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On the number of Hamiltonian cycles in the generalized Petersen graph

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ABSTRACT

The generalized Petersen graph $G(n,k)$ is a cubic graph with vertex set $V(G(n,k)) = \{v_i\}_{0\le i<n} \cup \{w_i\}_{0\le i<n}$ and edge set $E(G(n,k)) = \{v_iv_{i+1}\}_{0\le i<n} \cup \{w_iw_{i+k}\}_{0\le i<n} \cup \{v_iw_i\}_{0\le i<n}$ where the indices are taken modulo n. Schwenk found the number of Hamiltonian cycles in $G(n,2)$, and in this article we present initial conditions and linear recurrence relations for the number of Hamiltonian cycles in $G(n,3)$ and $G(n,4)$. This is attained by introducing $G'(n,k)$, which is a modified version of $G(n,k)$, and a subset of its subgraphs which we call admissible, and which are partitioned into different classes in such a manner that we can find relations between the number of admissible subgraphs of each class. The classes and their relations define a directed graph such that each strongly connected component is of a manageable size for $k=3$ and $k=4$, which allows us to find linear recurrence relations for the number of admissible subgraphs in each class in these cases. The number of Hamiltonian cycles in $G(n,k)$ is a sum of the number of admissible subgraphs of $G'(n,k)$ over a certain subset of the classes.

Keywords: Hamiltonian cycle, generalized Petersen graph, enumeration, linear recurrence relation, characteristic polynomial, strongly connected component

1. Introduction

Let n,k be positive integers. The generalized Petersen graph $G(n,k)$ is a cubic graph with vertex set

$$V(G(n,k)) = \{v_i\}_{0\le i<n} \cup \{w_i\}_{0\le i<n},$$

and edge set

$$E(G(n,k)) = \{v_iv_{i+1}\}_{0\le i<n} \cup \{w_iw_{i+k}\}_{0\le i<n} \cup \{v_iw_i\}_{0\le i<n},$$

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where the indices are taken modulo n. A Hamiltonian cycle of a graph is a cycle of edges that visits each vertex exactly once. Schwenk [1] found the number of Hamiltonian cycles in $G(n,k)$ for $k=2$, and asked for a corresponding enumeration for higher values of k. The objective of this paper is to give initial conditions and linear recurrence relations in the cases $k=3$ and $k=4$, from which explicit formulae can be constructed based on the roots of the characteristic polynomials.

To this end, we introduce the graph $G'(n,k)$ with the same vertex set and edge set as $G(n,k)$, except for the additional vertices $L_0,\dots,L_k$, $R_0,\dots,R_k$, and the edges v_0v_{n-1}, $w_0w_{n-k},\dots,w_{k-1}w_{n-1}$ replaced by v_0L_0, $w_0L_1,\dots,w_{k-1}L_k$, $v_{n-1}R_0$, $w_{n-k}R_1,\dots,w_{n-1}R_k$. Figure 1 shows this graph in the case $k=3$.

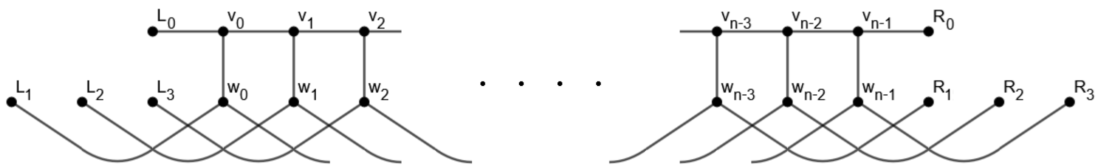


Fig. 1. The graph $G'(n,3)$

The visible edges in the figure are called the *outer* edges. Generally, those that are incident with at least one of $v_0,v_1,\dots,v_{k-1}$, $w_0,w_1,\dots,w_{k-1}$ are called the outer edges on the left hand side, and those that are incident with at least one of $v_{n-k},v_{n-k+1},\dots,v_{n-1}$, $w_{n-k},w_{n-k+1},\dots,w_{n-1}$ are called the outer edges on the right hand side.

Consider a subgraph Γ of $G'(n,k)$ with the same vertex set as that of $G'(n,k)$ for which each vertex of degree 3 in $G'(n,k)$ has degree 2 in Γ, and containing no cycles except possibly one that constitutes the whole of Γ. Such subgraphs are called *admissible* subgraphs of $G'(n,k)$. Each admissible subgraph Γ is associated with a list of pairs of the vertices among $L_0,\dots,L_k$, R_0, $R_1,\dots,R_k$ of degree 1 in Γ (called *loose ends* henceforth) that are in the same connected component. This list is called a *list of connections.*

For any given k, there is a finite number of possible intersections of an admissible subgraph and the outer edges on either side, which we label $S_1,S_2,\dots$. For $k=3$, there are 33 possibilities, as seen in Table 1. For $k=4$ and $k=5$, the number of possibilities is 85 and 217, respectively.

