ENUMERATION RISES ACCORDING TO PARITY IN COMPOSITIONS

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ABSTRACT. Let s, t be any numbers in $\{0, 1\}$ and let $\pi = \pi_1\pi_2\cdots\pi_m$ be any word, we say that $i \in [m-1]$ is an (s,t) *parity-rise* if $\pi_i \equiv s \pmod 2$, $\pi_{i+1} \equiv t \pmod 2$ whenever $\pi_i < \pi_{i+1}$. We denote the number occurrences of (s,t) parity-rises in π by $rise_{st}(\pi)$. Also, we denote the total sizes of the (s,t) parity-rises in π by $size_{st}(\pi)$, that is, $size_{st}(\pi) = \sum_{\pi_i<\pi_{i+1}}(\pi_{i+1} - \pi_i)$. A composition $\pi = \pi_1\pi_2\cdots\pi_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. The number of summands, namely m, is called the number of parts of π. In this paper, by using tools of linear algebra, we found the generating function that count the number of all compositions of n with m parts according to the statistics $rise_{st}$ and $size_{st}$, for all s, t.

1. INTRODUCTION

A *composition* $\pi = \pi_1\pi_2\cdots\pi_m$ of a positive integer $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n, i.e., π is a partition of n where the parts are ordered. The number of summands, namely m, is called the number of *parts* of π. Let $\mathcal{C}_n$ ($\mathcal{C}_{n,m}$, $\mathcal{C}_{n,m}^{[d]}$, respectively) be the set of all compositions of n (with exactly m parts, with exactly m parts in $[d] = \{1, 2, \ldots, d\}$, respectively). Clearly, the number of compositions of n is given by $|\mathcal{C}_n| = 2^{n-1}$ (for example, see [14]).

Let $\pi = \pi_1\pi_2\cdots\pi_m$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ be any two words of length m and s with $m \geq s$. An *occurrence* of σ in π is a subword $\pi_i\pi_{i+1}\cdots\pi_{i+s-1}$ such that $\pi_{i-1+a} < \pi_{i-1+b}$ if and only if $\sigma_a < \sigma_b$, for all $1 \leq a < b \leq s$. Here, σ is called a *subword pattern* of length s (or *s-letter pattern*). We denote the the number of the occurrences of σ in π by $occr_\sigma(\pi)$. We define $size_\sigma(\pi)$, the *total size* of σ in π, to be the sum over all occurrences $\pi_i\pi_{i+1}\cdots\pi_{i+s-1}$ of σ in π of the difference $\sum_{j=i}^{i+s-2} \pi_{j+1} - \pi_j$.

The subject statistics on compositions has been received a lot of attention (for instance, see [14] and references therein). For instance, Alladi and Hoggatt [1] found the average of rises (number occurrences of 12), descents (number occurrences of 21) and levels (number occurrences of 11) in compositions of n with parts in $\{1, 2\}$. This work has been extended by Heubach and Mansour [13], where they studied the generating function for the number of compositions of n with exactly m parts according to the number of occurrences of the patterns 11, 12 and 21. More recently, Blecher, Brennan

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and Knopfmacher [6] obtained asymptotic expressions for the average size of the descent immediately following the first and the last maximum. Heubach, Knopfmacher, Mays and Munagi [11] considered the generating function for the number of all compositions of n with exactly m parts according to the number of the inversions (an *inversion* in $\pi_1\pi_2\cdots\pi_m$ is a pair $\pi_i\pi_j$ of summands such that $1 \le i < j \le m$ and $\pi_i > \pi_j$). More recently, the authors [3] found the mean and the average of the total size of the rises, the levels and the descents taken over all compositions of n (see [2, 4, 5]).

Let s, t be any numbers in $\{0, 1\}$ and let $\pi = \pi_1\pi_2\cdots\pi_m$ be any word, we say that $i \in [m-1]$ is an (s, t) *parity-rise* if

$$\pi_i \equiv s \pmod 2,\ \pi_{i+1} \equiv t \pmod 2 \quad \text{whenever} \quad \pi_i < \pi_{i+1}. \tag{1}$$

We denote the number occurrences of (s, t) parity-rises in π by $rise_{st}(\pi)$. Also, we denote the total sizes of the (s, t) parity-rises in π by $size_{st}(\pi)$, that is,

$$size_{st}(\pi) = \sum_{\pi_i<\pi_{i+1}} (\pi_{i+1} - \pi_i).$$

For example, if $\pi = 12346263$ then $occr_{00}(\pi) = 2$ and $size_{00}(\pi) = 6$. We denote the generating function for the number of compositions of n with exactly m parts according to the number of (s, t) parity-rises and the statistic $size_{st}$ by $C_{st} = C_{st}(x, y, q, u)$, that is,

$$C_{st} = \sum_{n,m\ge 0} \sum_{\pi\in\mathcal{C}_{n,m}} x^n y^m q^{rise_{st}(\pi)} u^{size_{st}(\pi)}.$$

In the case that the m parts are related to the set $[d]$, we define

$$C^{[d]}_{st} = \sum_{n,m\ge 0} \sum_{\pi\in\mathcal{C}^{[d]}_{n,m}} x^n y^m q^{rise_{st}(\pi)} u^{size_{st}(\pi)}.$$

In this paper, we will derive explicit formulas for the generating functions C_{st}, where $s, t \in \{0, 1\}$. As consequence, we find an explicit formula for the average of the statistic $size_{st}$ in the set of compositions of n, see Table 1.

