A RELATIVE ROTH THEOREM IN DENSE SUBSETS OF SPARSE PSEUDORANDOM FRACTALS

MARC CARNOVALE

Abstract. We extend the main result of the paper "Arithmetic progressions in sets of fractional dimension" ([12]) in two ways. Recall that in [12], Łaba and Pramanik proved that any measure μ with Hausdorff dimension $\alpha \in (1-\epsilon_0, 1)$ (here ϵ_0 is a small constant) large enough depending on its Fourier dimension $\beta \in (2/3, \alpha]$ contains in its support three-term arithmetic progressions (3APs). In the present paper, we adapt an approach introduced by Green in "Roth's Theorem in the Primes" to both lower the requirement on β to $\beta > 1/2$ (and ϵ_0 to $1/10$) and perhaps more interestingly, extend the result to show for any $\delta > 0$, if α is large enough depending on δ then μ gives positive measure to the (basepoints of the) non-trivial 3APs contained within any set A for which $\mu(A) > \delta$.

1. Introduction

This paper seeks to employ the technology of restriction/transference results in additive combinatorics developed in [7] for the prime numbers to extend results of Łaba and Pramanik [12] demonstrating three-term arithmetic progressions in certain fractal sets.

We will be concerned with a measure μ on $\mathbb{T}$ satisfying the following conditions for appropriate α, β, C_H, C_F in the case that $d = 1$

(*a*) $|\mu|(B(x,r)) \leq C_H r^\alpha$ for all $x \in \mathbb{T}^d$

(*b*) $|\widehat{\mu}(\xi)| \leq C_F(1+|\xi|)^{-\frac{\beta}{2}}$ for all $\xi \in \mathbb{Z}^d$. [1]

In [12], it was shown that a measure μ satisfying (*a*) and (*b*) for α sufficiently close to 1 depending on $\beta > 2/3$, C_H, and C_F must contain in its support 3-term arithmetic progressions (for short, 3APs). Consequently, any closed set supporting a measure satisfying (*a*) and (*b*) with appropriate constants must contain 3APs. In the present paper, we show that in fact, dense subsets of such sets still contain 3APs, in particular providing a condition for a measurable set to contain progressions. Our main result is

---

Date: March 23, 2016.

1991 *Mathematics Subject Classification.* 28A78, 42A32, 42A38, 42A45, 11B25, 28A75.

Key words and phrases. Fourier dimension, Salem set, arithmetic progression, Roth's theorem, Hausdorff dimension.

[1] The largest α and $\beta \in (0,1)$ for which (*a*) and (*b*) hold are referred to as the Hausdorff and Fourier dimensions of the measure μ. For more on these, see, e.g., [13].

Theorem 1.1. *Let $\delta > 0$, and suppose that the probability measure μ on $\mathbb{T}$ satisfies* (b) *with $\beta > \frac{1}{2}$ and* (a) *with α sufficiently close to 1 depending on β, δ, and the implicit constants C_H and C_F in* (a) *and* (b)*.*

Then any measurable set A with $\mu(A) > \delta$ contains 3APs [2] ; indeed, μ gives positive measure to the set of x such that $x, x+r, x+2r$ is a non-trivial 3AP contained with A.

Remark 1.2. *Notice that it follows from the above Theorem that the total measure with respect to μ of the set of x which are not the basepoint of a 3AP entirely contained within supp μ is less than δ.*

The proof of Proposition 5.1 (upon which the proof of the above theorem rests) requires that $\alpha > \frac{9-2\beta}{8}$. By Remark 4.3, in the case that $\delta = 1$ and $C_H, C_F \approx 1$, it is sufficient to take $\alpha > 1 - \beta/6$ for Lemma 4.2. Thus if $\beta = \alpha$ in the full-density case, as is nearly-achieved by Salem sets, then we need only take $\alpha = \beta > 9/10$. It is unclear whether the dependency of α on the parameter $\delta < 1$ may be removed. For details, see Section 7.

This result even applied to the full support of the measure is new both in that we show μ gives positive measure to the basepoints of non-trivial 3APs contained in its support, and that it is valid for sets with Fourier dimension equal to anything greater than $1/2$, whereas [12] required a lower bound of $\beta > 2/3$.