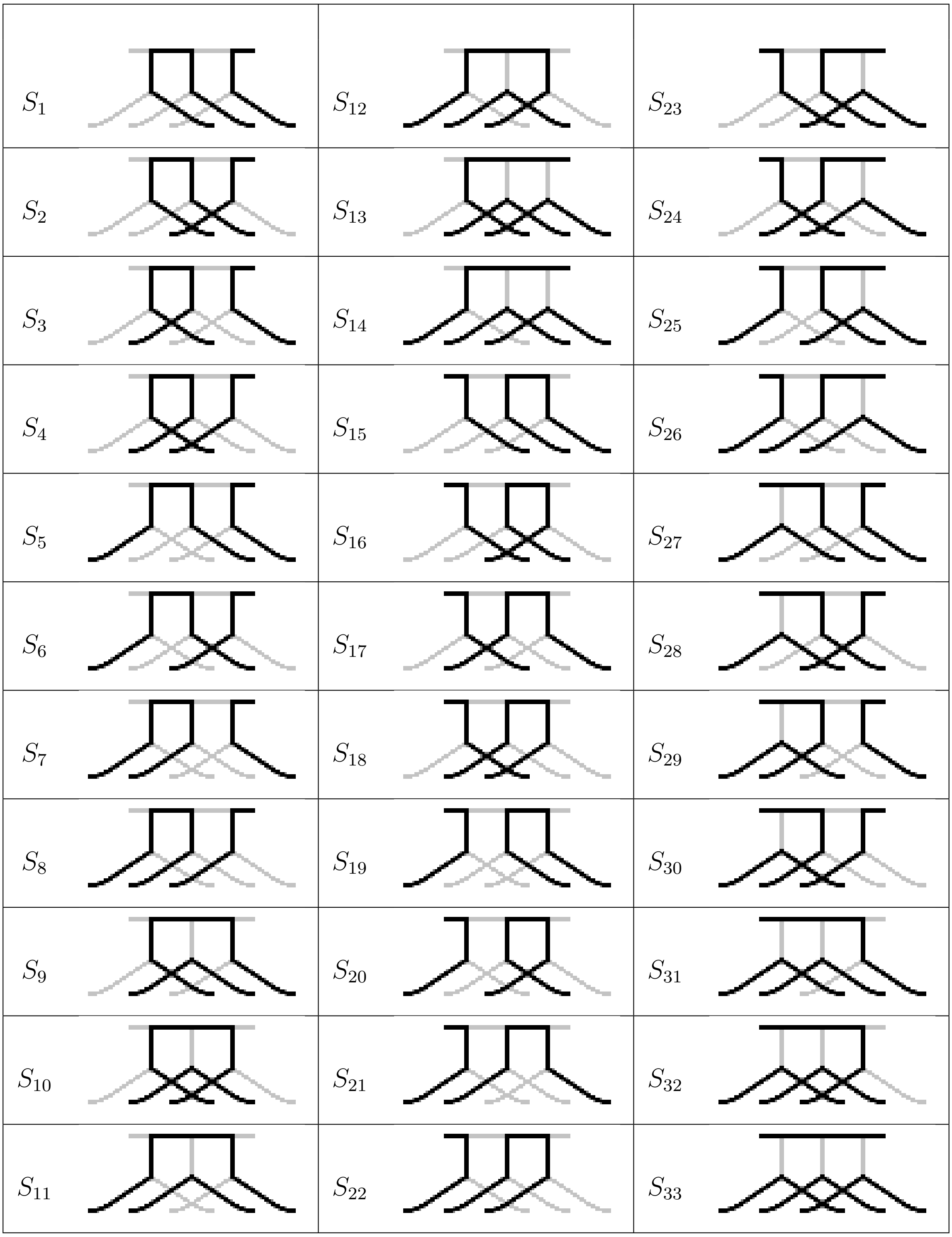
Suppose Γ is an admissible subgraph of $G'(n,k)$. If the intersection of Γ and the outer edges on the left hand side is S_{n_L}, the intersection of Γ and the outer edges on the right hand side is S_{n_R} and the list of connections is λ, we say that the *signature* of Γ is (n_L,n_R,λ). For $k=3$, there are 1705 distinct signatures in which all the loose ends are accounted for in the list of connections, and two loose ends are in the same pair if they are connected through the outer edges on one side. For example, if $n_L=8$, then necessarily $(L_1,L_2)\in\lambda$. Not all of them can be realized; for example, no admissible subgraph can have signature $(1,7,\{(R_0,R_3)\})$. For $k=4$ and $k=5$, the number of signatures is 25675 and 455835, respectively.

For a fixed k, let

$$u_n^{(n_L,n_R,\lambda)}, \tag{1}$$

denote the number of admissible subgraphs of $G'(n,k)$ with signature (n_L,n_R,λ). We are going to find a linear recurrence relation satisfied by all these sequences for $k=3$ and for $k=4$.

Table 1. Possible intersections of an admissible subgraph and the outer edges on either side for $k = 3$



Let H_k denote the set of signatures that pass the following test.

Checking the hamiltonicity induced by a signature
Start with any loose end. After an even number of steps, move from the current loose end to the one it is connected to according to the list of connections. After an odd number of steps, move from L_i to R_i or from R_i to L_i, depending on the current loose end. If this procedure traverses all loose ends, then identifying L_0 with v_{n-1}, L_1 with w_{n-k}, $\ldots$, L_k with w_{n-1}, R_0 with v_0, R_1 with w_0, $\ldots$, and R_k with w_{k-1} yields a Hamiltonian cycle in $G(n,k)$.

Thus, the number of Hamiltonian cycles in $G(n,k)$ is given by

$$h_k(n) = \sum_{(n_L,n_R,\lambda)\in H_k} u_n^{(n_L,n_R,\lambda)}. \qquad (2)$$

As a sum of (1) over a subset of the possible signatures, $h_k(n)$ also satisfies any linear recurrence relation satisfied by all the individual sequences.

2. The graph σ_k of signatures

The directed graph σ_k is defined as follows. $V(\sigma_k)$ is the set of all signatures. Thus, the terms "vertices" and "signatures" in σ_k can be used interchangeably, but we will primarily use the latter. Given a signature (n_L, n_R, λ), let Γ be a hypothetical (not necessarily realizable) admissible subgraph of $G'(n,k)$ having that signature, for some n. The graph $G'(n-1,k)$ can be obtained by removing $v_{n-1}R_0$, $v_{n-1}w_{n-1}$ and $w_{n-1}R_k$ from $E(G'(n,k))$ and R_0 and R_k from $V(G'(n,k))$, and renaming v_{n-1} to R_0, R_{k-1} to R_k, R_{k-2} to $R_{k-1}, \ldots, R_1$ to R_2 and w_{n-1} to R_1. If (n_L, n'_R, λ') is a possible signature of the intersection of Γ and $G'(n-1,k)$ thus constructed, then the arc $((n_L, n'_R, \lambda'), (n_L, n_R, \lambda))$ is in $E(\sigma_k)$.

For the remainder of this Section, we outline the possible signatures of the intersection of Γ and $G'(n-1,k)$.

$v_{n-k-1}w_{n-k-1}$ can not be in $E(\Gamma)$ if either $v_{n-k-1}v_{n-k}, \ldots, v_{n-2}v_{n-1}$, $v_{n-1}w_{n-1}$ and $w_{n-k-1}w_{n-1}$ are all in $E(\Gamma)$, as a cycle not constituting the whole of Γ would be formed, or if v_{n-k-1} and w_{n-k-1} are connected through the outer edges on the right hand side to two loose ends x and y such that $(x,y) \notin \lambda$.

There are five possibilities for which of the edges $v_{n-1}R_0$, $v_{n-1}w_{n-1}$, $w_{n-1}R_k$ are in $E(\Gamma)$: All three, any two (which covers three possibilities), or only $v_{n-1}w_{n-1}$. Here is how to construct λ' from the list of connections λ in Γ in each case, before the renaming step (recall that v_{n-1} is to be renamed to R_0, R_{k-1} to R_k etc.).

- If all three edges are in $E(\Gamma)$ and λ contains other pairs than (R_0, R_k), then remove (R_0, R_k). Note that if λ does not contain other pairs than (R_0, R_k), then the edges other than those three form a cycle.
- If only $v_{n-1}R_0$ and $v_{n-1}w_{n-1}$ are in $E(\Gamma)$, replace R_0 by w_{n-1}.
- If only $v_{n-1}w_{n-1}$ and $w_{n-1}R_k$ are in $E(\Gamma)$, replace R_k by v_{n-1}.
- If only $v_{n-1}R_0$ and $w_{n-1}R_k$ are in $E(\Gamma)$, replace R_0 by v_{n-1} and R_k by w_{n-1}.
- Suppose only $v_{n-1}w_{n-1}$ is in $E(\Gamma)$. If λ is empty, then add (v_{n-1}, w_{n-1}). Otherwise, some $(x,y) \in \lambda$ is to be replaced by (x, v_{n-1}), (y, w_{n-1}). Any loose end that is connected to v_{n-1}

through the edges on the right hand side other than $v_{n-1}w_{n-1}$ must take the role of x, and any loose end that is connected to w_{n-1} through the edges on the right hand side other than $v_{n-1}w_{n-1}$ must take the role of y. If one of x and y is determined in this way, the other one is determined by λ. If neither of them is determined in this way, any $(x, y) \in \lambda$ for which x and y are not connected through the outer edges on one side will do.