(s,t)	$\frac{1}{2^{n-1}}\sum_{\pi\in\mathcal{C}_n} size_{st}(\pi)$
$(0,0)$	$4\left(\frac{5n-23}{675}\right) + \frac{1}{2^{n+2}} + (-1)^n\left(\frac{6n^2-20n+5}{27\cdot 2^{n+2}}\right) + (-i)^n\left(\frac{-3i-4}{25\cdot 2^{n+1}}\right) + i^n\left(\frac{3i-4}{25\cdot 2^{n+1}}\right)$, $n \ge 6$
$(1,1)$	$16\left(\frac{5n-13}{675}\right) + \frac{1}{2^{n+2}} + (-1)^n\left(\frac{6n^2+4n-11}{27\cdot 2^{n+2}}\right) + (-i)^n\left(\frac{4+3i}{25\cdot 2^{n+1}}\right) + i^n\left(\frac{4-3i}{25\cdot 2^{n+1}}\right)$, $n \ge 4$
$(0,1)$	$\frac{n-4}{27} + \frac{1}{2^{n+2}} + (-1)^{n+1}\left(\frac{6n^2-20n+11}{27\cdot 2^{n+2}}\right)$, $n \ge 5$
$(1,0)$	$4\left(\frac{n-2}{27}\right) + \frac{1}{2^{n+2}} + (-1)^{n+1}\left(\frac{6n^2+4n-5}{27\cdot 2^{n+2}}\right)$, $n \ge 3$

TABLE 1. Explicit formulas for the average $\frac{1}{2^{n-1}}\sum_{\pi\in\mathcal{C}_n} size_{st}(\pi)$.

2. Main results

In order to study the generating function C_{st}, $s,t \in \{0,1\}$, we need the following general notation. We denote the generating function for the number of compositions $\pi = \pi_1\pi_2\cdots\pi_m$ of n with exactly m parts such that $\pi_j = a_j$ for all $j = 1,2,\ldots,\ell$ according to the statistics $rise_{st}$ and $size_{st}$ by

$$C_{st}(a_1\cdots a_\ell) = C_{st}(x,y,q,u|a_1\cdots a_\ell) = \sum_{n,m\geq 0}\sum_{\pi=a_1\cdots a_\ell\pi_{\ell+1}\cdots\pi_m\in C_{n,m}} x^n y^m q^{rise_{st}(\pi)} u^{size_{st}(\pi)}.$$

In the case that the m parts are related to the set $[d]$, we define

$$C^{[d]}_{st}(a_1\cdots a_\ell) = C^{[d]}_{st}(x,y,q,u|a_1\cdots a_\ell) = \sum_{n,m\geq 0}\sum_{\pi=a_1\cdots a_\ell\pi_{\ell+1}\cdots\pi_m\in C^{[d]}_{n,m}} x^n y^m q^{rise_{st}(\pi)} u^{size_{st}(\pi)}.$$

Now, we consider each case of counting (s,t) parity-rises by the following four subsections.

2.1. **Counting $(0,0)$ parity-rises.**

By the definitions, we have

$$C_{00}(x,y,q,u) = 1 + \sum_{a\geq 1} C_{00}(a). \tag{2}$$

The recurrence relation for the generating function $C_{00}(a)$ can be obtained as follows:

$$\begin{aligned}
C_{00}(a) &= x^a y + \sum_{b=1}^{a} C_{00}(ab) + \sum_{b\geq a+1} C_{00}(ab)\\
&= x^a y + x^a y\sum_{b=1}^{a} C_{00}(b) + \delta_a x^a yq \sum_{b\geq a+1}\delta_b C_{00}(b)u^{b-a} + \delta_a x^a y \sum_{b\geq a+1}(1-\delta_a)C_{00}(b)\\
&+ (1-\delta_a)x^a y \sum_{b\geq a+1} C_{00}(b),
\end{aligned}$$

where $\delta_a = 1$ when a is even, and $\delta_a = 0$ otherwise. By (2), we obtain that

$$C_{00}(a) = x^a y C_{00} + \delta_a x^a y \sum_{b\geq a+1}\delta_b C_{00}(b)(qu^{b-a}-1). \tag{3}$$

Now, we focus in studying the generating function $C^{[d]}_{00}(a)$. In order to obtain an explicit formula for the generating function $C^{[d]}_{00}(a)$, we need the following lemma.

Lemma 2.1. *Let $1 \leq i \leq d$. Then the determinant*

$$\begin{vmatrix}
\beta_i & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,d-1} & a_{i,d}\\
\beta_{i+1} & 1 & a_{i+1,i+2} & \cdots & a_{i+1,d-1} & a_{i+1,d}\\
 & \ddots & \ddots & \ddots & & \vdots\\
\beta_{d-1} & 0 & 0 & \cdots & 1 & a_{d-1,d}\\
\beta_d & 0 & 0 & \cdots & 0 & 1
\end{vmatrix}$$

is given by

$$\sum_{j=0}^{d-i}\beta_{i+j}\left(\sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j}(-1)^s\prod_{\ell=1}^{s}a_{k_{\ell-1},k_\ell}\right).$$

