vertices in G have equal m-codes, then W is called an m-resolving set of G. The multiset dimension md(G) of G is the minimum possible cardinality of an m-resolving set of G, if such a set exists. If G does not possess an m-resolving set, then we say that G has infinite multiset dimension. In this paper, we show that all cylindrical graphs $P_m \square C_n$, where $m, n \geq 3$, have finite multiset dimension. In particular, we show that md$(P_m \square C_n) \leq 4$ if $m \geq 6$ and $n \geq 3$, or if $m \geq 3$ and $n \geq 12$. Moreover, if $m \geq 3$ and $n \geq 8m + 1$, we show that $P_m \square C_n$ has multiset dimension 3.

Keywords: m-resolving set, ID-coloring, multiset dimension, cylindrical graphs

1. Introduction

Let $G = (V, E)$ be a connected graph. The distance $d(u, v)$ between vertices u and v in G is the length of a shortest path that joins u and v. A subset $W = \{w_1, w_2, \ldots, w_k\} \subseteq V$ is said to be a *resolving set* of G if

$$(d(u, w_1), d(u, w_2), \ldots, d(u, w_k)) \neq (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

for any pair of distinct vertices u and v of G. The smallest cardinality of a resolving set W of G is called the *metric dimension* of G. The concepts of resolving set and metric dimension were introduced independently by Slater [14], and Harary and Melter [3]. For works on these concepts and related topics, refer to the survey articles [9] and [15].

Saenpholphat [12] and Simanjuntak et al. [13] independently introduced a stronger version of a resolving set called an m-resolving set. Given a connected graph $G = (V, E)$ and a vertex $v \in V$,

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the *representation multiset* or *m-code* $r_m(v|W)$ (or simply $r_m(v)$) of v with respect to W is defined to be the multiset of distances between v and the vertices in W; that is $r_m(v) = \{d(v,w) \mid w \in W\}$. If no two vertices in G have equal m-codes, then W is called an *m-resolving set* of G. Hence, $W = \{w_1, w_2, \ldots, w_k\}$ is an m-resolving set of G if and only if

$$\{d(u,w_1), d(u,w_2), \ldots, d(u,w_k)\} \neq \{(d(v,w_1), d(v,w_2), \ldots, d(v,w_k)\},$$

for any pair of distinct vertices u and v of G.

The *multiset dimension* $\mathrm{md}(G)$ of G is the minimum possible cardinality of an m-resolving set of G, if such a set exists. If G does not possess an m-resolving set, then we say that G has *infinite multiset dimension*; that is, $\mathrm{md}(G) = \infty$. If $\mathrm{md}(G)$ is finite, then an m-resolving set with cardinality $\mathrm{md}(G)$ is called a *multiset basis* of G.

An m-resolving set of a graph G induces a vertex ID-coloring [1] of G. Let c be a red-white coloring of a graph G that has diameter d. For each vertex v, we construct the d-vector $\vec{d}(v) = (a_1, a_2, ..., a_d)$, where for each $i \in \{1, 2, ..., d\}$, a_i is equal to the number of red vertices of distance i from v. Then c is called an *ID-coloring* of G if $\vec{d}(v) \neq \vec{d}(w)$ for any pair of distinct vertices v, w. Hence, c is an ID-coloring if and only if the set of vertices colored red is m-resolving.

Various studies on m-resolving sets and multiset dimension of graphs have appeared recently [2, 4, 5, 6, 7, 11, 10].

Some of the known results that are used in the study are enumerated below.

Proposition 1.1 ([1, 12, 13]). *No connected graph has multiset dimension 2.*

Theorem 1.2 ([1, 12, 13]). *A nontrivial connected graph G has $\mathrm{md}(G) = 1$ if and only if G is a path.*

For the next result, let P_n be the path $(0, 1, 2, \ldots, n-1)$ of order $n \geq 4$. We define a *symmetric* subset W of $V(P_n)$ to be one with the property $i \in W$ if and only if $n-1-i \in W$, for each $i \in V(P_n)$.

Lemma 1.3 ([10]). *Let $n \geq 4$. If $W \subset V(P_n)$ contains 0 and $n-1$ and is not symmetric, then W is an m-resolving set of P_n.*

Theorem 1.4 ([11]). *Let $n \geq 9$ and G be the cycle C_n with $V(G) = \mathbb{Z}_n := \{0, 1, ..., n-1\}$ and $E(G)$ consisting of edges $\{i, i+1\}$, for all i, $0 \leq i \leq n-1$, with addition done modulo n. Then $W = \{0, \lfloor n/4 \rfloor, 2\lfloor n/4 \rfloor + 1\}$ is an m-resolving set of G.*

Now, let W be a subset of V with cardinality k. For any $v \in V$, we can order the elements of $r_m(v)$ as $\mathscr{O} = (d_1(v), d_2(v), \ldots, d_k(v))$, where $d_i(v) \leq d_{i+1}(v)$, for $i = 1, 2, \ldots, k-1$. For convenience, we also define $\mathrm{sum}_\ell(v) = \sum_{i=1}^{\ell} d_i(v)$; that is, $\mathrm{sum}_\ell(v)$ is the sum of the first ℓ elements of $\mathscr{O}$. When $\ell = k$, we write $\mathrm{sum}(v)$ for $\mathrm{sum}_k(v)$. Consequently, we have the following observation, which is an extension of Observation 1 in [11].

Observation 1.5. *Let $G = (V, E)$ be a connected graph and W a subset of V with cardinality k, and $1 \leq \ell \leq k$. If $u, v \in V$ with $\mathrm{sum}_\ell(u) \neq \mathrm{sum}_\ell(v)$, then $r_m(u) \neq r_m(v)$.*

2. Preliminaries

The next observation gives a sufficient condition for the m-code of a vertex in a subgraph to be equal to the m-code of the same vertex in a larger graph. We use the notation $d_G(u,v)$ for the distance $d(u,v)$ in the graph G.

Observation 2.1 ([11]). *Let $G=(V,E)$ be a connected graph, $W \subseteq V(G)$, and H a subgraph of G that satisfies the following:*

1. $W \subseteq V(H)$
2. $d_G(u,w) = d_H(u,w)$ *for any* $u \in V(H)$ *and* $w \in W$.

Then the m*-code of each* $u \in V(H)$ *in* H *is equal to the* m*-code of* u *in* G.

Let $m \geq 2$ and $n \geq 3$ be positive integers, and $G = P_m \,\square\, C_n$. Suppose

$$\begin{aligned} V(G) &= \{(i,j) \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}, \text{ and} \\ E(G) &= \{\{(i,j),(i,j+1)\} \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \\ &\quad \cup \{\{(i,j),(i+1,j)\} \mid 0 \leq i \leq m-2, 0 \leq j \leq n-1\}. \end{aligned}$$

For $0 \leq i \leq m-1$, we denote by $C_n(i)$ the cycle induced by $\{(i,j) \mid 0 \leq j \leq n-1\}$. Similarly, for $0 \leq j \leq n-1$, we denote by $P_m(j)$ the path induced by $\{(i,j) \mid 0 \leq i \leq m-1\}$. Figure 1 shows the graph $P_3 \,\square\, C_5$ with the labels of the vertices.