3. Linear recurrence relations and characteristic polynomials

If $k = 3$, then simply by inspecting Table 1, we find that the initial value of (1) for $n = k$ is 1 when (n_L, n_R, λ) is one of $(1, 1, \{(R_0, R_3), (R_1, R_2)\})$, $(2, 2, \{(L_3, R_0), (R_1, R_2)\})$, $(3, 3, \{(L_2, R_1), (R_0, R_3)\})$ and so on, and 0 in all other cases. Similarly, the initial values follow directly from the corresponding set $\{S_i\}$ for other values of k. For $n > k$, we have

$$u_n^{(n_L,n_R,\lambda)} = \sum_{((n_L,n'_R,\lambda'),(n_L,n_R,\lambda))\in E(\sigma_k)} u_{n-1}^{(n_L,n'_R,\lambda')}, \tag{3}$$

from which we we would like to obtain a linear recurrence relation

$$u_n^{(n_L,n_R,\lambda)} + p_{d-1}u_{n-1}^{(n_L,n_R,\lambda)} + \ldots + p_0 u_{n-d}^{(n_L,n_R,\lambda)} = 0,$$

that holds for all $n \geq n_0$ for some n_0. Such a recurrence relation is associated with a characteristic polynomial

$$p(x) = p_d x^d + p_{d-1}x^{d-1}d + \ldots + p_1 x + p_0,$$

with $p_d = 1$. We say that $p(x)$ is a characteristic polynomial for $\{u_n\}$ for $n \geq n_0$ if

$$p_d u_n + p_{d-1}u_{n-1} + \ldots + p_0 u_{n-d} = 0,$$

holds for all $n \geq n_0$.

Lemma 3.1. *Let $\{u_n\}_{n\geq 1}$ be a sequence. The sum of two or more characteristic polynomials for $\{u_n\}$ is also a characteristic polynomial for $\{u_n\}$, and any product of a characteristic polynomial for $\{u_n\}$ by any polynomial is also a characteristic polynomial for $\{u_n\}$, for sufficiently large n.*

Corollary 3.2. *Let $\{u_n\}_{n\geq 1}$ be a sequence. The greatest common divisor of two or more characteristic polynomials for $\{u_n\}$ is also a characteristic polynomial for $\{u_n\}$, for sufficiently large n.*

Generally, for any directed graph D, we can define sequences $\{u_n^v\}_{v\in E(D)}$ by a set of initial conditions and the relation

$$u_n^v = \sum_{(v',v)\in E(D)} u_{n-1}^{v'}, \tag{4}$$

corresponding to (3).

The monic polynomial of smallest degree that is a characteristic polynomial for all these sequences for a particular set I of initial conditions and sufficiently large n is called the *minimal* characteristic polynomial on (D, I). A polynomial that is a characteristic polynomial for all these sequences for *any* set of initial conditions and sufficiently large n is called a *universal* characteristic polynomial on D, and the monic one with the smallest degree is called the *minimal universal* characteristic polynomial on D. By Corollary 3.2, the minimal characteristic polynomial on (D, I) divides the minimal universal characteristic polynomial on D regardless of I. Let I_k denote the set of initial conditions for the sequences (1).

In order to prove that a linear recurrence relation is valid for all of the sequences for all sufficiently large n, it suffices to verify that it holds for all the sequences for one value of n.

Lemma 3.3. *Suppose $u_n^{(i)} = f^{(i)}(u_{n-1}^{(1)}, u_{n-1}^{(2)}, \ldots, u_{n-1}^{(N)})$ for $n \geq n_0$, $1 \leq i \leq N$ where each $f^{(i)}$ is a linear function in N variables, and that g is a linear function such that*

$$g(u_n^{(i)}, u_{n-1}^{(i)}, \ldots, u_{n-d}^{(i)}) = 0, \tag{5}$$

for some $n = n_1 \geq n_0 + d$ and all i, $1 \leq i \leq N$. Then (5) also holds for all $n \geq n_1$ and all i, $1 \leq i \leq N$.

Proof. By induction on n. If (5) holds for all i, then

$$\begin{aligned} g(u_{n+1}^{(i)}, u_n^{(i)}, \ldots, u_{n-d+1}^{(i)}) =& g(f^{(i)}(u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(N)}), f^{(i)}(u_{n-1}^{(1)}, u_{n-1}^{(2)}, \ldots, u_{n-1}^{(N)}), \ldots, \\ & f^{(i)}(u_{n-d}^{(1)}, u_{n-d}^{(2)}, \ldots, u_{n-d}^{(N)})) \\ =& f^{(i)}(g(u_n^{(1)}, u_{n-1}^{(1)}, \ldots, u_{n-d}^{(1)}), g(u_n^{(2)}, u_{n-1}^{(2)}, \ldots, u_{n-d}^{(2)}), \ldots, \\ & g(u_n^{(N)}, u_{n-1}^{(N)}, \ldots, u_{n-d}^{(N)})) \\ =& f^{(i)}(0, 0, \ldots, 0) \\ =& 0, \end{aligned}$$

for all i. □

A directed graph D can be partitioned into *strongly connected components* (SCCs), i.e., maximal subgraphs in which any vertex can reach any other. An SCC $\xi \subseteq D$ for which there is no arc from a vertex in $D \backslash \xi$ to a vertex in ξ is called a source SCC, and one for which there is no arc from a vertex in ξ to a vertex in $D \backslash \xi$ is called a sink SCC.

Suppose that for each SCC $\xi \subseteq D$, the minimal universal characteristic polynomial on ξ is known, called $p^{(\xi)}(x)$ of degree d_ξ. Then Algorithm 1 produces a characteristic polynomial on (D, I). In view of Lemma 3.1, the step in the bottom of the flowchart does not invalidate the universal characteristic polynomials on the remaining SCCs.