Proof. We proceed the proof by induction on $d \geq i$. For $d = i$, the determinant equals β_i, which agrees with the given formula. Assume that the claim holds for d and let us prove it for $d+1$. By the induction hypothesis we have that the determinant

$$D_i = \begin{vmatrix} \beta_i & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,d} & a_{i,d+1} \\ \beta_{i+1} & 1 & a_{i+1,i+2} & \cdots & a_{i+1,d} & a_{i+1,d+1} \\ & \ddots & \ddots & \ddots & & \vdots \\ \beta_d & 0 & 0 & \cdots & 1 & a_{d,d+1} \\ \beta_{d+1} & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

equals (evaluating by the leftmost column) $D_i = \sum_{j=0}^{d+1-i}(-1)^j\beta_{i+j}D_{ij}$, where D_{ij} is the determinant that obtained from D_i by removing the leftmost column and the $(j+1)$-st row. By induction hypothesis, we have that D_i

$$D_{ij} = \sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j}(-1)^{j-s}\prod_{\ell=1}^{s}a_{k_{\ell-1},k_\ell},$$

for $j = 0, 1, \ldots, d-i$. Thus, it remains to find $D_{i(d+1-i)}$, namely,

$$D_{i(d+1-i)} = \begin{vmatrix} a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,d} & a_{i,d+1} \\ 1 & a_{i+1,i+2} & \cdots & a_{i+1,d} & a_{i+1,d+1} \\ 0 & 1 & \ddots & a_{i+2,d} & a_{i+2,d+1} \\ \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{d,d+1} \end{vmatrix}.$$

Therefore, by induction hypothesis, we obtain that

$$\begin{aligned} D_{i(d+1-i)} &= a_{i,i+1}D_{(i+1)(d-i)} - \begin{vmatrix} a_{i,i+2} & a_{i,i+3} & \cdots & a_{i,d} & a_{i,d+1} \\ 1 & a_{i+2,i+3} & \cdots & a_{i+2,d} & a_{i+2,d+1} \\ \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{d,d+1} \end{vmatrix} \\ &= a_{i,i+1}\sum_{k_0=i+1<k_1<k_2<\cdots<k_s=d+1}(-1)^{d+1-i-s}\prod_{\ell=1}^{s}a_{k_{\ell-1},k_\ell} \\ &\qquad - \sum_{k_0=i<k_1=i+2<k_2<\cdots<k_s=d+1}(-1)^{d+1-i-s}\prod_{\ell=1}^{s}a_{k_{\ell-1},k_\ell} \\ &= \sum_{k_0=i<k_1<k_2<\cdots<k_s=d+1}(-1)^{d+1-i-s}\prod_{\ell=1}^{s}a_{k_{\ell-1},k_\ell}. \end{aligned}$$

Hence,

$$D_i = \sum_{j=0}^{d+1-i} (-1)^j \beta_{i+j} \left(\sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j} (-1)^{j-s} \prod_{\ell=1}^{s} a_{k_{\ell-1},k_\ell} \right),$$

which completes the induction step. □

Theorem 2.2. *Let $i = 1, 2, \ldots, d$. Then*

$$C_{00}^{[d]}(x,y,q,u|i) = p_i C_{00}^{[d]}(x,y,q,u) \text{ and } C_{00}^{[d]}(x,y,q,u) = \frac{1}{1-\sum_{i=1}^{d} p_i},$$

where

$$p_i = x^i y \sum_{j=0}^{d-i} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} \delta_{k_\ell} \prod_{\ell=1}^{s} (qu^{k_\ell - k_{\ell-1}} - 1) \right).$$

Proof. By (3) we have

$$\begin{cases} C_{00}^{[d]}(1) & = \beta_1 - \sum_{j=2}^{d} C_{00}(j)\widehat{\alpha}_{1,j} \\ C_{00}^{[d]}(2) & = \beta_2 - \sum_{j=3}^{d} C_{00}(j)\widehat{\alpha}_{2,j} \\ & \vdots \\ C_{00}^{[d]}(d-1) & = \beta_{d-1} - \sum_{j=d}^{d} C_{00}(j)\widehat{\alpha}_{d-1,j} \\ C_{00}^{[d]}(d) & = \beta_d. \end{cases}$$

The above system of equations can be written in a matrix form as follows

$$A \begin{pmatrix} C_{00}^{[d]}(1) \\ C_{00}^{[d]}(2) \\ \vdots \\ C_{00}^{[d]}(d) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_d \end{pmatrix}, \ A = \begin{pmatrix} 1 & \widehat{\alpha}_{1,2} & \widehat{\alpha}_{1,3} & \widehat{\alpha}_{1,4} & \widehat{\alpha}_{1,5} & \widehat{\alpha}_{1,6} & \widehat{\alpha}_{1,7} & \cdots & \widehat{\alpha}_{1,d} \\ 0 & 1 & \widehat{\alpha}_{2,3} & \widehat{\alpha}_{2,4} & \widehat{\alpha}_{2,5} & \widehat{\alpha}_{2,6} & \widehat{\alpha}_{2,7} & \cdots & \widehat{\alpha}_{2,d} \\ & \ddots & \ddots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \widehat{\alpha}_{d-1,d} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We solve this system by Cramer's method and we obtain

$$C_{00}^{[d]}(i) = \begin{vmatrix} \beta_i & \widehat{\alpha}_{i,i+1} & \widehat{\alpha}_{i,i+2} & \cdots & \widehat{\alpha}_{i,d-1} & \widehat{\alpha}_{i,d} \\ \beta_{i+1} & 1 & \widehat{\alpha}_{i+1,i+3} & \widehat{\alpha}_{i+1,i+4} & \cdots & \widehat{\alpha}_{i+1,d} \\ & \ddots & \ddots & \vdots & & \\ \beta_{d-1} & 0 & 0 & \cdots & 1 & \widehat{\alpha}_{d-1,d} \\ \beta_d & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Lemma 2.1 gives