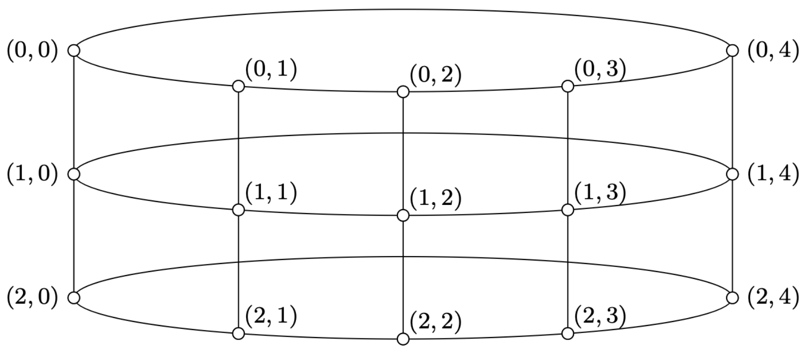

Fig. 1. The cylindrical graph $P_3 \,\square\, C_5$

Kono and Zhang [8] showed that prisms $P_2 \,\square\, C_n$ have a finite multiset dimension if and only if $n \geq 6$. In [11], we found a multiset basis for the prisms that have finite multiset dimension. In particular, we have the following result that we will use later.

Theorem 2.2 ([11]). *Let $G = P_2 \,\square\, C_n$, where $n \geq 6$. Then*

$$\mathrm{md}(G) = \begin{cases} 3, & \textit{if } n \geq 11, n \neq 13, \\ 4, & \textit{if } n = 8, 9, 10, 13, \\ 5, & \textit{if } n = 6, 7. \end{cases}$$

Moreover, for all $n \geq 11$, except $n = 12, 13, 16$, the set $\{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ is an m*-resolving set of G.*

We extend an observation made in [11].

Observation 2.3. *Let $G = P_m \,\square\, C_n$, and $W \subseteq V(C_n(0))$ be an* m*-resolving set of $C_n(0)$ with k elements. Suppose $1 \le i \le m-1$, $0 \le j \le n-1$, $1 \le \ell \le k$, $w \in W$. Then,*

(i) $d((i,j),w) = d((i-1,j),w) + 1$;

(ii) $\mathrm{sum}_\ell(i,j) = \mathrm{sum}_\ell(i-1,j) + \ell$;

(iii) *the* m*-codes of the vertices of $C_n(i)$ are distinct.*

We state two lemmas that will be used later.

Lemma 2.4. *Let a_1, a_2, b_1, b_2 be positive integers such that $a_1 + a_2 = b_1 + b_2$ and the multisets $\{a_1, a_1-2, a_2, a_2+1\}$ and $\{b_1, b_1-2, b_2, b_2+1\}$ are distinct. Then $\{a_1+1, a_1-1, a_2+1, a_2\} \neq \{b_1+1, b_1-1, b_2+1, b_2\}$.*

Proof. Suppose a_1, a_2, b_1, b_2 satisfy the assumptions but not the conclusion. Let $d = a_1+a_2 = b_1+b_2$. For any integer x, let $M(x)$ be the multiset $\{x+1, x-1, d-x+1, d-x\}$. Then $M(a_1) = M(b_1)$. The proof is complete if we can show that $a_1 = b_1$. To do this, we observe that in general, $M(x)$ can only have the following types.

1: $d-x+1 < x-1$

2: $d-x+1 = x-1$

3: $d-x+1 = x$ (and so $d-x = x-1$)

4: $d-x+1 = x+1$

5: $d-x+1 = x+2$ (and so $d-x = x+1$)

6: $d-x+1 > x+2$

We show that $M(x)$ can belong only to one type. Among the six types, only Type 1 and Type 6 result in four distinct elements for $M(x)$. But in Type 1, the two smallest elements differ by 1, while in Type 6, they differ by 2. Among the four remaining types, only Type 3 has the smallest element repeated, and only Type 4 has the largest element repeated. Type 2 and Type 5 have the second smallest element repeated. But in Type 2 (resp., Type 5), the difference between the smallest element and the repeated element is 1 (resp., 2). Since they are equal, both $M(a_1)$ and $M(a_2)$ are of the same type; hence, the result follows. □

Lemma 2.5. *Let a_1, a_2, b_1, b_2 be positive integers such that $a_1 + a_2 = b_1 + b_2$ and the multisets $\{a_1, a_1-2, a_2, a_2-1\}$ and $\{b_1, b_1-2, b_2, b_2-1\}$ are distinct. Then $\{a_1, a_1-2, a_2, a_2\} \neq \{b_1, b_1-2, b_2, b_2\}$.*

Proof. Suppose a_1, a_2, b_1, b_2 satisfy the assumptions but not the conclusion. Let $d = a_1+a_2 = b_1+b_2$. For any integer x, let $M'(x)$ be the multiset $\{x, x-2, d-x, d-x\}$. Then $M'(a_1) = M'(b_1)$. We observe that in general, $M'(x)$ can only have the following types.

1: $d-x < x-2$

2: $d-x = x-2$

3: $d-x = x-1$

4: $d-x = x$

5: $d-x > x$

We show that $M'(x)$ can only belong to one type. We observe that among the five types, only Type 2 and Type 4 have an element with multiplicity 3. In Type 2 (resp., Type 4), the repeated element is minimum (resp., maximum) of the multiset. For the remaining types, $M'(x)$ has exactly three distinct elements. Only Type 3 has the three distinct elements consecutive. In Type 1 (resp., Type 5), the minimum (resp., maximum) element is repeated. □

3. Main results

For the first main result, we need two lemmas.

Lemma 3.1. *Let $G = P_m \square C_n$, $m \geq 6, n \geq 3$, $W = \{(0,0),(2,0),(m-1,0),(m-1,1)\}$, $X = \{(i,j) \mid 0 \leq i \leq m-1, 0 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$, and $Y = \{(i,j) \mid 0 \leq i \leq m-1, \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-1\}$. Then the following hold.*

(i) $\text{sum}(i,0) = \begin{cases} 2m+1, & \text{if } i = 0, \\ 2m-1, & \text{if } i = 1, \\ 2m-3, & \text{if } 2 \leq i \leq m-1. \end{cases}$

(ii) $\text{sum}(i,1) = \begin{cases} 2m+3, & \text{if } i = 0, \\ 2m+1, & \text{if } i = 1, \\ 2m-1, & \text{if } 2 \leq i \leq m-1. \end{cases}$

(iii) *Between any $u \in X$ and $w \in W$, there is a shortest path that consists of vertices from X alone.*

(iv) *For any k, i, w with $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $0 \leq i \leq m-1$, and $w \in W$, we have $d((i,k),w) = d((i,k-1),w)+1$. Hence, $\text{sum}(i,k) = \text{sum}(i,k-1)+4$.*

(v) *Between any $u = (i,k) \in Y$ and $w \in W$, there exists a shortest path that contains the edge $\{(i,n-1),(i,0)\}$.*

(vi) *For any k, i, w with $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n-1$, $0 \leq i \leq m-1$, and $w \in W$, we have $d((i,k),w) = d((i,k+1),w)+1$ (where vertex (i,n) is the same as $(i,0)$). Hence, $\text{sum}(i,k) = \text{sum}(i,k+1)+4$.*