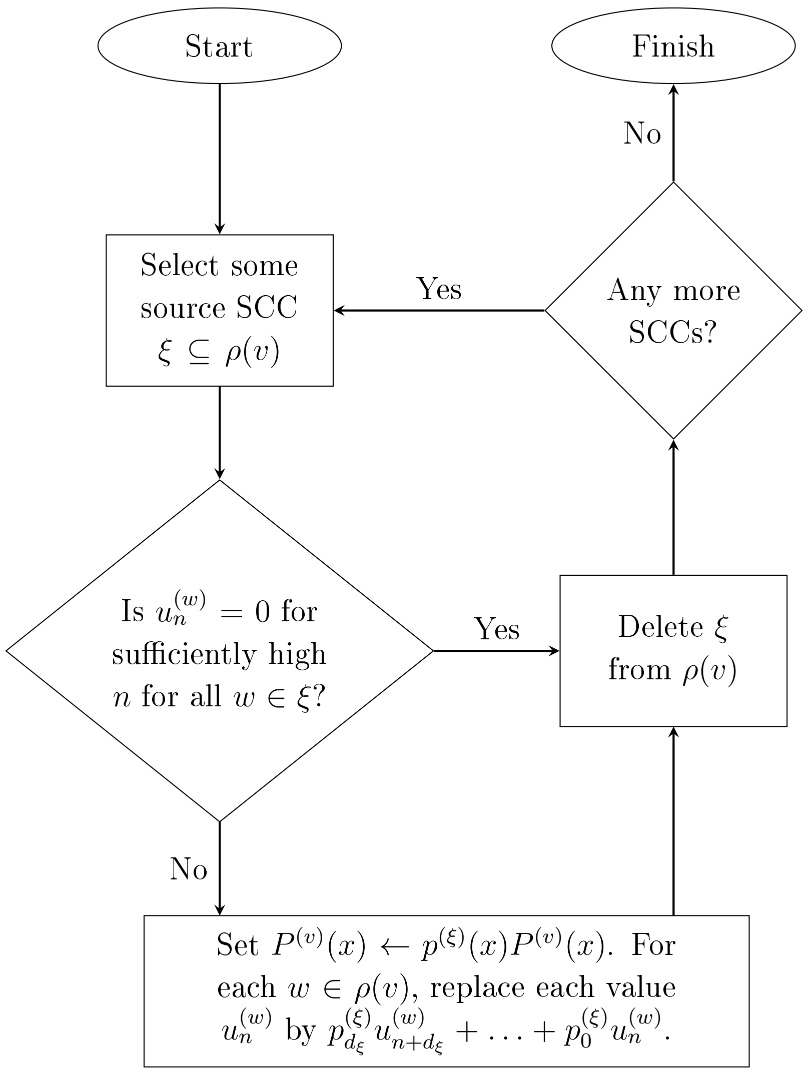
Analysis of σ_k has shown that for $k = 3$, each SCC is isomorphic to one out of 7 directed graphs with at most 24 vertices, and for $k = 4$, each SCC is isomorphic to one out of 11 directed graphs with at most 122 vertices. These are small enough that we can find universal characteristic polynomials using iterated changes of variables. In order to find the minimal one for an SCC $\xi \subseteq D$, we can run the iterated changes of variables procedure several times and take the greatest common divisor, and/or we can check if some divisor of any given universal characteristic polynomial is also a characteristic polynomial if we set

$$u_k^{(v)} = \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{otherwise,} \end{cases}$$

regardless of $v_0 \in V(\xi)$. We set the initial values for $n = k$, as (1) is only defined for $n \geq k$.

Algorithm 1

For one vertex v from each sink SCC of D, set $P^{(v)}(x) = 1$, let $\rho(v) \subseteq D$ be induced by all vertices that can reach v, define the sequences $\{u_n^{(w)}\}_{w \in \rho(v)}$ by I and (4), and execute this flowchart. Return the least common multiple of the polynomials it produces.



4. Examples

As an example of the analysis for a single SCC, consider $\xi \subset \sigma_3$ consisting of

$$
\begin{aligned}
&\{(4, 32, \{(L_2, R_2), (L_3, R_1)\}), (4, 30, \{(L_2, R_1), (L_3, R_0)\}),\\
&(4, 26, \{(L_2, R_3), (L_3, R_0)\}), (4, 11, \{(L_2, R_2), (L_3, R_3)\}),\\
&(4, 27, \{(L_2, R_1), (L_3, R_2), (R_0, R_3)\}), (4, 16, \{(L_2, R_2), (L_3, R_1)\}),\\
&(4, 4, \{(L_2, R_1), (L_3, R_0)\}), (4, 23, \{(L_2, R_3), (L_3, R_1), (R_0, R_2)\}),\\
&(4, 10, \{(L_2, R_2), (L_3, R_1)\}), (4, 13, \{(L_2, R_2), (L_3, R_3), (R_0, R_1)\}),\\
&(4, 33, \{(L_2, R_1), (L_3, R_2), (R_0, R_3)\}), (4, 33, \{(L_2, R_3), (L_3, R_1), (R_0, R_2)\}),\\
&(4, 33, \{(L_2, R_2), (L_3, R_3), (R_0, R_1)\})\},
\end{aligned}
$$

and the corresponding sequences $\{u_n^{(1)}\}$ through $\{u_n^{(13)}\}$. The relations for these sequences, based on (3), are as follows for $n \geq 4$.

$$\begin{aligned}
u_n^{(1)} &= u_{n-1}^{(11)} + u_{n-1}^{(12)}, & u_n^{(8)} &= u_{n-1}^{(5)},\\
u_n^{(2)} &= u_{n-1}^{(1)} + u_{n-1}^{(9)}, & u_n^{(9)} &= u_{n-1}^{(8)},\\
u_n^{(3)} &= u_{n-1}^{(2)} + u_{n-1}^{(7)}, & u_n^{(10)} &= u_{n-1}^{(8)},\\
u_n^{(4)} &= u_{n-1}^{(3)}, & u_n^{(11)} &= u_{n-1}^{(10)} + u_{n-1}^{(13)},\\
u_n^{(5)} &= u_{n-1}^{(4)}, & u_n^{(12)} &= u_{n-1}^{(11)},\\
u_n^{(6)} &= u_{n-1}^{(5)}, & u_n^{(13)} &= u_{n-1}^{(12)}.\\
u_n^{(7)} &= u_{n-1}^{(6)}, & &
\end{aligned}$$

Using only $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\{u_n^{(3)}\}$ and $\{u_n^{(11)}\}$ (those with more than one term in their relation), we have $u_n^{(4)} = u_{n-1}^{(3)}$, $u_n^{(5)} = u_{n-2}^{(3)}$, $u_n^{(6)} = u_{n-3}^{(3)}$, $u_n^{(7)} = u_{n-4}^{(3)}$, $u_n^{(8)} = u_{n-3}^{(3)}$, $u_n^{(9)} = u_{n-4}^{(3)}$, $u_n^{(10)} = u_{n-4}^{(3)}$, $u_n^{(12)} = u_{n-1}^{(11)}$, $u_n^{(13)} = u_{n-2}^{(11)}$, and the relations

$$u_n^{(1)} = u_{n-1}^{(11)} + u_{n-2}^{(11)}, \tag{6}$$

$$u_n^{(2)} = u_{n-1}^{(1)} + u_{n-5}^{(3)}, \tag{7}$$

$$u_n^{(3)} = u_{n-1}^{(2)} + u_{n-5}^{(3)}, \tag{8}$$

$$u_n^{(11)} = u_{n-5}^{(3)} + u_{n-3}^{(11)}. \tag{9}$$

(6) and (9) gives

$$\begin{aligned}
u_n^{(1)} &= u_{n-6}^{(3)} + u_{n-4}^{(11)} + u_{n-7}^{(3)} + u_{n-5}^{(11)}\\
&= u_{n-3}^{(1)} + u_{n-6}^{(3)} + u_{n-7}^{(3)}.
\end{aligned} \tag{10}$$