$$C_{00}^{[d]}(i) = \sum_{j=0}^{d-i} \beta_{i+j} \left(\sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j} (-1)^s \prod_{\ell=1}^{s} \widehat{\alpha}_{k_{\ell-1},k_\ell} \right),$$

for all $i = 1, 2, \ldots, d$, where $\beta_i = x^i y C_{00}^{[d]}$ and $\widehat{\alpha}_{i,j} = -\delta_i x^i y \delta_j (qu^{j-i} - 1)$. Thus,

$$C_{00}^{[d]}(i) = \sum_{j=0}^{d-i} x^{i+j} y C_{00}^{[d]} \left(\sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j} \prod_{\ell=1}^{s} \delta_{k_{\ell-1}} x^{k_{\ell-1}} y \delta_{k_\ell} (qu^{k_\ell - k_{\ell-1}} - 1) \right),$$

which is equivalent to $C_{00}^{[d]}(i) = p_i C_{00}^{[d]}$. By the fact that $C_{00}^{[d]} = 1 + \sum_{i=1}^{d} C_{00}^{[d]}(i)$, we complete the proof. □

Theorem 2.3. *The generating function $C_{00}(x, y, q, u)$ is given by*

$$C_{00}(x, y, q, u) = \frac{1}{1 - \sum_{i \geq 1} p_i},$$

where

$$p_i = x^i y \sum_{j \geq 0} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} \delta_{k_\ell} \prod_{\ell=1}^{s} (qu^{k_\ell - k_{\ell-1}} - 1) \right).$$

Proof. By taking $d \to \infty$ in Theorem 2.2, we obtain the result. □

Example 2.4. *By substituting $q = u = 1$ in Theorem 2.3, we get $C_{00}(x, y, 1, 1) = \frac{1-x}{1-x-y}$ which is the generating function of all the compositions of n with m parts.*

Corollary 2.5. *The mean of $size_{00}$, taken over all compositions of n, for $n \geq 6$, is given by*

$$\begin{aligned}
&\frac{1}{2^{n-1}} \sum_{\pi \in \mathcal{C}_n} size_{00}(\pi) \\
&= 4\left(\frac{5n-23}{675}\right) + \frac{1}{2^{n+2}} + (-1)^n \left(\frac{6n^2 - 20n + 5}{27 \cdot 2^{n+2}}\right) \\
&+ (-i)^n \left(\frac{-3i-4}{25 \cdot 2^{n+1}}\right) + i^n \left(\frac{3i-4}{25 \cdot 2^{n+1}}\right),
\end{aligned}$$

where $i^2 = -1$.

Proof. By differentiating the generating function $C_{00}(x, y, q, u)$ with respect to u and evaluating it at $u = 1$, we obtain

$$\frac{d}{du} C_{00}(x, 1, 1, u) \mid_{u=1} = \frac{-\frac{d}{du} A}{A^2} \mid_{u=1} = \frac{-(1-x)^2 \frac{d}{du} A}{(1-2x)^2} \mid_{u=1},$$

where

$$A = 1 - \sum_{i \geq 1} \left(x^i \sum_{j \geq 0} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} \delta_{k_\ell} \prod_{\ell=1}^{s} (qu^{k_\ell - k_{\ell-1}} - 1) \right) \right).$$

We denote A_j to be

$$\sum_{j \geq 0} x^j \sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} \delta_{k_\ell} \prod_{\ell=1}^{s} (qu^{k_\ell - k_{\ell-1}} - 1),$$

and $A_0 = 1$, which leads to

$$A = 1 - \sum_{i\geq 1} x^i \left(A_0 + \sum_{j\geq 1} A_j \right) = 1 - \sum_{i\geq 1} x^i \left(1 + \sum_{j\geq 1} A_j \right).$$

Clearly,

$$\begin{aligned} A_j = & \sum_{k_0=i<k_1=i+j} x^{k_0} \delta_{k_0} \delta_{k_1} (u^j - 1) \\ & + \sum_{s\geq 2} \sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} \delta_{k_\ell} \prod_{\ell=1}^{s} (q u^{k_\ell - k_{\ell-1}} - 1). \end{aligned}$$

By differentiating A_j with respect to u and substituting $u = 1$, we get

$$\frac{d}{du} A_j \mid_{u=1} = x^i \delta_i \delta_{i+j} j,$$

which leads to

$$\frac{d}{du} A \mid_{u=1} = -\sum_{i\geq 1} \sum_{j\geq 1} x^{2i+j} \delta_i \delta_{i+j} j = -\sum_{i\geq 1} \sum_{j\geq 1} x^{4i+2j} 2j = -\frac{2x^6}{(1-x^4)(1-x^2)^2}.$$

Thus

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{C}_n} size_{00}(\pi) x^n = \frac{(1-x)^2 2x^6}{(1-2x)^2(1-x^2)^2(1-x^4)}.$$