Proof. Let us label the vertices in W as follows:

$$w_1 = (0,0), w_2 = (2,0), w_3 = (m-1,0), \text{and } w_4 = (m-1,1).$$

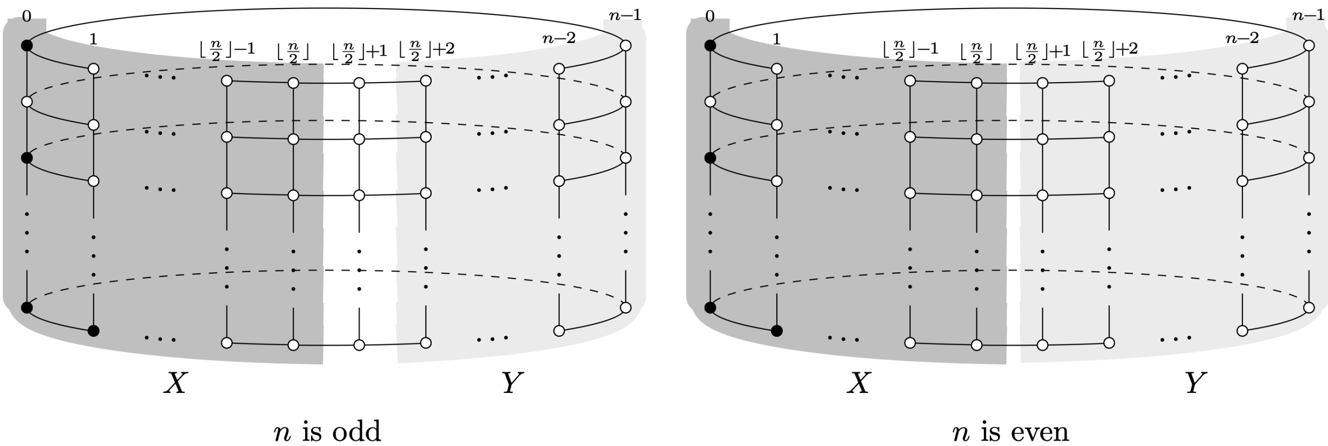

Fig. 2. The sets X and Y in $P_m \square C_n$, defined in the proof of Lemma 3.1 and Theorem 3.3

We note that,

$$r_m(0,0) = \{0,2,m-1,m\},\ r_m(1,0) = \{1,1,m-2,m-1\},\ \text{and } r_m(2,0) = \{2,0,m-3,m-2\},$$

so $\mathrm{sum}(0,0) = 2m+1$, $\mathrm{sum}(1,0) = 2m-1$, and $\mathrm{sum}(2,0) = 2m-3$. Now, whenever $2 \le i \le m-2$, we have $d((i+1,0),w_j) = d((i,0),w_j)+1$, if $j=1$ or 2; while $d((i+1,0),w_j) = d((i,0),w_j)-1$, if $j=3$ or 4. Hence, $\mathrm{sum}(i,0) = 2m-3$ for each i, $2 \le i \le m-1$. Hence, (i) is proved.

Next, whenever $0 \le i \le m-1$, we have

$$d((i,1),w_k) = d((i,0),w_k)+1 \text{ if } 1 \le k \le 3, \quad \text{while } d((i,1),w_4) = d((i,0),w_4)-1. \tag{1}$$

Therefore, $\mathrm{sum}(i,1) = \mathrm{sum}(i,0)+2$, and so $\mathrm{sum}(0,1) = 2m+3$, $\mathrm{sum}(1,1) = 2m+1$, and $\mathrm{sum}(i,1) = 2m-1$ if $i \ge 2$, proving (ii).

Let $0 \le i \le m-1$. Property (iii) follows from the fact that for any $2 \le k \le \lfloor \frac{n}{2} \rfloor$, we have $d((i,k),(i,0)) = k$ and $d((i,k),(i,1)) = k-1$. Similarly, (v) is true because whenever $\lceil \frac{n}{2} \rceil + 1 \le k \le n-1$, we have $d((i,k),(i,0)) = n-k$ and $d((i,k),(i,1)) = n-k+1$.

Properties (iv) and (vi) are immediate consequences of (iii) and (v), respectively. □

Lemma 3.2. *Let $m \ge 6$ and $W = \{(0,0),(2,0),(m-1,0),(m-1,1)\}$. Then W is an* m*-resolving set of $G = P_m \,\square\, C_3$.*

Proof.

We define $w_i, 1 \le i \le 4$, as in the proof of Lemma 3.1. Let P be the path induced by the vertices in $\{(i,0) \mid 0 \le i \le m-1\} \cup \{(m-1,1)\}$. By Lemma 1.3 and Observation 2.1, the vertices of $P_m(0)$ have distinct codes. We also note that by Lemma 3.1(i), $\mathrm{sum}(i,0) = 2m+1$, $2m-1$, or $2m-3$, depending on whether $i=0$, 1, or $i \ge 2$.

We now show that the vertices of $P_m(1)$ have distinct m-codes. First, by Lemma 3.1(ii), we have $\mathrm{sum}(i,1) = 2m+3$, $2m+1$, or $2m-1$, depending on whether $i=0$, 1, or $i \ge 2$. By Observation 1.5, $(0,1)$ and $(1,1)$ have different m-codes, which are both distinct from that of $(i,1)$ for each $i \ge 2$. To show that the m-codes of $(i,1), 2 \le i \le m-1$, are distinct, we use (1) in the proof of Lemma 3.1, the fact that the m-codes of $(i,0), 2 \le i \le m-1$ are distinct, and Lemma 2.4. Particularly, Lemma 2.4 is applicable since when $2 \le i \le m-1$, the following statements hold: $d((i,0),w_1) + d((i,0),w_3)$ is constant, $d((i,0),w_2) = d((i,0),w_1) - 2$, and $d((i,0),w_4) = d((i,0),w_3)+1$.

We can show that the m-codes of the vertices in $V(P_m(2))$ are distinct by following a similar strategy as in $V(P_m(1))$ and applying Lemma 2.5. For any i, $0 \le i \le m-1$, note that $d((i,2),w_k) = d((i,1),w_k)$, for $k = 1,2$ or 3; while $d((i,2),w_4) = d((i,1),w_4)+1$. For these vertices, note that the sum of the elements of the m-codes are $\mathrm{sum}(0,2) = 2m+4$, $\mathrm{sum}(1,2) = 2m+2$, and $\mathrm{sum}(i,2) = 2m$ for $i \ge 2$.

We are now left to show that no two vertices (i,j) and (k,ℓ), where $j \ne \ell$, have equal m-codes. By Observation 1.5, it is enough to consider vertices (a) $(0,0)$ and $(1,1)$ that both have sum equal to $2m+1$, and (b) $(1,0)$ and $(i,1)$, $i \ge 2$, that have sum $2m-1$. The m-code of $(0,0)$ has an element 0 while that of $(1,1)$ has none. For (b), $r_m(1,0)$ contains 1 twice, but none among $(i,1), i \ge 2$, has the same property. □

Theorem 3.3. *Let $m \ge 6$ and $n \ge 3$. Then $W = \{(0,0),(2,0),(m-1,0),(m-1,1)\}$ is an* m*-resolving set of $P_m \,\square\, C_n$. Hence,* $\mathrm{md}(P_m \,\square\, C_n) \le 4$.

Proof. (By induction on n.) By Lemma 3.2, the statement holds if $n = 3$. Now, suppose the statement holds for some $n \ge 3$. Let $H = P_m \,\square\, C_n$; by the inductive assumption, W is an m-resolving set of H, so the vertices of H have distinct m-codes. Let the vertices in H be denoted by

$(i,j)_H$ for $0 \le i \le m-1, 0 \le j \le n-1$. Let us construct $G = P_m \,\square\, C_{n+1}$ from H as follows. First, the vertices of G will be denoted as usual by (i,j), where $0 \le i \le m-1, 0 \le j \le n$. We split each edge $\{(i, \lfloor \frac{n}{2} \rfloor)_H, (i, \lfloor \frac{n}{2} \rfloor + 1)_H\}$, $0 \le i \le m-1$, and insert a new vertex which we denote by $(i, \lfloor \frac{n}{2} \rfloor + 1)$, and join the new vertices by a path as shown in Figure 3. Then the vertices in G and H are related as follows for $0 \le i \le m-1$ and $0 \le j \le n$:

$$(i,j) = \begin{cases} (i,j)_H & \text{if } 0 \le j \le \lfloor \frac{n}{2} \rfloor, \\ (i,j-1)_H & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \le j \le n. \end{cases}$$

