A remark on Bourgain's distributional inequality on the Fourier spectrum of Boolean functions

Hamed Hatami
Department of Computer Science
University of Toronto
E-mail: hamed@cs.toronto.edu

Submitted: September 24, 2005; Accepted: May 2, 2006; Published: May 5, 2006

Abstract

Bourgain's theorem says that under certain conditions a function $f : \{0,1\}_2^n \to \{0,1\}$ can be approximated by a function g which depends only on a small number of variables. By following his proof we obtain a generalization for the case that there is a nonuniform product measure on the domain of f.

1 Introduction

Fix $0 < \alpha < 1$. Consider the measure space $(\mathbb{F}_2^n, \mu_\alpha)$ where μ_α is the product measure defined as $\mu_\alpha(x) = \alpha^{|x|}(1-\alpha)^{n-|x|}$ with $|x| = \sum_{i=1}^n x_i$. Let $f : (\mathbb{F}_2^n, \mu_\alpha) \to \{0,1\}$ be a Boolean function. We want to show that under certain conditions f can be approximated by a function g which depends only on a small number of variables. More formally, there exist indices $1 \leq i_1 < \ldots < i_m \leq n$ such $g(x) = g(y)$, if $x_{i_j} = y_{i_j}$ for every $1 \leq j \leq m$. Moreover by g approximates f we mean that

$$\|f - g\|_2^2 \leq \epsilon. \tag{1}$$

Here m and ϵ are parameters which depend on each other and the conditions on f. We are interested in the conditions that come from the Fourier-Walsh spectrum of f. The results of this type have many applications in combinatorics and computer science [1, 7, 3, 4, 8, 6]. Let

$$h = \sum_{|S| \geq k} |\widehat{f}(S)|^2, \tag{2}$$

denote the second norm squared of the Fourier transform of f on large frequencies, i.e., $|S| \geq k$. It is usually the case that when h is small, f can be approximated by a function g which depends only on a few number of variables. The result of this type for $k = 2$, $\alpha = 1/2$ has been proven in [4], and for $k = 2$ in the more general setting of the uniform measure on $\mathbb{F}_r^n$ in [1] and [5]. So far, the most general known result is Bourgain's distributional inequality [2] which deals with $\alpha = 1/2$ and the general k (See Khot and Naor [6] for a quantitative version).

In all the mentioned results the measure is assumed to be uniform. In [7] Kindler and Safra tried to generalize these results to arbitrary values of α, and proved a theorem which deals with the general values of k and α. That result requires h to be very small, and for $\alpha = 1/2$ is not as strong as Bourgain's theorem. In the present note we show that by following

Bourgain's proof one can obtain a theorem (Corollary 2.2) for general values of α which does not require h to be as small as in [7]. However we should mention that this theorem does not completely cover their result, and for sufficiently small h, their approximation is stronger.

One notion that has been used in both [2] and [7] is the hypercontractivity of the Bonami-Beckner operator. Remember that the Bonami-Beckner operator can be defined as $T_\delta f = \sum \delta^{|S|}\widehat{f}(S)w_s$ where w_S are the bases of the Fourier-Walsh expansion. For $1 \leq p \leq q < \infty$ and $0 < \eta < 1$, we say that a function $f : (\mathbb{F}_2^n, \mu_\alpha) \to \mathbb{F}$ is (p, q, η)-hypercontractive, if

$$\|T_\eta f\|_q \leq \|f\|_p.$$

The classic Bonami-Beckner Theorem says that for $\alpha = 1/2$, f is $(2, q, \frac{1}{\sqrt{q-1}})$-hypercontractive. Recently P. Wolff proved the following theorem (see also [9] and [10]).

Theorem 1.1. [11] *Let $f : (\mathbb{F}_2^n, \mu_\alpha) \to \mathbb{R}$, $q \geq 2$, $1/q + 1/q' = 1$, and $A = \frac{1-\alpha}{\alpha}$. Define*

$$\eta_{q'}(\alpha) = \eta_q(\alpha) = \left(\frac{A^{1/q'} - A^{-1/q'}}{A^{1/q} - A^{-1/q}}\right)^{-1/2}.$$

Then

1. *f is $(2, q, \eta_q(\alpha))$-hypercontractive.*
2. *f is $(q', 2, \eta_{q'}(\alpha))$-hypercontractive.*

Remark. Since T_δ is self-adjoint, by duality of L_p spaces, (1) and (2) are equivalent. Note that in Theorem 1.1, $\eta_q(1/2)$ and $\eta_{q'}(1/2)$ are not defined. However having Bonami-Beckner Theorem in mind, we define $\eta_q(1/2) = \eta_{q'}(1/2) = \frac{1}{\sqrt{q-1}}$.