By using partial fraction decompositions, we have

$$\begin{aligned} \sum_{n\geq 0} \sum_{\pi\in\mathcal{C}_n} size_{00}(\pi) x^n = & \frac{2}{135(1-2x)^2} - \frac{56}{675(1-2x)} + \frac{1}{8(1-x)} + \frac{1}{18(1+x)^3} \\ & - \frac{19}{108(1+x)^2} + \frac{31}{216(1+x)} - \frac{4+3i}{100(1+ix)} + \frac{3i-4}{100(1-ix)}, \end{aligned}$$

then by comparing the coefficients of x^n, we obtain

$$\begin{aligned} \sum_{\pi\in\mathcal{C}_n} size_{00}(\pi) = & \frac{2^{n+1}(5n-23)}{675} + \frac{1}{8} + (-1)^n \left(\frac{6n^2 - 20n + 5}{216} \right) \\ & + (-i)^n \left(\frac{-3i-4}{100} \right) + i^n \left(\frac{3i-4}{100} \right) \end{aligned}$$

with $i^2 = -1$, which completes the proof. □

2.2. **Counting (1,1) parity-rises.** By the definitions, we have

$$C_{11}(x,y,q,u)=1+\sum_{a\geq 1}C_{11}(a). \tag{4}$$

The recurrence relation for the generating function $C_{11}(a)$ can be obtained as follows:

$$\begin{aligned}C_{11}(a)&=x^ay+\sum_{b=1}^{a}C_{11}(ab)+\sum_{b\geq a+1}C_{11}(ab)\\&=x^ay+x^ay\sum_{b=1}^{a}C_{11}(b)+(1-\delta_a)x^ayq\sum_{b\geq a+1}(1-\delta_b)C_{11}(b)u^{b-a}\\&\quad+(1-\delta_a)x^ay\sum_{b\geq a+1}\delta_bC_{11}(b)+\delta_ax^ay\sum_{b\geq a+1}C_{11}(b).\end{aligned}$$

By (4), we obtain that

$$C_{11}(a)=x^ayC_{11}+(1-\delta_a)x^ay\sum_{b\geq a+1}(1-\delta_b)C_{11}(b)(qu^{b-a}-1). \tag{5}$$

Now, we restrict our attention to study the generating function $C_{11}^{[d]}(a)$.

Theorem 2.6. *For all $i=1,2,\ldots,d$,*

$$C_{11}^{[d]}(x,y,q,u|i)=p_iC_{11}^{[d]}(x,y,q,u) \text{ and } C_{11}^{[d]}(x,y,q,u)=\frac{1}{1-\sum_{i=1}^{d}p_i},$$

where p_i is given by

$$x^iy\sum_{j=0}^{d-i}x^j\left(\sum_{k_0=i<k_1<\cdots<k_s=i+j}y^sx^{k_0+\cdots+k_{s-1}}\prod_{\ell=0}^{s-1}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1)\right).$$

Proof. By (5) we have

$$\begin{cases}C_{11}^{[d]}(1)&=\beta_1-\sum_{j=2}^{d}C_{11}(j)\widehat{\alpha}_{1,j}\\C_{11}^{[d]}(2)&=\beta_2-\sum_{j=3}^{d}C_{11}(j)\widehat{\alpha}_{2,j}\\&\vdots\\C_{11}^{[d]}(d-1)&=\beta_{d-1}-\sum_{j=d}^{d}C_{11}(j)\widehat{\alpha}_{d-1,j}\\C_{11}^{[d]}(d)&=\beta_d.\end{cases}$$

The above system of equations can be written in a matrix form as follows

$$A\begin{pmatrix}C_{11}^{[d]}(1)\\C_{11}^{[d]}(2)\\\vdots\\C_{11}^{[d]}(d)\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\\\vdots\\\beta_d\end{pmatrix},\ A=\begin{pmatrix}1&\widehat{\alpha}_{1,2}&\widehat{\alpha}_{1,3}&\widehat{\alpha}_{1,4}&\widehat{\alpha}_{1,5}&\widehat{\alpha}_{1,6}&\widehat{\alpha}_{1,7}&\cdots&\widehat{\alpha}_{1,d}\\0&1&\widehat{\alpha}_{2,3}&\widehat{\alpha}_{2,4}&\widehat{\alpha}_{2,5}&\widehat{\alpha}_{2,6}&\widehat{\alpha}_{2,7}&\cdots&\widehat{\alpha}_{2,d}\\&\ddots&\ddots&\vdots&&&&&\\0&0&0&0&0&\cdots&0&1&\widehat{\alpha}_{d-1,d}\\0&0&0&0&0&0&\cdots&0&1\end{pmatrix}.$$

We solve this system by Cramer's method and we obtain

$$C_{11}^{[d]}(i)=\begin{vmatrix} \beta_i & \widehat{\alpha}_{i,i+1} & \widehat{\alpha}_{i,i+2} & \cdots & \widehat{\alpha}_{i,d-1} & \widehat{\alpha}_{i,d} \\ \beta_{i+1} & 1 & \widehat{\alpha}_{i+1,i+3} & \widehat{\alpha}_{i+1,i+4} & \cdots & \widehat{\alpha}_{i+1,d} \\ & \ddots & \ddots & \vdots & & \\ \beta_{d-1} & 0 & 0 & \cdots & 1 & \widehat{\alpha}_{d-1,d} \\ \beta_d & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Lemma 2.1 gives

$$C_{11}^{[d]}(i)=\sum_{j=0}^{d-i}\beta_{i+j}\left(\sum_{k_0=i<k_1<k_2<\cdots<k_s=i+j}(-1)^s\prod_{\ell=1}^{s}\widehat{\alpha}_{k_{\ell-1},k_\ell}\right),$$

for all $i=1,2,\ldots,d$, where $\beta_i=x^iyC_{11}^{[d]}$ and $\widehat{\alpha}_{i,j}=-(1-\delta_i)x^iy(1-\delta_j)(qu^{j-i}-1)$. Thus,

$$C_{11}^{[d]}(i)=\sum_{j=0}^{d-i}x^{i+j}yC_{11}^{[d]}\left(\sum_{k_0=i<k_1<\cdots<k_s=i+j}\prod_{\ell=1}^{s}(1-\delta_{k_{\ell-1}})x^{k_{\ell-1}}y(1-\delta_{k_\ell})(qu^{k_\ell-k_{\ell-1}}-1)\right),$$

which is equivalent to $C_{11}^{[d]}(i)=p_iC_{11}^{[d]}$. By using (4), we complete the proof. □

By taking $d\to\infty$ in Theorem 2.6, we obtain the main result of this subsection.