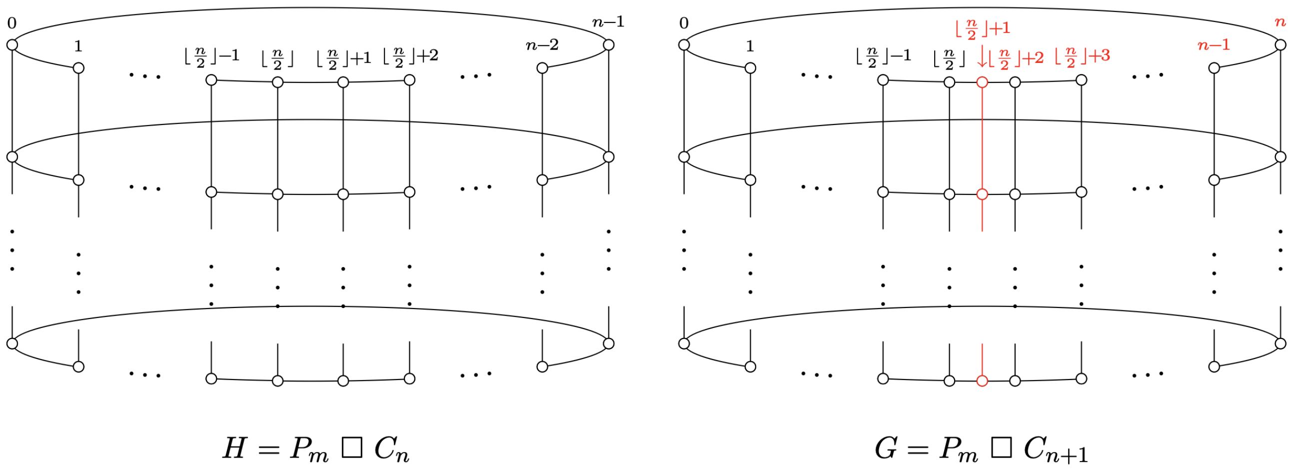

Fig. 3. Construction of $P_m \,\square\, C_{n+1}$ from $P_m \,\square\, C_n$

Let $X = \{(i,j) \mid 0 \le i \le m-1, 0 \le j \le \lfloor \frac{n}{2} \rfloor\}$ and $Y = \{(i,j) \mid 0 \le i \le m-1, \lceil \frac{n}{2} \rceil + 2 \le j \le n\}$. By Lemma 3.1 (iii) and (v), it follows that the m-codes of $X \cup Y \subset V(G)$ are distinct.

We define $w_i, 1 \le i \le 4$, as in the proof of Lemma 3.1. We consider two cases depending on the parity of n. First, suppose n is even. Note that for any i, $0 \le i \le m-1$, we have

$$d\left(\left(i, \lfloor \tfrac{n}{2} \rfloor + 1\right), w_k\right) = \begin{cases} d\left(\left(i, \lfloor \frac{n}{2} \rfloor\right), w_k\right), & \text{if } 1 \le k \le 3, \\ 1 + d\left(\left(i, \lfloor \frac{n}{2} \rfloor\right), w_k\right), & \text{if } k = 4. \end{cases}$$

Hence, by Lemma 3.1(iv),

$$\operatorname{sum}\left(i, \lfloor \tfrac{n}{2} \rfloor + 1\right) = \begin{cases} 2m + 2n, & \text{if } i = 0, \\ 2m + 2n - 2, & \text{if } i = 1, \\ 2m + 2n - 4, & \text{if } 2 \le i \le m-1. \end{cases}$$

Hence, for any $u \in V(H)$, $\operatorname{sum}(u)$ is odd, while for any $v \in V(G \setminus H)$, $\operatorname{sum}(v)$ is even; thus, u and v have distinct m-codes. Finally, by Lemma 2.5, it follows that no two vertices in $V(G \setminus H)$ have the same m-code. This completes the proof for the case where n is even.

Now, suppose n is odd. In this case, note that $Y = \{(i,j) \mid 0 \le i \le m-1, \lfloor \frac{n}{2} \rfloor + 3 \le j \le n\}$. By Lemma 3.1, the two largest values of $\operatorname{sum}(u)$, for $u \in X \cup Y$, are

$$\operatorname{sum}\left(0, \lfloor \tfrac{n}{2} \rfloor\right) = 2m + 2n - 3, \text{ and } \operatorname{sum}\left(0, \lfloor \tfrac{n}{2} \rfloor + 3\right) = \operatorname{sum}\left(1, \lfloor \tfrac{n}{2} \rfloor\right) = 2m + 2n - 5.$$

Now, for any i, $0 \le i \le m-1$, we have

$$d\left(\left(i, \lfloor \tfrac{n}{2} \rfloor + 1\right), w_k\right) = 1 + d\left(\left(i, \lfloor \tfrac{n}{2} \rfloor\right), w_k\right),$$

and

$$d\left(\left(i, \lfloor \tfrac{n}{2} \rfloor + 2\right), w_k\right) = 1 + d\left(\left(i, \lfloor \tfrac{n}{2} \rfloor + 3\right), w_k\right),$$

for each k, $1 \leq k \leq 4$. Then the m-codes of $\left(i, \lfloor \frac{n}{2} \rfloor + 1\right), 0 \leq i \leq m-1$, are distinct. Similarly, the m-codes of $\left(i, \lfloor \frac{n}{2} \rfloor + 2\right), 0 \leq i \leq m-1$, are distinct. Moreover, from Lemma 3.1,

$$\operatorname{sum}\left(i, \lfloor \tfrac{n}{2} \rfloor + 1\right) = \begin{cases} 2m+2n+1, & \text{if } i = 0, \\ 2m+2n-1, & \text{if } i = 1, \\ 2m+2n-3, & \text{if } 2 \leq i \leq m-1, \end{cases}$$

and

$$\operatorname{sum}\left(i, \lfloor \tfrac{n}{2} \rfloor + 2\right) = \begin{cases} 2m+2n-1, & \text{if } i = 0, \\ 2m+2n-3, & \text{if } i = 1, \\ 2m+2n-5, & \text{if } 2 \leq i \leq m-1. \end{cases}$$

In light of Observation 1.5, we can complete the proof by showing that if $u = (i,j), v = (i',j') \in V(G)$ with $\operatorname{sum}(u) = \operatorname{sum}(v)$, where $j = \lfloor \frac{n}{2} \rfloor + 1$ or $\lfloor \frac{n}{2} \rfloor + 2$, and $j' \neq j$, then u and v have different m-codes. We take cases.

If $\operatorname{sum}(u) = \operatorname{sum}(v) = 2m+2n-1$, then we can take $u = \left(1, \lfloor \frac{n}{2} \rfloor + 1\right)$ and $v = \left(0, \lfloor \frac{n}{2} \rfloor + 2\right)$. But $d_1(v) = \lfloor \frac{n}{2} \rfloor < d(u,w)$ for any $w \in W$. So $r_m(u) \neq r_m(v)$.