(8) and (10) gives

$$\begin{aligned}
u_n^{(1)} &= u_{n-3}^{(1)} + u_{n-7}^{(2)} + u_{n-11}^{(3)} + u_{n-8}^{(2)} + u_{n-12}^{(3)}\\
&= u_{n-3}^{(1)} + u_{n-7}^{(2)} + u_{n-8}^{(2)} + u_{n-5}^{(1)} - u_{n-8}^{(1)}\\
&= u_{n-3}^{(1)} + u_{n-5}^{(1)} - u_{n-8}^{(1)} + u_{n-7}^{(2)} + u_{n-8}^{(2)},
\end{aligned} \tag{11}$$

and (7) and (8) gives

$$\begin{aligned}
u_n^{(2)} &= u_{n-1}^{(1)} + u_{n-6}^{(2)} + u_{n-10}^{(3)}\\
&= u_{n-1}^{(1)} + u_{n-6}^{(2)} + u_{n-5}^{(2)} - u_{n-6}^{(1)}\\
&= u_{n-1}^{(1)} - u_{n-6}^{(1)} + u_{n-5}^{(2)} + u_{n-6}^{(2)}.
\end{aligned} \tag{12}$$

Finally, (11) and (12) gives

$$\begin{aligned}
u_n^{(1)} =& u_{n-3}^{(1)} + u_{n-5}^{(1)} - u_{n-8}^{(1)} + u_{n-8}^{(1)} - u_{n-13}^{(1)} + u_{n-12}^{(2)}\\
&+ u_{n-13}^{(2)} + u_{n-9}^{(2)} - u_{n-14}^{(1)} + u_{n-13}^{(1)} + u_{n-14}^{(2)}\\
=& u_{n-3}^{(1)} + u_{n-5}^{(1)} + u_{n-9}^{(1)} - u_{n-13}^{(1)} - u_{n-1}^{14} + u_{n-5}^{(1)} - u_{n-8}^{(1)}\\
&- u_{n-10}^{(1)} + u_{n-13}^{(1)} + u_{n-6}^{(1)} - u_{n-9}^{(1)} - u_{n-11}^{(1)} + u_{n-14}^{(1)}\\
=& u_{n-3}^{(1)} + 2u_{n-5}^{(1)} + u_{n-6}^{(1)} - u_{n-8}^{(1)} - u_{n-10}^{(1)} - u_{n-11}^{(1)},
\end{aligned}$$

which gives us a universal characteristic polynomial $x^{11} - x^8 - 2x^6 - x^5 + x^3 + x + 1$. However, it factorizes as

$$x^{11} - x^8 - 2x^6 - x^5 + x^3 + x + 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)(x^6 - x^3 - x - 1),$$

and we can check whether some proper divisor is also a universal characteristic polynomial. Specifically, we can let $j \in \{1, \dots, 13\}$ and set $u_3^{(j)} = 1$ and $u_3^{(i)} = 0$ for $i \neq j$. It turns out that $u_{13}^{(i)} - u_{10}^{(i)} - u_8^{(i)} - u_7^{(i)} = 0$ for all $i \in \{1, \dots, 13\}$ regardless of j. By Lemma 3.3, we then have $u_n^{(i)} - u_{n-3}^{(i)} - u_{n-5}^{(i)} - u_{n-6}^{(i)} = 0$ for all $n \geq 13$ for all i, and since any set of initial values is a linear combination of the ones we have tried out, $x^6 - x^3 - x - 1$ is a universal characteristic polynomial on (ξ, I_3). Since it is irreducible and ξ is nontrivial, it must be minimal.

As an example of an execution of the flowchart in Algorithm 1, let

$$v = (4, 16, \{(L_2, L_3), (R_1, R_2)\}).$$

The subgraph $\rho(v) \subset \sigma_3$ consists of the SCCs

$$\begin{aligned}
\xi_1 = \{&(4, 33, \{(L_2, R_0), (L_3, R_2), (R_1, R_3)\}), (4, 33, \{(L_2, R_0), (L_3, R_1), (R_2, R_3)\}), \\
&(4, 33, \{(L_2, R_0), (L_3, R_3), (R_1, R_2)\})\} \\
\xi_2 = \{&(4, 33, \{(L_2, R_1), (L_3, R_0), (R_2, R_3)\}), (4, 33, \{(L_2, R_3), (L_3, R_0), (R_1, R_2)\}), \\
&(4, 33, \{(L_2, R_2), (L_3, R_0), (R_1, R_3)\})\} \\
\xi_3 = \{&(4, 4, \{(L_2, R_0), (L_3, R_1)\}), (4, 26, \{(L_2, R_0), (L_3, R_3)\}), \\
&(4, 11, \{(L_2, R_3), (L_3, R_2)\}), (4, 27, \{(L_2, R_2), (L_3, R_1), (R_0, R_3)\}), \\
&(4, 16, \{(L_2, R_1), (L_3, R_2)\}), (4, 23, \{(L_2, R_1), (L_3, R_3), (R_0, R_2)\}), \\
&(4, 10, \{(L_2, R_1), (L_3, R_2)\}), (4, 30, \{(L_2, R_0), (L_3, R_1)\}), \\
&(4, 13, \{(L_2, R_3), (L_3, R_2), (R_0, R_1)\}), (4, 33, \{(L_2, R_2), (L_3, R_1), (R_0, R_3)\}), \\
&(4, 33, \{(L_2, R_1), (L_3, R_3), (R_0, R_2)\}), (4, 33, \{(L_2, R_3), (L_3, R_2), (R_0, R_1)\}), \\
&(4, 32, \{(L_2, R_1), (L_3, R_2)\})\} \\
\xi_4 = \{&(4, 32, \{(L_2, R_2), (L_3, R_1)\}), (4, 30, \{(L_2, R_1), (L_3, R_0)\}), \\
&(4, 26, \{(L_2, R_3), (L_3, R_0)\}), (4, 11, \{(L_2, R_2), (L_3, R_3)\}), \\
&(4, 27, \{(L_2, R_1), (L_3, R_2), (R_0, R_3)\}), (4, 16, \{(L_2, R_2), (L_3, R_1)\}), \\
&(4, 4, \{(L_2, R_1), (L_3, R_0)\}), (4, 23, \{(L_2, R_3), (L_3, R_1), (R_0, R_2)\}), \\
&(4, 10, \{(L_2, R_2), (L_3, R_1)\}), (4, 13, \{(L_2, R_2), (L_3, R_3), (R_0, R_1)\}), \\
&(4, 33, \{(L_2, R_1), (L_3, R_2), (R_0, R_3)\}), (4, 33, \{(L_2, R_3), (L_3, R_1), (R_0, R_2)\}), \\
&(4, 33, \{(L_2, R_2), (L_3, R_3), (R_0, R_1)\})\} \\
\xi_5 = \{&(4, 22, \{(L_2, L_3)\})\} \\
\xi_6 = \{&(4, 7, \{(L_2, L_3), (R_0, R_3)\})\} \\
\xi_7 = \{&(4, 19, \{(L_2, L_3), (R_2, R_3)\})\} \\
\xi_8 = \{&(4, 1, \{(L_2, L_3), (R_0, R_3), (R_1, R_2)\})\} \\
\xi_9 = \{&(4, 16, \{(L_2, L_3), (R_1, R_2)\}), (4, 4, \{(L_2, L_3), (R_0, R_1)\}), \\
&(4, 26, \{(L_2, L_3), (R_0, R_3)\}), (4, 11, \{(L_2, L_3), (R_2, R_3)\}),
\end{aligned}$$