For simplicity we will write η_p for $\eta_p(\alpha)$. Next we want to state Kindler and Safra's theorem. Let

$$I_\kappa = \left\{ i \in \{1, \ldots, n\} : \sum_{i \in S, |S| \leq k} \widehat{f}(S)^2 \geq \kappa \right\}. \tag{3}$$

The goal is to show that for small values of h and κ, f essentially depends only on the variables with indices in I_κ. First note that

$$\kappa |I_\kappa| \leq \sum_{i=1}^{n} \sum_{S : i \in S, |S| \leq k} \widehat{f}(S)^2 \leq k$$

which follows

$$|I_\kappa| \leq k/\kappa. \tag{4}$$

Let $g = \sum_{S \subseteq I_\kappa} \widehat{f}(S) w_S$. Note that g depends only on the variables with indices in I_κ. Moreover

$$\|f - g\|_2^2 = \sum_{S \not\subseteq I_\kappa} \widehat{f}(S)^2. \tag{5}$$

So we have to bound the right hand side of (5).

Theorem 1.2. (Kindler and Safra [**7**]) *There exists a global constant C such that for $h, \kappa \le \eta_4^{16k}/C$, we have*
$$\sum_{S \not\subseteq I_\kappa} \widehat{f}(S)^2 \le h(1 + 1266\eta_4^{-4k} h^{1/4}).$$

the next lemma estimates $\eta_p(\alpha)$. In the following we write $x \lesssim y$ to indicate that there is a universal constant $c > 0$ such that $x \le cy$.

Lemma 1.3. *For $\alpha \le 1/2$, and $1 \le p = 1 + x \le 2$, we have*

- *If $\alpha \le e^{-1/x}$, then*
$$\eta_p(\alpha) \gtrsim \alpha^{\frac{2-p}{2p}}.$$
- *If $\alpha > e^{-1/x}$, then*
$$\eta_p(\alpha) \gtrsim \alpha^{\frac{2-p}{2p}} \sqrt{\ln(1/\alpha)x}.$$

Proof. Notice that
$$\begin{aligned} \eta_p(\alpha) = A^{\frac{p-2}{2p}} \left(\frac{1 - A^{-2/p}}{1 - A^{-2/p'}} \right)^{-1/2} & \ge \alpha^{\frac{2-p}{2p}} \left(\frac{1 - e^{-2\ln(A)/p}}{1 - e^{-2\ln(A)/p'}} \right)^{-1/2} \\ & \ge \alpha^{\frac{2-p}{2p}} \sqrt{1 - e^{-2\ln(A)/p'}}. \end{aligned} \tag{6}$$

First assume that $\alpha \le e^{-1/x}$. Then since $p' = 1 + 1/x$, we have
$$\ln(A)/p' \ge \frac{\ln(1/(2\alpha))}{p'} \ge \frac{1/x - \ln(2)}{1/x + 1} \ge 1/10.$$

So
$$(6) \gtrsim \alpha^{\frac{2-p}{2p}}.$$

Next consider $\alpha > e^{-1/x}$. We can assume that $-2\ln(A)/p' > -1$ as otherwise the theorem becomes clear. Using the fact that for $0 < y < 1$, $e^{-y} \le 1 - y/2$, we get
$$(6) \gtrsim \alpha^{\frac{2-p}{2p}} \sqrt{\ln(A)/p'} \gtrsim \alpha^{\frac{2-p}{2p}} \sqrt{\ln(1/\alpha)x}.$$

□

2 Main Result

In this section we state our main result.