If $\operatorname{sum}(u) = \operatorname{sum}(v) = 2m+2n-3$, then $u, v \in \{u_1, u_2, v_1\}$, where $u_1 = \left(1, \lfloor \frac{n}{2} \rfloor + 2\right)$, $u_2 = \left(i, \lfloor \frac{n}{2} \rfloor + 1\right)$, for some $2 \leq i \leq m-1$, and $v_1 = \left(0, \lfloor \frac{n}{2} \rfloor\right)$. We note that $d_1(u_1) = d(u_1, w_1) = \lfloor \frac{n}{2} \rfloor + 1 = d(u_1, w_2) = d_2(u_1)$. On the other hand, $d_1(v_1) = d(v_1, w_1) = \lfloor \frac{n}{2} \rfloor \leq d_2(v_1) - 2$, so $r_m(u_1) \neq r_m(v_1)$. If $d_1(u_2) = d_1(v_1)$, then $d_1(u_2) = d(u_2, w_4)$ and $u_2 = \left(m-1, \lfloor \frac{n}{2} \rfloor + 1\right)$, so $d_2(u_2) = d(u_2, w_3) = \lfloor \frac{n}{2} \rfloor + 1$; hence, $r_m(v_1) \neq r_m(u_2)$. However, if $d_1(u_2) = d_1(u_1) = \lfloor \frac{n}{2} \rfloor + 1$, then $u_2 = \left(m-2, \lfloor \frac{n}{2} \rfloor + 1\right)$ or $\left(2, \lfloor \frac{n}{2} \rfloor + 1\right)$. In either case, $d_2(u_2) > \lfloor \frac{n}{2} \rfloor + 1$, so $r_m(u_2) \neq r_m(u_1)$.

Finally, if $\operatorname{sum}(u) = \operatorname{sum}(v) = 2m+2n-5$, then $u = \left(i, \lfloor \frac{n}{2} \rfloor + 2\right)$, for some $2 \leq i \leq m-1$, and $v = v_1 = \left(0, \lfloor \frac{n}{2} \rfloor + 3\right)$ or $v = v_2 = \left(1, \lfloor \frac{n}{2} \rfloor\right)$. Note that $d_1(v_1) = \lfloor \frac{n}{2} \rfloor - 1$ while $d_1(u) \geq \lfloor \frac{n}{2} \rfloor$, so $r_m(u) \neq r_m(v_1)$. Now, $d_1(v_2) = d_2(v_2) = d(v_2, w_1) = d(v_2, w_2) = \lfloor \frac{n}{2} \rfloor + 1$. If $d_1(u) = d_1(v_2)$, then $u = \left(m-2, \lfloor \frac{n}{2} \rfloor + 1\right)$ or $\left(3, \lfloor \frac{n}{2} \rfloor + 1\right)$. In either case, $d_2(u) > d_1(u)$, so $r_m(u) \neq r_m(v_2)$. □

In the next two results, we identify cylindrical graphs $P_m \,\square\, C_n$, where $m, n \geq 3$, that have multiset dimension equal to 3. The case $n = 3$ has been previously studied in [11].

Theorem 3.4 ([11])**.** *Let $m \geq 3$. Then $W = \{(0,0),(0,1),(2,0)\}$ is a multiset basis of $P_m \,\square\, C_3$. Hence,* $\operatorname{md}(P_m \,\square\, C_3) = 3$.

Our next result is a generalization of Theorem 2.2.

Theorem 3.5. *Let $G = P_m \,\square\, C_n$ with $m \geq 2$ and $n \geq 8m+1$. Then the set*

$$W = \{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\},$$

is an m*-resolving set of G. Hence,* $\operatorname{md}(G) = 3$.

Proof. We use induction on m. By Theorem 2.2, the statement holds when $m = 2$. Let $m \geq 3$, and suppose the theorem's statement is true for $P_{m-1} \,\square\, C_n$, for all $n \geq 8(m-1)+1$. Let $G = P_m \,\square\, C_n$.

Let us label the elements of W as follows: $w_1 = (0,0)$, $w_2 = (0,\ell)$ and $w_3 = (0, 2\ell+1)$, where $\ell = \lfloor \frac{n}{4} \rfloor$. We construct the following subsets of $V(C_n(0))$ as shown in Figure 4. In the figure, note that if the vertices are equally spaced along a circle, the location of $(0, 2\ell+1)$ relative to $(0,0)$ will change depending on the value of n. The representation shown is when $n = 4\ell$.

$$
\begin{aligned}
A &= \{(0,j) \mid 0 \le j \le \ell\}\,,\\
B &= \{(0,j) \mid \ell+1 \le j \le 2\ell+1\}\,,\\
C &= \left\{(0,j) \mid 2\ell+2 \le j \le 2\ell+1+\lfloor \tfrac{n-3\ell-3}{2} \rfloor\right\},\\
D &= \left\{(0,j) \mid 2\ell+2+\lfloor \tfrac{n-3\ell-3}{2} \rfloor \le j \le n-1-\lfloor \tfrac{n-3\ell-2}{2} \rfloor\right\},\\
E &= \left\{(0,j) \mid n-\lfloor \tfrac{n-3\ell-2}{2} \rfloor \le j \le n-1\right\}.
\end{aligned}
$$

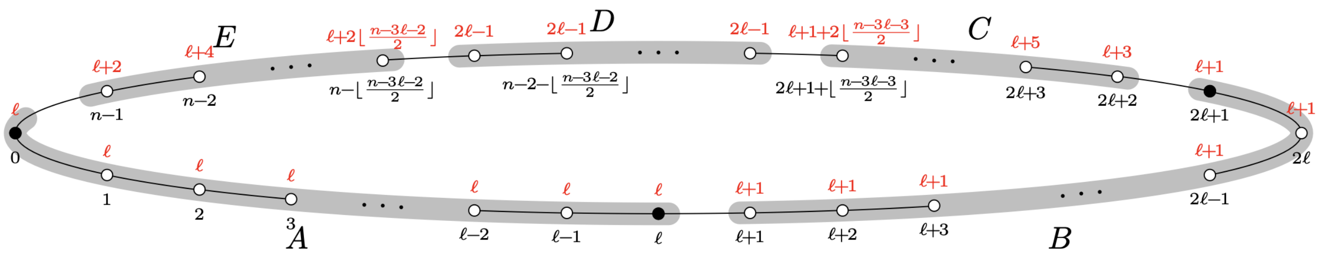

Fig. 4. The values of sum_2 (in red) of the vertices of $C_n(0)$ of $P_m \,\square\, C_n$, where $n = 4\ell$, relative to the set W in Theorem 3.5. The numbers under the vertices are the second coordinates of the vertices

These sets satisfy the following properties:

(a) For any $w \in A$, $\mathrm{sum}_2(w) = \ell$ and $d_3(w) = d(w, w_3)$.

(b) For any $w \in B$, $\mathrm{sum}_2(w) = \ell+1$ and $d_3(w) = d(w, w_1)$.

(c) For any $w \in C$, $\mathrm{sum}_2(w) = \ell+1+2h$, where $1 \le h \le \lfloor \frac{n-3\ell-3}{2} \rfloor$, $\mathrm{sum}_2(w) < n-2\ell-1$, $d_1(w) = d(w, w_3)$, and $d_3(w) = d(w, w_1)$.

(d) For any $w \in D$, $\mathrm{sum}_2(w) = n-2\ell-1$ and $d_3(w) = d(w, w_2)$.

(e) For any $w \in E$, $\mathrm{sum}_2(w) = \ell+2h$, where $1 \le h \le \lfloor \frac{n-3\ell-2}{2} \rfloor$, $\mathrm{sum}_2(w) < n-2\ell-1$, $d_1(w) = d(w, w_1)$, and $d_3(w) = d(w, w_3)$.