$$(4, 27, \{(L_2, L_3), (R_0, R_3), (R_1, R_2)\}), (4, 23, \{(L_2, L_3), (R_0, R_2), (R_1, R_3)\}),$$
$$(4, 10, \{(L_2, L_3), (R_1, R_2)\}), (4, 30, \{(L_2, L_3), (R_0, R_1)\}),$$
$$(4, 13, \{(L_2, L_3), (R_0, R_1), (R_2, R_3)\}), (4, 33, \{(L_2, L_3), (R_0, R_3), (R_1, R_2)\}),$$
$$(4, 33, \{(L_2, L_3), (R_0, R_2), (R_1, R_3)\}), (4, 33, \{(L_2, L_3), (R_0, R_1), (R_2, R_3)\}),$$
$$(4, 32, \{(L_2, L_3), (R_1, R_2)\})\},$$

where ξ_4 is the SCC we analyzed earlier, and $v \in \xi_9$. The *condensantion* of $\rho(v)$ (the resulting graph if each SCC is contracted to a single vertex) is shown in Figure 2.

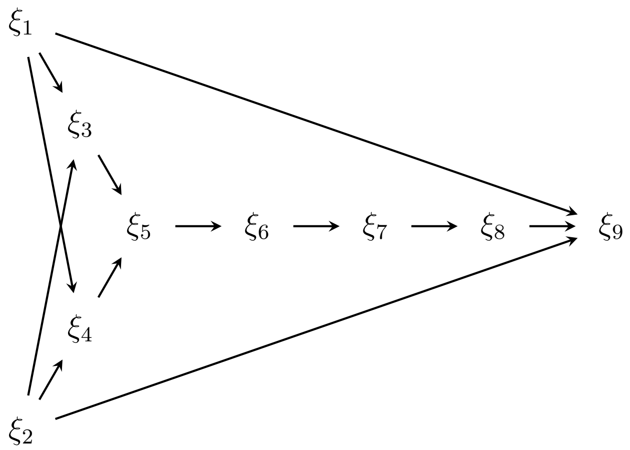


Fig. 2. Condensation of $\rho((4, 16, \{(L_2, L_3), (R_1, R_2)\}))$

The initial values in the source SCCs ξ_1 and ξ_2 are all zero, and they are deleted accordingly. ξ_3 then becomes a source SCC, and since all the initial values are zero here as well, it is also deleted, and we are left with

$$\xi_4 \longrightarrow \xi_5 \longrightarrow \xi_6 \longrightarrow \xi_7 \longrightarrow \xi_8 \longrightarrow \xi_9.$$

We have just seen that the minimal universal characteristic polynomial on ξ_4 is $x^6 - x^3 - x - 1$. ξ_9 is isomorphic to ξ_4, and therefore has the same minimal universal characteristic polynomial. The remaining SCCs are trivial, and this gives $P^{(v)}(x) = (x^6 - x^3 - x - 1)^2$.

Generally, checking divisors ought to be done after executing the flowchart like we did in the example with a single SCC.

5. Results

For $k = 3$, let the polynomials $P^{(1)} \ldots, P^{(8)}$ be given by

$$P^{(1)}(x) = x - 1,$$
$$P^{(2)}(x) = x + 1,$$
$$P^{(3)}(x) = x^2 + 1,$$
$$P^{(4)}(x) = x^2 - x + 1,$$
$$P^{(5)}(x) = x^2 + x + 1,$$
$$P^{(6)}(x) = x^4 - x^2 + 1,$$

$$P^{(7)}(x) = x^6 - x^3 - x - 1,$$
$$P^{(8)}(x) = x^{14} - x^{12} - x^8 + x^6 - x^4 - 2x^2 - 1.$$

Table 2 gives the minimal universal characteristic polynomials for each class of SCCs under graph isomorphism.

Table 2. Minimal universal characteristic polynomials for SCCs in σ_3

SCC size	Characteristic polynomial
1	1
2	$P^{(1)}(x)P^{(2)}(x)$
3	$P^{(1)}(x)P^{(5)}(x)$
4	$P^{(1)}(x)P^{(2)}(x)P^{(3)}(x)$
12	$P^{(1)}(x)P^{(2)}(x)P^{(3)}(x)P^{(4)}(x)P^{(5)}(x)P^{(6)}(x)$
13	$P^{(7)}(x)$
24	$P^{(8)}(x)$

Algorithm 1 yields

$$P(x) = P_1(x)P_2(x)P_3(x)P_4(x)P_5(x)P_6(x)\left(P_7(x)\right)^2 P_8(x),$$

of degree 38 as a characteristic polynomial on (σ_3, I_3), which we have verified to be minimal by checking all maximal divisors. The corresponding recurrence relation

$$P_{38}u_n + P_{37}u_{n-1} + \ldots + P_1 u_{n-37} + P_0 u_{n-38} = 0,$$

holds for each signature for $n = 45$, and thus, by Lemma 3.3, for all $n \geq 45$. It follows that it holds for $u_n = h_3(n)$ for $n \geq 45$. However, we have $h_3(1) = 1$ and $h_3(2) = 4$ and can verify that the recurrence relation also holds for $39 \leq n < 45$, from which we can conclude that it holds for all $n \geq 39$.

Similarly for $k = 4$, let

$$\begin{aligned}
Q^{(1)}(x) &= x - 1,\\
Q^{(2)}(x) &= x + 1,\\
Q^{(3)}(x) &= x^2 + 1,\\
Q^{(4)}(x) &= x^4 + 1,\\
Q^{(5)}(x) &= x^4 - x^3 + x^2 - x + 1,\\
Q^{(6)}(x) &= x^4 + x^3 + x^2 + x + 1,\\
Q^{(7)}(x) &= x^8 - x^6 + x^4 - x^2 + 1,\\
Q^{(8)}(x) &= x^8 - x^4 - x^2 + x - 1,\\
Q^{(9)}(x) &= x^8 - x^4 - x^2 - x - 1,\\
Q^{(10)}(x) &= x^{10} + x^8 - 2x^6 - x^4 + x^2 - 1,\\
Q^{(11)}(x) &= x^{21} - x^{20} + x^{19} - 2x^{18} + 2x^{17} - 3x^{16} + 3x^{15} - 3x^{14} + 3x^{13} - 3x^{12}\\
&\quad + 2x^{11} - 2x^{10} + x^9 - 2x^8 + 2x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1,\\
Q^{(12)}(x) &= x^{22} - 2x^{18} - x^{15} - 2x^{14} - x^{12} + x^{11} + 3x^{10} - x^9 - x^8 + 2x^6 + x^5 + x^4 - x^3 + x + 1,
\end{aligned}$$

$$Q^{(13)}(x) = x^{37} + x^{34} - x^{29} + x^{27} - x^{25} + x^{20} + 3x^{17} + 3x^{14} + 8x^{12} + x^9 + 3x^7 - 3x^5 - 1.$$

Table 3 gives the minimal universal characteristic polynomials for each class of SCCs.