Theorem 2.7. *The generating function $C_{11}(x,y,q,u)$ is given by*

$$C_{11}(x,y,q,u)=\frac{1}{1-\sum_{i\geq1}p_i},$$

where p_i is given by

$$x^iy\sum_{j\geq0}x^j\left(\sum_{k_0=i<k_1<\cdots<k_s=i+j}y^sx^{k_0+\cdots+k_{s-1}}\prod_{\ell=0}^{s-1}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1)\right).$$

Corollary 2.8. *The mean of $size_{11}$, taken over all compositions of n, for $n\geq4$, is given by*

$$\frac{1}{2^{n-1}}\sum_{\pi\in C_n}size_{11}(\pi)=$$

$$16\left(\frac{5n-13}{675}\right)+\frac{1}{2^{n+2}}+(-1)^n\left(\frac{6n^2+4n-11}{27\cdot2^{n+2}}\right)+(-i)^n\left(\frac{4+3i}{25\cdot2^{n+1}}\right)+i^n\left(\frac{4-3i}{25\cdot2^{n+1}}\right),$$

where $i^2=-1$.

Proof. By differentiating the generating function $C_{11}(x,y,q,u)$ with respect to u and evaluating it at $u=1$, we obtain

$$\frac{d}{du}C_{11}(x,1,1,u)\mid_{u=1}=\frac{-\frac{d}{du}A}{A^2}\mid_{u=1}=\frac{-(1-x)^2\frac{d}{du}A}{(1-2x)^2}\mid_{u=1},$$

where $A = 1 - \sum_{i\geq 1} x^i A_j$,

$$A_j = \sum_{j\geq 0} x^j \sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1),$$

and $A_0 = 1$, which leads to

$$A = 1 - \sum_{i\geq 1} x^i \left(A_0 + \sum_{j\geq 1} A_j\right) = 1 - \sum_{i\geq 1} x^i \left(1 + \sum_{j\geq 1} A_j\right).$$

Clearly, A_j is equal to

$$\sum_{k_0=i<k_1=i+j} x^{k_0}(1-\delta_{k_0})(1-\delta_{k_1})(u^j-1)$$
$$+ \sum_{s\geq 2} \sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1),$$

by differentiating A_j with respect to u and substituting $u = 1$ we get

$$\frac{d}{du}A_j \mid_{u=1} = x^i(1-\delta_i)(1-\delta_{i+j})j,$$

which leads to

$$\begin{aligned}\frac{d}{du}A \mid_{u=1} &= -\sum_{i\geq 1}\sum_{j\geq 1} x^{2i+j}(1-\delta_i)(1-\delta_{i+j})j \\ &= -\sum_{i\geq 1}\sum_{j\geq 1} x^{4i-2+2j}2j = -\frac{2x^4}{(1-x^4)(1-x^2)^2}.\end{aligned}$$

So

$$\sum_{n\geq 0}\sum_{\pi\in C_n} size_{11}(\pi)x^n = \frac{(1-x)^2 2x^4}{(1-2x)^2(1-x^2)^2(1-x^4)}.$$

By comparing the coefficients of x^n, we have

$$\begin{aligned}\sum_{\pi\in C_n} size_{11}(\pi) = {} & \frac{2^{n+3}(5n-13)}{675} + \frac{1}{8} + (-1)^n\left(\frac{6n^2+4n-11}{216}\right) \\ & + (-i)^n\left(\frac{3i+4}{100}\right) + i^n\left(\frac{4-3i}{100}\right)\end{aligned}$$

with $i^2 = -1$, which completes the proof. □

2.3. **Counting (0,1) parity-rises.** By the definitions, we have

$$C_{01}(x,y,q,u) = 1 + \sum_{a\geq 1} C_{01}(a). \tag{6}$$

The recurrence relation for the generating function $C_{01}(a)$ can be obtained as follows:

$$\begin{aligned} C_{01}(a) &= x^a y + \sum_{b=1}^{a} C_{01}(ab) + \sum_{b\geq a+1} C_{01}(ab) \\ &= x^a y + x^a y \sum_{b=1}^{a} C_{01}(b) + \delta_a x^a y q \sum_{b\geq a+1} (1-\delta_b) C_{01}(b) u^{b-a} + \delta_a x^a y \sum_{b\geq a+1} \delta_a C_{01}(b) \\ &+ (1-\delta_a) x^a y \sum_{b\geq a+1} C_{01}(b). \end{aligned}$$

By (6), we obtain that

$$C_{01}(a) = x^a y C_{01} + \delta_a x^a y \sum_{b\geq a+1} (1-\delta_b) C_{01}(b)(qu^{b-a}-1). \tag{7}$$

Again, we focus on the generating function $C_{01}^{[d]}(a)$.