Then by inductive assumption, and Observation 2.3, we just need to show that $r_m(u) \neq r_m(v)$ for all vertices $u \in V(C_n(0))$ and $v \in V(C_n(m-1))$. By Observation 1.5, we may assume that $\mathrm{sum}_2(u) = \mathrm{sum}_2(v)$. By (a)-(e), we have $\mathrm{sum}_2(u) \in \{\ell, \ell+1, \ldots, n-2\ell-1\}$. By Observation 2.3(ii),

$$
\mathrm{sum}_2(v) \in \{\ell+2(m-1), \ell+1+2(m-1), \ldots, n-2\ell-1+2(m-1)\}\,.
$$

We consider cases according to the value of $\mathrm{sum}_2(u) = k \in \{\ell+2m-2, \ell+2m-1, \ldots, n-2\ell-1\}$. Let us define E' to consist of the vertices of the form $(m-1, j)$ whenever $(0,j) \in E$. We define B' and C' similarly.

1: If $k = \ell+2m-2 = \ell+2(m-1)$, then $u = (0, n-(m-1)) \in E$ (since $m-1 \le \lfloor \frac{n-3\ell-2}{2} \rfloor$) and $d_1(u) = m-1$. Hence, $v = v_1 = (m-1, 0)$ or $v = v_2 = (m-1, \ell)$, both in A. Now, $d_3(u) = d(u, w_3) = n-(m-1)-(2\ell+1) = n-2\ell-m$, while $d_3(v_2) = d(v_2, w_3) = (2\ell+1)-\ell+(m-1) = \ell+m$, so $d_3(u) - d_3(v_2) = n-3\ell-2m > 0$. On the other hand, $d_3(v_1) = d(v_1, w_3) = \min\left((2\ell+1)+m-1, m-1+n-(2\ell+1)\right)$. Hence, $d_3(v_1) - d_3(u) = \min(4\ell+2m-n, 2m-2) > 0$.

2: Suppose $k = \ell + 1 + 2h$ where $m - 1 \leq h \leq \lfloor \frac{n-3\ell-3}{2} \rfloor$ (so $k < n - 2\ell - 1$). Then $u = (0, 2\ell + 1 + h) \in C$, $d_1(u) = h$, and $d_3(u) = d(u, w_1) = n - (2\ell + 1 + h) = n - h - 2\ell - 1$. Now, $k = \ell + 1 + 2(h - m + 1) + 2(m - 1)$, so $v = (m - 1, 2\ell + 1 + h - m + 1)$. Note that $v \in B'$ if $h = m - 1$, and $v \in C'$, otherwise. In either case, $d_3(v) = d(v, w_1) = n - (2\ell + 1 + h - m + 1) + (m - 1)$, so $d_3(v) - d_3(u) = 2m - 2 > 0$.

3: Suppose $k = \ell + 2h$ where $m - 1 < h \leq \lfloor \frac{n-3\ell-2}{2} \rfloor$ (so $k < n - 2\ell - 1$). Then $u = (0, n - h) \in E$, and $d_3(u) = d(u, w_3) = (n - h) - (2\ell + 1) = n - 2\ell - h - 1$. Since $k = \ell + 2(h - m + 1) + 2(m - 1)$, we have $v = (m - 1, n - (h - m + 1)) \in E'$. Hence, $d_3(v) = d(v, w_3) = (n - h + m - 1) - (2\ell + 1) + (m - 1) = n - 2\ell - h + 2m - 3$, so $d_3(v) - d_3(u) = 2m - 2 > 0$.

4: Let $k = n - 2\ell - 1 = \ell + (n - 3\ell - 1)$. Then $u \in D$. First, suppose $n - 3\ell - 1$ is odd. Then $k = \ell + 1 + 2\lfloor \frac{n-3\ell-2}{2} \rfloor = \ell + 1 + 2(m - 1) + 2\left(\lfloor \frac{n-3\ell-2}{2} \rfloor - m + 1\right)$. Hence, $v = \left(m - 1, 2\ell + 1 + \lfloor \frac{n-3\ell-2}{2} \rfloor - m + 1\right) \in C'$, $d_1(v) = \lfloor \frac{n-3\ell-2}{2} \rfloor = 1 + \lfloor \frac{n-3l-3}{2} \rfloor$, and $d_3(v) = d(v, w_1) = n - \left(2\ell + 1 + \lfloor \frac{n-3\ell-2}{2} \rfloor - m + 1\right) + (m - 1)$. On the other hand, we need only to consider when $u = \left(0, 2\ell + 1 + 1 + \lfloor \frac{n-3\ell-3}{2} \rfloor\right)$. Hence, $d_3(u) = d(u, w_2) = \ell + 1 + \lfloor \frac{n-3\ell-2}{2} \rfloor$, and $d_3(v) - d_3(u) = 2m - 2 > 0$.

Now, suppose $n - 3\ell - 1$ is even. Then $k = \ell + 2\lfloor \frac{n-3\ell-1}{2} \rfloor = \ell + 2\left(\lfloor \frac{n-3\ell-1}{2} \rfloor - m + 1\right) + 2(m - 1)$. Hence, $v = \left(m - 1, n - \left(\lfloor \frac{n-3\ell-1}{2} \rfloor - m + 1\right)\right) \in E'$, $d_1(v) = \lfloor \frac{n-3\ell-1}{2} \rfloor = 1 + \lfloor \frac{n-3\ell-2}{2} \rfloor = 1 + \lfloor \frac{n-3\ell-3}{2} \rfloor$. Hence, we need only to consider $u = u_1 = \left(0, n - 1 - \lfloor \frac{n-3\ell-2}{2} \rfloor\right)$ or $u = u_2 = \left(0, 2\ell + 1 + 1 + \lfloor \frac{n-3\ell-3}{2} \rfloor\right)$. Then $d_3(u_1) = d(u_1, w_2) = \ell + 1 + \lfloor \frac{n-3\ell-2}{2} \rfloor$ while $d_3(u_2) = d(u_2, w_2) = d_3(u_1) + 1$. Now, $d_3(v) = (m - 1) + n - \left(\lfloor \frac{n-3\ell-1}{2} \rfloor - m + 1\right) - 2\ell - 1$. Therefore, $d_3(v) - d_3(u_1) > d_3(v) - d_3(u_2) = 2m - 3 > 0$.

In all cases, we were able to show that $r_m(u) \neq r_m(v)$. Therefore, the result follows. □

To cover additional cases of $P_m \,\square\, C_n$ that are not included in the previous results, we have the following theorem.

Theorem 3.6. *If $m \geq 1$ and $n \geq 12$, then the set $\{(0,0), (0,1), (0,3), (0, 2\lfloor n/4 \rfloor + 1 + (n \mod 2))\}$ is an* m*-resolving set of $P_m \,\square\, C_n$.*

To prove the theorem, we use the notion of "strong ID-coloring" introduced in [10], which is equivalent to having a "strong" m-resolving set, defined below. Given a multiset T and an integer k, we denote by $T + k$ the multiset $\{t + k \mid t \in T\}$.

Definition 3.7 ([10])**.** An m-resolving set W of a graph G is *strong* if for any two distinct vertices u and v of G, $r_m(u) \neq r_m(v) + k$ for any integer k.

Lemma 3.8 ([10])**.** *Let W be a strong* m*-resolving set of G and m any positive integer. Then $W' = \{(0, w) \mid w \in W\}$ is an* m*-resolving set of $P_m \,\square\, G$.*

We now define Δd-codes of vertices. Given a graph G and a set $W \subset V(G)$ with cardinality ℓ, suppose for $v \in V(G)$,

$$r_m(v) = \{d_1, d_2, \ldots, d_\ell\},$$

where $d_i \leq d_{i+1}$ for each i, $1 \leq i \leq \ell - 1$. Let

$$\Delta d(v) = [d_2 - d_1, d_3 - d_2, \ldots, d_\ell - d_{\ell-1}],$$

a sequence with $\ell - 1$ entries.

If $u, v \in V(G)$ and $r_m(u) = r_m(v) + k$ for some integer k, then it is clear that $\Delta d(u) = \Delta d(v)$. We have thus shown the following.