Theorem 2.1. *Let $0 < \alpha \le 1/2$, and $f : (\mathbb{F}_2^n, \mu_\alpha) \to \{0, 1\}$ be a Boolean function. For $2 < k \le n$ and $0 < \kappa < 1$, h and I_κ are defined as in (2) and (3), respectively. If*
$$\phi = \frac{\log_2(1/\alpha) + \sqrt{\log_2(1/h)\log_2\log_2 k}}{16\log_2(1/h)},$$
then
$$\gamma = \sum_{|S|<k, S \not\subseteq I_\kappa} \widehat{f}(S)^2 \lesssim \sqrt{k} 2^{\frac{\log_2\log_2 k}{\phi}} \left(h/\alpha + \alpha^{-\frac{k+1}{2}} \kappa^{1/4} \right) + (\log_2 k)\sqrt{h}. \tag{7}$$

Proof. To prove the theorem in the special case of $\alpha = 1/2$, [2] used the following facts:

$$\|f\|_p \geq \left(\sum_{A \subseteq \{1,\ldots,n\}} \eta_p(\alpha)^{2|A|} \widehat{f}(A)^2 \right)^{1/2} \geq \eta_p(\alpha) \left(\sum_{i=1}^{n} \widehat{f}(\{i\})^2 \right)^{1/2}, \tag{8}$$

and if a function f satisfies $\widehat{f}(S) = 0$, for every $|S| \geq k$, then

$$\|f\|_4 = \|T_{\eta_4} T_{\eta_4^{-1}} f\|_4 \leq \|T_{\eta_4^{-1}} f\|_2 \leq \eta_4^{-k} \|f\|_2,$$

or

$$\|f\|_4^4 \leq \eta_4^{-4k} \|f\|_2^4. \tag{9}$$

By substituting (8) and (9) for general value of α in the proof, we obtain the following inequality instead of Equation (20) in [2]:

$$\delta^{p/2} \rho_{t_0} \lesssim \frac{\eta_p^{-p} \delta}{2^{t_0}} \left(\sum_{t < \log_2 k} 2^t \rho_t \right) + \eta_p^{-p} h + (\delta h)^{p/2} + (\eta_4^{-2k} \sqrt{\kappa})^{p/2}, \tag{10}$$

for every $1 \leq t_0 < \log_2 k$, $0 < \delta < 1$, and $1 \leq p \leq 2$, where

$$\rho_t = \sum_{2^{t-1} \leq |S \setminus I_\kappa| < 2^t} \widehat{f}(S)^2. \tag{11}$$

We distinguish two cases:
Case 1:

$$\sum_{t < \log_2 k} 2^t \rho_t \geq \gamma \sqrt{k},$$

where γ is defined in (7). Choose t_0 to satisfy

$$2^{t_0} \rho_{t_0} \geq \frac{\sum_{t < \log_2 k} 2^t \rho_t}{\log_2 k}.$$

It follows that $\rho_{t_0} \geq \frac{\gamma}{\sqrt{k} \log_2 k}$. Assume $p \in (3/2, 2)$ so that we can use Lemma 1.3 and obtain $\eta_p \gtrsim \alpha^{\frac{2-p}{2p}}$. Substituting these in (10) we get

$$\left(\delta^{p/2} - \alpha^{\frac{p-2}{2}} \delta \log_2 k \right) \frac{\gamma}{\sqrt{k} \log_2 k} \lesssim \alpha^{\frac{p-2}{2}} h + (\delta h)^{p/2} + (\eta_4^{-2k} \sqrt{\kappa})^{p/2}. \tag{12}$$

Now taking $\delta = \frac{\alpha (\log_2 k)^{\frac{2}{p-2}}}{4}$ we obtain

$$\delta^{p/2} \frac{\gamma}{\sqrt{k} \log_2 k} \lesssim \alpha^{\frac{p-2}{2}} h + (\delta h)^{p/2} + (\eta_4^{-2k} \sqrt{\kappa})^{p/2}. \tag{13}$$

We can assume that

$$\sqrt{k} h / \alpha < 1, \tag{14}$$

as otherwise (7) becomes clear. Let $p = 2 - 4\phi$. Note that (14) implies $p > 3/2$, and by a straightforward calculation that the first term on the right hand side of (13) is greater than the second term. So

$$\gamma \lesssim 2^{\frac{\log_2 \log_2 k}{2\phi}} \left(\sqrt{k} h / \alpha + \sqrt{k} (\eta_4^{-2k} \sqrt{\kappa} / \alpha)^{1 - 2\phi} \right). \tag{15}$$