Table 3. Minimal universal characteristic polynomials for SCCs in σ_4

SCC size	Characteristic polynomial
1A	1
1B	$Q^{(1)}(x)$
2	$Q^{(1)}(x)Q^{(2)}(x)$
4	$Q^{(1)}(x)Q^{(2)}(x)Q^{(3)}(x)$
5	$Q^{(1)}(x)Q^{(6)}(x)$
10	$Q^{(1)}(x)Q^{(2)}(x)Q^{(5)}(x)Q^{(6)}(x)$
20	$Q^{(1)}(x)Q^{(2)}(x)Q^{(3)}(x)Q^{(5)}(x)Q^{(6)}(x)Q^{(7)}(x)$
46	$Q^{(2)}(x)Q^{(11)}(x)$
66	$Q^{(1)}(x)Q^{(2)}(x)Q^{(3)}(x)Q^{(12)}(x)$
114	$Q^{(1)}(x)Q^{(2)}(x)Q^{(4)}(x)Q^{(8)}(x)Q^{(9)}(x)Q^{(10)}(x)$
122	$Q^{(2)}(x)Q^{(3)}(x)Q^{(5)}(x)Q^{(7)}(x)Q^{(11)}(x)Q^{(13)}(x)$

Again using Algorithm 1 and checking divisors, the minimal characteristic polynomial on (σ_4, I_4) was found to be

$$\begin{aligned}&\left(Q^{(1)}(x)Q^{(2)}(x)Q^{(3)}(x)Q^{(5)}(x)Q^{(6)}(x)Q^{(7)}(x)Q^{(11)}(x)\right)^2\\&Q^{(4)}(x)Q^{(8)}(x)Q^{(9)}(x)Q^{(10)}(x)Q^{(12)}(x)Q^{(13)}(x),\end{aligned} \tag{13}$$

of degree 171.

Let

$$\begin{aligned}Q(x) =&Q^{(1)}(x)\left(Q^{(2)}(x)Q^{(3)}(x)Q^{(5)}(x)Q^{(7)}(x)Q^{(11)}(x)\right)^2\\&Q^{(4)}(x)Q^{(8)}(x)Q^{(9)}(x)Q^{(10)}(x)Q^{(12)}(x)Q^{(13)}(x),\end{aligned}$$

of degree 162, such that (13) is equal to

$$Q(x)Q^{(1)}(x)\left(Q^{(6)}(x)\right)^2.$$

There are only 25 signatures (n_L, n_R, λ) for which $Q(x)Q^{(1)}(x)Q^{(6)}(x)$ is *not* a characteristic polynomial for $\{u_n^{(n_L,n_R,\lambda)}\}$ for $n = 176$, and none of them can reach any of the signatures corresponding to Hamiltonian cycles in $G(n, 4)$. Therefore, $Q(x)Q^{(1)}(x)Q^{(6)}(x)$ is a characteristic polynomial for $h_4(n)$ for $n \geq 176$ by Lemma 3.3, and since $Q^{(1)}(x)Q^{(6)}(x) = x^5 - 1$, it follows that

$$Q_{162}h_4(n) + Q_{161}h_4(n-1) + \ldots + Q_1h_4(n-161) + Q_0h_4(n-162),$$

is periodic with period 5, starting from $n = 171$. But all the first five terms are zero, and thus, $Q(x)$ is a characteristic polynomial for $h_4(n)$ for $n \geq 171$. Moreover, with $h_4(1) = 1$, $h_4(2) = 0$ and $h_4(3) = 3$, we can verify that the recurrence relation also holds for $163 \leq n < 171$ and conclude that it holds for all $n \geq 163$.

Theorem 5.1. *With P and Q defined as above, we have*

$$P_{38}h_3(n) + P_{37}h_3(n-1) + \ldots + P_1h_3(n-37) + P_0h_3(n-38) = 0 \text{ for } n \geq 39,$$

$$Q_{162}h_4(n) + Q_{161}h_4(n-1) + \ldots + Q_1h_4(n-161) + Q_0h_4(n-162) = 0 \text{ for } n \geq 163.$$

Appendix A gives the initial values of $h_3(n)$, $h_4(n)$ and the coefficients of $P(x)$ and $Q(x)$.

It might be desirable to write $h_k(n)$ as a sum of the nth terms of several recursive sequences, each with a characteristic polynomial of relatively small degree. One possible first step is to distinguish between admissible subgraphs of $G'(n,k)$ with an even and an odd number of loose ends on each side. Partitioning the graph of signatures σ_k into $\sigma_k^{(e)}$ and $\sigma_k^{(o)}$ accordingly, we can replace (2) by

$$h_k^{(e)}(n) = \sum_{(n_L,n_R,\lambda)\in H_k\cap V\left(\sigma_3^{(e)}\right)} u_n^{(n_L,n_R,\lambda)},$$

and

$$h_k^{(o)}(n) = \sum_{(n_L,n_R,\lambda)\in H_k\cap V\left(\sigma_3^{(o)}\right)} u_n^{(n_L,n_R,\lambda)},$$

such that $h_k(n) = h_k^{(e)}(n) + h_k^{(o)}(n)$. Furthermore, as touched upon when we considered characteristic polynomials for $h_4(n)$ above, a divisor of the characteristic polynomial of the form (or dividing) $x^j - 1$ can be eliminated if we introduce a periodic function of period j on the right hand side of the recurrence relation.

Here is a brief summary of how this plays out for $k = 3$. $V\left(\sigma_3^{(e)}\right)$ consists of signatures with $n_L, n_R \in \{1,4,6,7,10,\ldots\}$ (confer Table 1). With the same analysis as before, we obtain the initial values $1,2,0,0,0,8,7,0,9,12,11,36$ of $h_k^{(e)}(n)$ for $1 \leq n \leq 12$, and the recurrence relation

$$\begin{aligned} h_k^{(e)}(n) - 2h_k^{(e)}(n-3) - 2h_k^{(e)}(n-5)& \\ -h_k^{(e)}(n-6) + 2h_k^{(e)}(n-8) + 2h_k^{(e)}(n-9)& = \begin{cases} 8 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \\ +h_k^{(e)}(n-10) + 2h_k^{(e)}(n-11) + h_k^{(e)}(n-12)& \end{aligned}$$

for $n \geq 13$. A periodic function satisfying the same recurrence relation is given by