Lemma 3.9. *Let G be a graph, $W \subset V(G)$ such that $\Delta d(u) \neq \Delta d(v)$ for all distinct $u, v \in V(G)$. Then W is a strong* m*-resolving set of G.*

Proof of Theorem 3.6. It is enough to show that the set $W = \{0, 1, 3, 2\lfloor n/4 \rfloor + 1 + (n \mod 2)\}$ is a strong m-resolving set of C_n using Lemma 3.9. Let $n = 4k + r$, $0 \leq r \leq 3$. Denote the elements of W by w_i, $1 \leq i \leq 4$, arranged in increasing order. Then $w_4 = 2k + 1$ if n is even, and $w_4 = 2k + 2$ if n is odd.

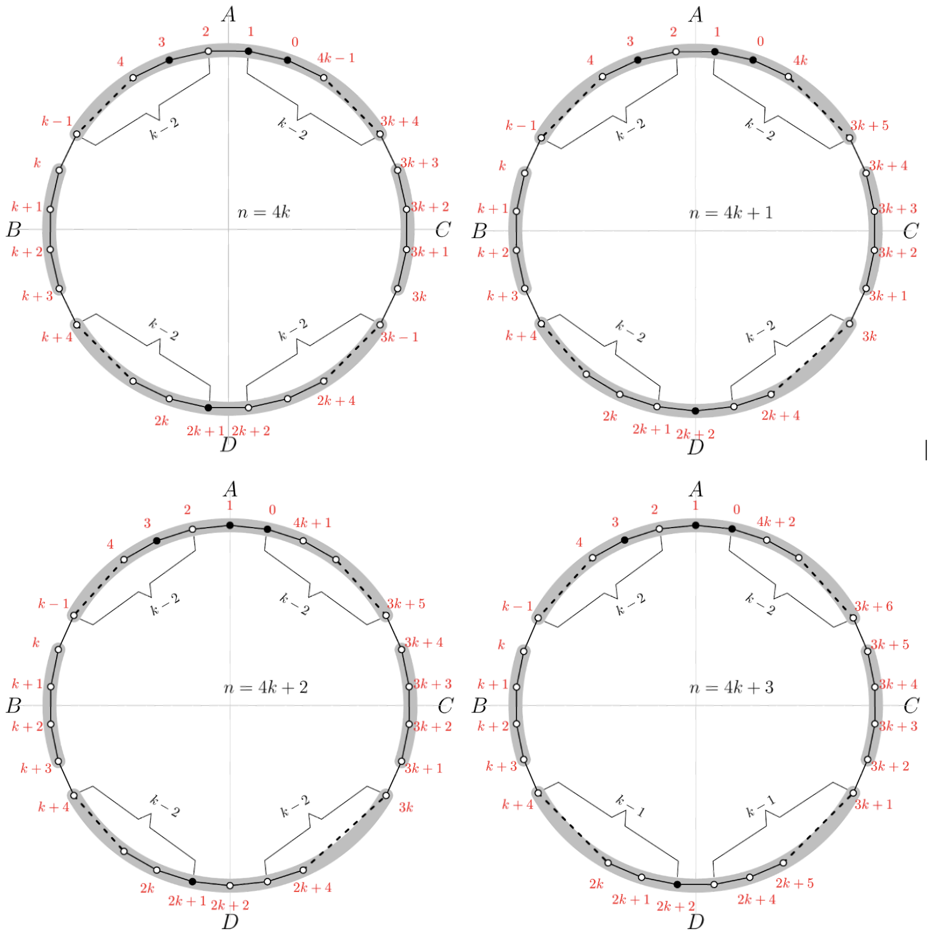

Fig. 5. The sets A, B, C, and D in $V(C_n)$, $n = 4k + r$, $0 \leq r \leq 3$

We identify C_n with a regular n-gon in the Euclidean plane as shown in Figure 5. Note that the vertical axis contains the vertex 1 if $r = 2$ or 3, while it bisects the edge joining vertices 1 and 2 in the other cases. In each case, we split the vertices of C_n into four sets: A, B, C, and D. B and C have four elements each. The cardinality of A and D range from $2k - 4$ to $2k - 2$, depending on r. Note that if $n = 12$ or 13, the vertex 0 is in C and 3 is in B; they are not in A.

Let $C = \{c_1, c_2, c_3, c_4\}$, arranged in increasing order. Note that for any r, $d(c_1, w_2) = d(c_1, w_1) + 1$ and $d(c_1, w_3) = d(c_1, w_2) + 2$. Moreover, if $r = 0$ or 1, then $d(c_1, w_1) = k$ and $d(c_1, w_4) = k - 1$; hence, $r_m(c_1) = \{k, k + 1, k + 3, k - 1\}$. On the other hand, if $r = 2$ or 3, then $d(c_1, w_1) = k + 1$ while

$d(c_1, w_4) = k$, so $r_m(c_1) = \{k+1, k+2, k+4, k\}$. Therefore, for any r, $r_m(c_1) = \{a+1, a+2, a+4, a\}$ for some a. And since for any i, $1 \leq i \leq 3$, c_i is closer to (resp., farther from) w_j relative to c_{i+1} if $j = 4$ (resp., $j = 1, 2$, or 3) by one unit, we obtain the following table.

u	$r_m(u)$	$\Delta d(u)$
c_1	$\{a+1, a+2, a+4, a\}$	$[1, 1, 2]$
c_2	$\{a, a+1, a+3, a+1\}$	$[1, 0, 2]$
c_3	$\{a-1, a, a+2, a+2\}$	$[1, 2, 0]$
c_4	$\{a-2, a-1, a+1, a+3\}$	$[1, 2, 2]$

Now, let $B = \{b_1, b_2, b_3, b_4\}$, arranged in increasing order. Then for any r, $d(b_1, w_1) = d(b_1, w_2) + 1$ and $d(b_1, w_2) = d(b_1, w_3) + 2$. If $r = 0$ or 2, then $d(b_1, w_3) = k - 3$ and $d(b_1, w_4) = k + 1$. On the other hand, if $r = 1$ or 3, then $d(b_1, w_3) = k - 3$ while $d(b_1, w_4) = k + 2$. Following the same argument above, we obtain $\Delta d(b)$ for any $b \in B$.

$r = 0$ or 2			$r = 1$ or 3		
u	$r_m(u)$	$\Delta d(u)$	u	$r_m(u)$	$\Delta d(u)$
b_1	$\{k, k-1, k-3, k+1\}$	$[2, 1, 1]$	b_1	$\{k, k-1, k-3, k+2\}$	$[2, 1, 2]$
b_2	$\{k+1, k, k-2, k\}$	$[2, 0, 1]$	b_2	$\{k+1, k, k-2, k+1\}$	$[2, 1, 0]$
b_3	$\{k+2, k+1, k-1, k-1\}$	$[0, 2, 1]$	b_3	$\{k+2, k+1, k-1, k\}$	$[1, 1, 1]$
b_4	$\{k+3, k+2, k, k-2\}$	$[2, 2, 1]$	b_4	$\{k+3, k+2, k, k-1\}$	$[1, 2, 1]$

From the above, we conclude that for any r, the vertices in $B \cup C$ have distinct Δd-codes. Note also that for any vertex u in $B \cup C$, the entries in $\Delta d(u)$ are all at most 2. To complete the proof, we take cases.