Theorem 2.9. *For all $i = 1,2,\ldots,d$,*

$$C_{01}^{[d]}(x,y,q,u|i) = p_i C_{01}^{[d]}(x,y,q,u) \text{ and } C_{01}^{[d]}(x,y,q,u) = \frac{1}{1-\sum_{i=1}^{d} p_i},$$

where

$$p_i = x^i y \sum_{j=0}^{d-i} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} (1-\delta_{k_\ell})(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

Proof. By (7) and Lemma 2.1, we obtain

$$C_{01}^{[d]}(i) = \sum_{j=0}^{d-i} x^{i+j} y C_{01}^{[d]} \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} \prod_{\ell=1}^{s} \delta_{k_{\ell-1}} x^{k_{\ell-1}} y (1-\delta_{k_\ell})(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

where, $\beta_i = x^i y C_{01}^{[d]}$ and $\widehat{\alpha}_{i,j} = -\delta_i x^i y (1-\delta_j)(qu^{j-i}-1)$. Thus $C_{01}^{[d]}(i) = p_i C_{01}^{[d]}(x,y,q,u)$. Hence, by the fact that $C_{01}^{[d]} = 1 + \sum_{i=1}^{d} C_{01}^{[d]}(i)$, we complete the proof. □

By taking $d \to \infty$ in Theorem 2.9, we obtain the main result of this subsection.

Theorem 2.10. *The generating function $C_{01}(x,y,q,u)$ is given by,*

$$C_{01}(x,y,q,u) = \frac{1}{1-\sum_{i\geq 1} p_i}$$

where

$$p_i = x^i y \sum_{j\geq 0} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1} \delta_{k_\ell} \prod_{\ell=1}^{s} (1-\delta_{k_\ell})(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

Corollary 2.11. *The mean of $size_{01}$, taken over all compositions of n, for $n \geq 5$ is given by*

$$\frac{1}{2^{n-1}} \sum_{\pi \in C_n} size_{01}(\pi) = \frac{n-4}{27} + \frac{1}{2^{n+2}} + (-1)^{n+1}\left(\frac{6n^2-20n+11}{27 \cdot 2^{n+2}}\right).$$

Proof. By differentiating the generating function $C_{01}(x,y,q,u)$ with respect to u and evaluating it at $u=1$, we obtain

$$\frac{d}{du}C_{01}(x,1,1,u) \mid_{u=1} = \frac{-\frac{d}{du}A}{A^2} \mid_{u=1} = \frac{-(1-x)^2\frac{d}{du}A}{(1-2x)^2} \mid_{u=1},$$

where $A = 1 - \sum_{i\geq 1} x^i A_j$,

$$A_j = \sum_{j\geq 0} x^j \sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}\delta_{k_\ell} \prod_{\ell=1}^{s}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1),$$

and $A_0 = 1$, which leads to

$$A = 1 - \sum_{i\geq 1} x^i \left(A_0 + \sum_{j\geq 1} A_j\right) = 1 - \sum_{i\geq 1} x^i \left(1 + \sum_{j\geq 1} A_j\right).$$

Obviously,

$$A_j = \sum_{k_0=i<k_1=i+j} x^{k_0}\delta_{k_0}(1-\delta_{k_1})(u^j-1)$$
$$+ \sum_{s\geq 2}\sum_{k_0=i<k_1<\cdots<k_s=i+j} x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}\delta_{k_\ell} \prod_{\ell=1}^{s}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1).$$

By differentiating A_j with respect to u and substituting $u=1$ we get

$$\frac{d}{du}A_j \mid_{u=1} = x^i\delta_i(1-\delta_{i+j})j,$$

which leads to

$$\frac{d}{du}A \mid_{u=1} = -\sum_{i\geq 1}\sum_{j\geq 1} x^{2i+j}\delta_i(1-\delta_{i+j})j = -\sum_{i\geq 1}\sum_{j\geq 1} x^{4i+2j-1}(2j-1) = -\frac{x^5}{(1-x^2)^3}.$$

Thus

$$\sum_{n\geq 0}\sum_{\pi\in C_n} size_{01}(\pi)x^n = \frac{(1-x)^2x^5}{(1-2x)^2(1-x^2)^3}.$$

By comparing the coefficients of x^n, we have

$$\sum_{\pi\in C_n} size_{01}(\pi) = \frac{2^{n-1}(n-4)}{27} + \frac{1}{8} + (-1)^{n+1}\left(\frac{6n^2-20n+11}{216}\right),$$

which completes the proof. □

2.4. **Counting (1,0) parity-rises.** By the definitions, we have

$$C_{10}(x,y,q,u) = 1 + \sum_{a\geq 1} C_{10}(a). \tag{8}$$

The recurrence relation for the generating function $C_{10}(a)$ can be obtained as follows:

$$\begin{aligned} C_{10}(a) &= x^a y + \sum_{b=1}^{a} C_{10}(ab) + \sum_{b\geq a+1} C_{10}(ab) \\ &= x^a y + x^a y \sum_{b=1}^{a} C_{10}(b) + (1-\delta_a)x^a yq \sum_{b\geq a+1} \delta_b C_{10}(b) u^{b-a} \\ &\quad + (1-\delta_a)x^a y \sum_{b\geq a+1}(1-\delta_a)C_{10}(b) + \delta_a x^a y \sum_{b\geq a+1} C_{10}(b). \end{aligned}$$

By (8), we obtain that

$$C_{10}(a) = x^a y C_{10} + (1-\delta_a)x^a y \sum_{b\geq a+1} \delta_b C_{10}(b)(qu^{b-a}-1). \tag{9}$$

Now, we consider the generating function $C_{10}^{[d]}(x,y,q,u)$.