Case 2:

$$\sum_{t<\log_2 k} 2^t \rho_t \leq \gamma\sqrt{k}.$$

In this case we choose t_0 such that $\rho_{t_0} \geq \frac{\gamma}{\log_2 k}$. Substituting these in (10) we get

$$\left(\frac{\delta^{p/2}}{\log_2 k} - \eta_p^{-p}\delta\sqrt{k}\right)\gamma \lesssim \eta_p^{-p}h + (\delta h)^{p/2} + (\eta_4^{-2k}\sqrt{\kappa})^{p/2}. \tag{16}$$

Taking $\delta \approx \frac{\alpha}{k(\log_2 k)^4}$ and $p = 1 + \frac{1}{6\log_2 k}$ we get

$$\eta_p \geq \alpha^{\frac{2-p}{2p}}(\log_2 k)^{-1/2},$$

and so

$$\gamma \lesssim \frac{1}{\alpha}\sqrt{k}(\log_2 k)^4 h + (\log_2 k)\sqrt{h} + \left(\frac{\eta_4^{-2k}\sqrt{\kappa}k(\log_2 k)^6}{\alpha}\right)^{1/2}. \tag{17}$$

From (15) and (17) we obtain

$$\gamma \lesssim 2^{\frac{\log_2\log_2 k}{\phi}}\left(\sqrt{k}h/\alpha + (\eta_4^{-2k}\sqrt{\kappa}k/\alpha)^{1/2}\right) + (\log_2 k)\sqrt{h}.$$

□

Corollary 2.2. *Let $f : (\mathbb{F}_2^n, \mu_\alpha) \to \{0,1\}$ be a Boolean function. Let $2 < k \leq n$, and h and I_κ be defined as in (2) and (3) respectively. If $0 < \kappa < h^4\alpha^{2k-2}$ and*

$$\phi = \frac{\log_2(1/\alpha) + \sqrt{\log_2(1/h)\log_2\log_2 k}}{16\log_2(1/h)},$$

then

$$\sum_{S\not\subseteq I_\kappa} \widehat{f}(S)^2 \lesssim \sqrt{k}2^{\frac{\log_2\log_2 k}{\phi}}h/\alpha + (\log_2 k)\sqrt{h}.$$

Proof. Note that $\sum_{S\not\subseteq I_\kappa}\widehat{f}(S)^2 \leq \gamma + h$, where γ is defined in Theorem 2.1. Moreover we can assume that $\frac{\sqrt{h}}{2} \geq h$ as otherwise the corollary becomes obvious. Now the assumption $\kappa < h^4\alpha^{2k-2}$ completes the proof. □

Acknowledgment

The author is grateful to Mahya Ghandehari for reading the draft version of this note, and her valuable comments.

References

[1] N. Alon, I. Dinur, E. Friedgut, and B. Sudakov, *Graph products, Fourier analysis and spectral techniques*, Geom. Funct. Anal., 14 (2004), pp. 913–940.

[2] J. Bourgain, *On the distributions of the Fourier spectrum of Boolean functions*, Israel J. Math., 131 (2002), pp. 269–276.

[3] E. Friedgut, *Boolean functions with low average sensitivity depend on few coordinates*, Combinatorica, 18 (1998), pp. 27–35.

[4] E. Friedgut, G. Kalai, and A. Naor, *Boolean functions whose Fourier transform is concentrated on the first two levels*, Adv. in Appl. Math., 29 (2002), pp. 427–437.

[5] M. Ghandehari and H. Hatami, *Fourier analysis and large independent sets in powers of complete graphs*, submitted, (2005).

[6] S. Khot and A. Naor, *Nonembeddability theorems via Fourier analysis*, in 46th IEEE Symposium on Foundations of Computer Science, 2005. to appear.

[7] G. Kindler and S. Safra, *Noise-resistant boolean-functions are juntas*, prerint.

[8] N. Linial, Y. Mansour, and N. Nisan, *Constant depth circuits, Fourier transform, and learnability*, J. Assoc. Comput. Mach., 40 (1993), pp. 607–620.

[9] E. Mossel, R. O'Donnell, and K. Oleszkiewicz, *Noise stability of functions with low influences: invariance and optimality*, in 46th IEEE Symposium on Foundations of Computer Science, 2005. to appear.

[10] K. Oleszkiewicz, *On a nonsymmetric version of the Khinchine-Kahane inequality*, in Stochastic inequalities and applications, vol. 56 of Progr. Probab., Birkhäuser, Basel, 2003, pp. 157–168.

[11] P. Wolff, *Hypercontractivity of simple random variables*, preprint, (2005).