1: Suppose $r = 0$. Observe that

$$d_2(x) - d_1(x) = \begin{cases} 0, & \text{if } x = 2, \\ 2, & \text{if } 3 \leq x \leq k-1, \\ 1, & \text{if } x = 0, 1, \text{ or } 3k+4 \leq x \leq 4k-1, \end{cases}$$

while

$$d_4(x) - d_3(x) = \begin{cases} 2k-4, & \text{if } x = 0, \\ 2k-2, & \text{if } x = 1, \\ 2k-2x+1, & \text{if } 2 \leq x \leq k-1, \\ 2x-6k-4, & \text{if } 3k+4 \leq x \leq 4k-1. \end{cases}$$

From these, we see that vertices in A have distinct Δd-codes. Recall that if $k = 3$, the vertex $0 \notin A$; also, no vertex $x \in A$ satisfies $3k + 4 \leq x \leq 4k - 1$. Hence, $d_4(x) - d_3(x) \geq 3$ for each $x \in A$. It follows that the vertices in $A \cup B \cup C$ have distinct Δd-codes.

Now, for any $x \in D$, we have $d_2(x) \geq d(k+4, 3) = k+1$ and $d_1(x) \leq d(3k-1, 2k+1) = k-2$, so $d_2(x) - d_1(x) \geq 3$. Hence, the Δd-codes of D are distinct from those in $A \cup B \cup C$. It now remains to show that no two elements of D have the same Δd-code, which follows from

$$d_2(x) - d_1(x) = \begin{cases} 2x - 2k - 4, & \text{if } k + 4 \le x \le 2k + 1, \\ 6k - 2x + 1, & \text{if } 2k + 2 \le x \le 3k - 1. \end{cases}$$

2: Suppose $r = 1$. We can show that no Δd-code is repeated in $V(C_n)$ using the same argument used in Case 1. We have for $x \in A$,

$$d_2(x) - d_1(x) = \begin{cases} 0, & \text{if } x = 2, \\ 2, & \text{if } 3 \le x \le k - 1, \\ 1, & \text{if } x = 0, 1, \text{ or } 3k + 5 \le x \le 4k, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 2k - 4, & \text{if } x = 0, \\ 2k - 2, & \text{if } x = 1, \\ 2k - 2x + 2, & \text{if } 2 \le x \le k - 1, \\ 2x - 6k - 6, & \text{if } 3k + 5 \le x \le 4k. \end{cases}$$

For $x \in D$,

$$d_2(x) - d_1(x) = \begin{cases} 2x - 2k - 5, & \text{if } k + 4 \le x \le 2k + 1, \\ 2k - 1, & \text{if } x = 2k + 2, \\ 6k - 2x + 3, & \text{if } 2k + 3 \le x \le 3k, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 1, & \text{if } k + 4 \le x \le 2k, \text{ or } 2k + 2 \le x \le 2k + 3, \\ 0, & \text{if } x = 2k + 1, \\ 2, & \text{if } 2k + 4 \le x \le 3k. \end{cases}$$

3: Suppose $r = 2$. As in the previous two cases, the result follows from the following formulas. For $x \in A$,

$$d_2(x) - d_1(x) = \begin{cases} 0, & \text{if } x = 2, \\ 2, & \text{if } 3 \le x \le k - 1, \\ 1, & \text{if } x = 0, 1, \text{ or } 3k + 5 \le x \le 4k + 1, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 2k - 2, & \text{if } x = 0 \text{ or } 1, \\ 2k - 2x + 1, & \text{if } 2 \le x \le k - 1, \\ 2x - 6k - 6, & \text{if } 3k + 5 \le x \le 4k + 1. \end{cases}$$

Note that the vertices 0 and 1 have equal values of $d_4 - d_3$ and $d_2 - d_1$. However, $d_3(0) - d_2(0) = 2 \ne 1 = d_3(1) - d_2(1)$.

For $x \in D$,

$$d_2(x) - d_1(x) = \begin{cases} 2x - 2k - 4, & \text{if } k+4 \leq x \leq 2k+1, \\ 6k - 2x + 3, & \text{if } 2k+2 \leq x \leq 3k, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 1, & \text{if } k+4 \leq x \leq 2k+2, \\ 0, & \text{if } x = 2k+3, \\ 2, & \text{if } 2k+4 \leq x \leq 3k. \end{cases}$$

4: Suppose $r = 3$. For $x \in A$,

$$d_2(x) - d_1(x) = \begin{cases} 0, & \text{if } x = 2, \\ 2, & \text{if } 3 \leq x \leq k-1, \\ 1, & \text{if } x = 0, 1, \text{ or } 3k+6 \leq x \leq 4k+2, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 2k-2, & \text{if } x = 0, \\ 2k-1, & \text{if } x = 1, \\ 2k-2x+2, & \text{if } 2 \leq x \leq k-1, \\ 2x-6k-8, & \text{if } 3k+6 \leq x \leq 4k+2. \end{cases}$$

For $x \in D$,

$$d_2(x) - d_1(x) = \begin{cases} 2x - 2k - 5, & \text{if } k+4 \leq x \leq 2k+2, \\ 6k - 2x + 5, & \text{if } 2k+3 \leq x \leq 3k+1, \end{cases}$$

and

$$d_4(x) - d_3(x) = \begin{cases} 1, & \text{if } k+4 \leq x \leq 2k+1, \text{ or } 2k+3 \leq x \leq 2k+4, \\ 0, & \text{if } x = 2k+2, \\ 2, & \text{if } 2k+5 \leq x \leq 3k+1. \end{cases}$$

Note that $d_4(2k+4) - d_3(2k+4) = 1 = d_4(2k+1) - d_3(2k+1)$ and $d_2(2k+4) - d_1(2k+4) = 2k-3 = d_2(2k+1) - d_1(2k+1)$, but $d_3(2k+4) - d_2(2k+4) = 1 \neq 2 = d_3(2k+1) - d_2(2k+1)$. □

To cover all cylindrical graphs, we identify m-resolving sets for the following remaining cases of n when $3 \leq m \leq 5$, as shown in Table 1.

Hence, we may now state the following result for all cylindrical graphs.

Theorem 3.10. *All cylindrical graphs $P_m \square C_n$ $(m, n \geq 3)$ have finite multiset dimension.*

4. Conclusions and recommendations

With the results of this paper, investigating the multiset dimension of $P_m \square C_n$, for all $m \geq 1$ and $n \geq 3$, is nearing completion. $P_1 \square C_n$ are cycles and their multiset dimension have been discussed extensively in [1]. For $P_2 \square C_n$ (or prisms), the multiset dimension is finite if and only if $n \geq 6$ [8]. Moreover, the multiset dimensions of such prisms have been previously determined (Theorem 2.2). For $P_m \square C_n$ ($m, n \geq 3$) or cylindrical graphs, we obtained m-resolving sets (Theorems 3.3 to Theorem 3.6, and Table 1), proving that the multiset dimension is always finite. In particular, $\text{md}(G) = 3$ for the following cases:

1. $P_m \square C_3$ (Theorem 3.4)
2. $P_m \square C_n$, $n \geq 8m + 1$ (Theorem 3.5)

Solving for the multiset dimension of the remaining cases remains an open problem.

Table 1. m-resolving sets of special cases of $P_m \square C_n$

n	m-resolving set
4,5	$\{(0,0),(0,1),(0,2),(1,0),(2,1)\}$
6,7	$\{(0,0),(0,2),(0,4),(1,0),(2,1)\}$
8,10	$\{(0,0),(0,1),(0,3),(0,5)\}$
9	$\{(0,0),(0,1),(0,2),(0,4),(0,6)\}$
11	$\{(0,0),(0,1),(0,3),(0,6)\}$

Aside from the multiset dimension, we can also consider the ID spectrum, as defined by Chartrand et al. [1]. In addition, the multiset dimension of other families of graphs borne out of other graph operations (e.g., strong direct product, corona) are also suggested for future research.

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