$$u_n^{(e)} = \begin{cases} 2 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $V\left(\sigma_3^{(o)}\right)$ consists of signatures with $n_L, n_R \in \{2,3,5,8,9,\ldots\}$, the initial values of $h_k^{(o)}(n)$ for $1 \leq n \leq 14$ are $0,2,0,6,0,8,0,6,0,12,0,32,0,58$, and

$$\begin{aligned} h_k^{(o)}(n) - h_k^{(o)}(n-2) - h_k^{(o)}(n-6)& \\ +h_k^{(o)}(n-8) - h_k^{(o)}(n-10)& = \begin{cases} 16 & \text{if } n \equiv 0 \text{ or } 2 \pmod{12} \\ -16 & \text{if } n \equiv 8 \pmod{12} \\ 0 & \text{otherwise} \end{cases} \\ -2h_k^{(o)}(n-12) - h_k^{(o)}(n-14)& \end{aligned}$$

for $n \geq 15$. A periodic function satisfying the same recurrence relation is given by

$$u_n^{(o)} = \begin{cases} -12 & \text{if } n \equiv 0 \pmod 4 \\ 4 & \text{if } n \equiv 4 \text{ or } 8 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, taking $h_3^{(\alpha)}(n) = h_k^{(e)}(n) - u_n^{(e)}$, $h_3^{(\beta)}(n) = h_k^{(o)}(n) - u_n^{(o)}$ and $h_3^{(\gamma)}(n) = u_n^{(e)} + u_n^{(o)}$ leads to the following alternative to Theorem 5.1 for $k = 3$.

Proposition 5.2. *$h_3(n) = h_3^{(\alpha)}(n) + h_3^{(\beta)}(n) + h_3^{(\gamma)}(n)$ for $n \geq 1$ where the initial values of $h_3^{(\alpha)}(n)$ are $1, 0, 0, 0, 0, 6, 7, 0, 9, 10, 11, 36$ and the characteristic polynomial is $(P^{(7)}(x))^2$, the initial values of $h_3^{(\beta)}(n)$ are $0, 2, 0, 2, 0, 8, 0, 2, 0, 12,\ 0, 44, 0, 58$ and the characteristic polynomial is $P^{(8)}(x)$, and $h_3^{(\gamma)}(n)$ is periodic with the terms $0, 2, 0, 4, 0, 2, 0, 4, 0, 2, 0, -12$ repeating.*

Conflicts of interest

The author declares that he has no conflicts of interest.

References

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Appendix A: Initial values for $h_3(n)$ and $h_4(n)$ and coefficients of their characteristic polynomials

n	$h_3(n)$	P_n	$h_4(n)$	Q_n
0		1		-1
1	1	2	1	2
2	4	3	0	-2
3	0	6	3	3
4	6	5	0	-10
5	0	6	5	12
6	16	3	6	-13
7	7	-2	7	30
8	6	0	0	-51
9	9	-6	3	38
10	24	-2	30	-56
11	11	-4	11	124
12	68	4	6	-125
13	26	0	26	91
14	88	-2	56	-205
15	75	-6	53	260
16	150	-3	80	-87
17	102	-10	34	150
18	316	-7	216	-395
19	152	0	152	131
20	436	2	240	165
21	399	6	192	205
22	664	2	462	7
23	667	6	598	-903
24	1352	-4	750	680
25	975	-2	580	-370
26	2344	-2	1300	1934
27	1998	0	1245	-2628
28	3618	-2	2156	1773
29	3683	4	2117	-3209
30	6440	4	3346	5354
31	5766	2	4123	-4076
32	11350	-2	6432	4494
33	10230	0	5822	-8135
34	18160	0	9418	7988
35	18865	-2	10400	-6156
36	30542	-1	18954	10028
37	31339	0	18944	-11486
38	53736	1	26144	7586
39			30826	-9113
40			51380	13095

n	$h_4(n)$	Q_n
41	53997	-8205
42	75718	5742
43	86344	-10580
44	148170	7544
45	158858	-266
46	212980	3517
47	249429	-3091
48	404094	-7967
49	454818	10326
50	619370	-7766
51	719698	18242
52	1133158	-25199
53	1287158	19498
54	1757538	-23046
55	2078521	31413
56	3138352	-24322
57	3669321	19086
58	5046406	-27751
59	5969030	26018
60	8756276	-18757
61	10333827	26544
62	14336756	-32452
63	17180271	22362
64	24489280	-21291
65	29332438	24233
66	40707530	-9473
67	49039712	-6804
68	68644708	10117
69	82999229	-23260
70	115391926	40604
71	140133552	-39327
72	192987894	33736
73	235458361	-36746
74	326022466	27198
75	398576453	-10072
76	543868768	7580
77	668462960	-7641
78	921571358	1552
79	1133200253	-8165
80	1534180524	24431
81	1898214186	-29303

n	$h_4(n)$	Q_n
82	2601335036	33015
83	3217850655	-40137
84	4336332804	34426
85	5393967609	-16912
86	7345848928	4473
87	9131563039	11028
88	12260475150	-31833
89	15327541552	42174
90	20748259282	-41404
91	25918078745	39960
92	34700887198	-31311
93	43555191769	13171
94	58629617490	787
95	73554425379	-9286
96	98250605966	17737
97	123781143317	-19638
98	165799513320	11823
99	208828178672	-3909
100	278309480680	-3465
101	351735275884	13053
102	469167706296	-18753
103	593050414339	17089
104	788699042066	-13954
105	999630887903	8952
106	1328619760504	-216
107	1684786193306	-7352
108	2235846181590	11071
109	2841083989352	-13733
110	3765130309158	14258
111	4788053484820	-10650
112	6341062586720	6458
113	8076169517439	-2911
114	10676699727520	-1068
115	13611490424942	4344
116	17990921225536	-5168
117	22962336522505	5384
118	30294726883974	-5326
119	38707246756884	4113
120	51065984371156	-2619
121	65302052798697	1677
122	86006615216140	-532

n	$h_4(n)$	Q_n
123	110103773896354	-854
124	145010577267434	1600
125	185761373321580	-2266
126	244298061980386	2975
127	313278174180143	-3287
128	411958560266624	3051
129	528566740244249	-2690
130	694237816258292	1946
131	891605743330819	-830
132	1170830974647758	-229
133	1504404142946959	1128
134	1973697694412892	-1880
135	2538183157473095	2277
136	3328991719090480	-2184
137	4282979219051053	1866
138	5613375501130086	-1325
139	7227354535798072	664
140	9468956256988646	-55
141	12196622849583337	-386
142	15970732181345526	686
143	20584346089684544	-804
144	26943410940175302	733
145	34740880947606620	-583
146	45454097795928882	387
147	58639873463485310	-184
148	76692548521685142	15
149	98978960938944210	82
150	129407167837177966	-136
151	167087245796774770	141
152	218370361987361104	-112
153	282059107544821099	80
154	368529000886276550	-46
155	476193350045990337	20
156	621960955996700508	0
157	803944918169270089	-9
158	1049795010065254566	12
159	1357402794266151258	-11
160	1771950050554423404	6
161	2291899313078463756	-3
162	2991220108882081740	1