Theorem 2.12. *For all $i = 1,2,\ldots,d$,*

$$C_{10}^{[d]}(x,y,q,u|i) = p_i C_{10}^{[d]}(x,y,q,u) \text{ and } C_{10}^{[d]}(x,y,q,u) = \frac{1}{1-\sum_{i=1}^{d} p_i},$$

where

$$p_i = x^i y \sum_{j=0}^{d-i} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s} \delta_{k_\ell}(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

Proof. By (9)and using Lemma 2.1 we obtain,

$$C_{10}^{[d]}(i) = \sum_{j=0}^{d-i} x^{i+j} y C_{10}^{[d]} \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} \prod_{\ell=1}^{s}(1-\delta_{k_{\ell-1}})x^{k_{\ell-1}} y \delta_{k_\ell}(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

where, $\beta_i = x^i y C_{10}^{[d]}$ and $\widehat{\alpha}_{i,j} = -(1-\delta_i)x^i y \delta_j (qu^{j-i}-1)$. Thus, $C_{10}^{[d]}(i) = p_i C_{10}^{[d]}$. Hence, by the fact that $C_{10}^{[d]} = 1 + \sum_{i=1}^{d} C_{10}^{[d]}(i)$, we complete the proof. □

By taking $d\to\infty$ in Theorem 2.12, we obtain the main result of this subsection.

Theorem 2.13. *The generating function $C_{10}(x,y,q,u)$ is given by,*

$$C_{01}(x,y,q,u) = \frac{1}{1-\sum_{i\geq 1} p_i}$$

where

$$p_i = x^i y \sum_{j\geq 0} x^j \left(\sum_{k_0=i<k_1<\cdots<k_s=i+j} y^s x^{k_0+\cdots+k_{s-1}} \prod_{\ell=0}^{s-1}(1-\delta_{k_\ell}) \prod_{\ell=1}^{s} \delta_{k_\ell}(qu^{k_\ell-k_{\ell-1}}-1) \right),$$

Corollary 2.14. *The mean of size$_{10}$, taken over all compositions of n, for $n \geq 3$, is given by*

$$\frac{1}{2^{n-1}}\sum_{\pi\in C_n} size_{10}(\pi) = 4\left(\frac{n-2}{27}\right)+\frac{1}{2^{n+2}}+(-1)^{n+1}\left(\frac{6n^2+4n-5}{27\cdot 2^{n+2}}\right).$$

Proof. By differentiating the generating function $C_{10}(x,y,q,u)$ with respect to u and evaluating it at $u=1$, we obtain

$$\frac{d}{du}C_{10}(x,1,1,u)\mid_{u=1} = \frac{-\frac{d}{du}A}{A^2}\mid_{u=1} = \frac{-(1-x)^2\frac{d}{du}A}{(1-2x)^2}\mid_{u=1},$$

where $A = 1-\sum_{i\geq 1}x^iA_j$,

$$A_j = \sum_{j\geq 0}x^j\sum_{k_0=i<k_1<\cdots<k_s=i+j}x^{k_0+\cdots+k_{s-1}}\prod_{\ell=0}^{s-1}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}\delta_{k_\ell}\prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1),$$

and $A_0=1$, which leads to

$$A = 1-\sum_{i\geq 1}x^i\left(A_0+\sum_{j\geq 1}A_j\right) = 1-\sum_{i\geq 1}x^i\left(1+\sum_{j\geq 1}A_j\right).$$

Evidently,

$$\begin{aligned}A_j = &\sum_{k_0=i<k_1=i+j}x^{k_0}(1-\delta_{k_0})\delta_{k_1}(u^j-1)\\ &+\sum_{s\geq 2}\sum_{k_0=i<k_1<\cdots<k_s=i+j}x^{k_0+\cdots+k_{s-1}}\prod_{\ell=0}^{s-1}(1-\delta_{k_\ell})\prod_{\ell=1}^{s}\delta_{k_\ell}\prod_{\ell=1}^{s}(qu^{k_\ell-k_{\ell-1}}-1).\end{aligned}$$

By differentiating A_j with respect to u and substituting $u=1$ we get

$$\frac{d}{du}A_j\mid_{u=1} = x^i(1-\delta_i)\delta_{i+j}j,$$

which gives

$$\frac{d}{du}A\mid_{u=1} = -\sum_{i\geq 1}\sum_{j\geq 1}x^{2i+j}(1-\delta_i)\delta_{i+j}j = -\sum_{i\geq 1}x^{4i-2}\sum_{j\geq 1}x^{2j-1} = -\frac{x^3}{(1-x^2)^3}.$$

Thus

$$\sum_{n\geq 0}\sum_{\pi\in C_n} size_{10}(\pi)x^n = \frac{(1-x)^2x^3}{(1-2x)^2(1-x^2)^3}.$$

Hence, by comparing the coefficients of x^n, we have

$$\sum_{\pi\in C_n} size_{10}(\pi) = 2^{n+1}\left(\frac{n-2}{27}\right)+\frac{1}{8}+(-1)^{n+1}\left(\frac{6n^2+4n-5}{216}\right),$$

which completes the proof. □

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