Our approach is based on [7], in which it was shown that 3APs are contained in any set taking up a positive proportion of the prime numbers. There, the main ingredients were the pseudorandomness of the prime numbers, a restriction theorem for pseudorandom sets (though in that paper stated only for the prime numbers), and the properties of Bohr sets. In the present context, all three ingredients are available. Namely, the Fourier decay condition (*b*) plays the role of pseudorandomness, we invoke Theorem 1.3 to obtain restriction estimates, and the properties of Bohr sets carry over unchanged to the continuous setting.

One reason to be interested in the arithmetic properties of fractional sets is their implication in the fine analytic behaviour of singular sets.

In 2002 (respectively, 2000) Mitsis [14] (independently Mockenhaupt [15]) obtained the following Stein-Tomas type restriction theorem for fractional sets.[3]

Theorem 1.3. *Let μ be a compactly supported positive measure on $\mathbb{R}^d$ which obeys* (a) *and* (b) *for some $\alpha, \beta \in (0, d)$.*

Then for all $p \geq p_\circ = p_\circ(d, \alpha, \beta) := \frac{2(2d-2\alpha+\beta)}{\beta}$, there is a $C(p) > 0$ such that

$$\|\widehat{fd\mu}\|_{\ell^p(\mathbb{Z}^d)} \leq C(p)\|f\|_{L^2(\mu)}.^4$$

[2]Note that by requiring that $\text{supp}(\mu) \subset [1/3, 2/3] \subset [0,1] \approx \mathbb{T}$, or equivalently by dilating, we guarantee that the progressions above are genuine progressions when $\text{supp}(\mu)$ is embedded in $\mathbb{R}$.

[3]Aside from the end point, which was obtained by Bak and Seeger in 2010.

[4]In [14] and [15] this theorem was stated with $\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)}$ in place of $\|\widehat{fd\mu}\|_{\ell^p(\mathbb{Z}^d)}$. By Lemma 4.4, the two are equivalent for $p \in (1, \infty]$

As noted by Mockenhaupt [15], obstructions to restriction results for the sphere rely on arithmetic properties of the surface measure on the sphere (more specifically, on the existence of arithmetic progressions in small neighborhoods of pieces of this measure). Recent work of Łaba and Hambrook [9] made more precise the analogy between arithmetic progressions in fractional sets and the classical Knapp example showing sharpness of the spherical Stein-Tomas theorem. They did so by constructing measures μ on $\mathbb{R}$ obeying (*a*) and (*b*) which nonetheless contained an abundance of long arithmetic progressions, and demonstrating that an extension of Theorem 1.3 to an improved L^p range fails for such μ via an argument which makes direct use of the many long arithmetic progressions contained in $\mathrm{supp}(\mu)$. Thus an understanding of deep properties of singular measures can be seen to begin with the easier problem of understanding their arithmetic properties.

This work is particularly motivated by a desire to directly connect to harmonic analysis a principle in additive combinatorics that a not too small pseudorandom subset of a space should possess many of the same properties as the full space, and results for subsets of the entire space should, under suitable hypotheses, hold for subsets of the pseudorandom subspace. The most well-known example of such a phenomenon is the celebrated Green-Tao theorem that the primes contain arbitrarily long arithmetic progressions [8], whose proof entailed first demonstrating that the prime numbers behave in a suitably pseudorandom fashion and second developing a "relative Szemerédi" theorem for subsets of appropriately pseudorandom subsets of the integers. Their result extends ideas present in Green's earlier *Roth's Theorem in the Primes* [7], from which we take inspiration in the present paper. The principle of such transference from a large global space to a nice but sparser subspace can be seen in earlier work, such as in Stein's Spherical maximal theorem [17], which finds that the measure on the sphere behaves almost "as well" as the measure on the unit ball for purposes of obtaining a maximal theorem, and similarly in Bourgain's ergodic theorem along the squares [1]. What many of these results have in common is that they rely on the pseudorandom set being close in a Fourier or spectral sense to the indicator function for the entire space. In [7], this is expressed by a Fourier restriction estimate which underlies the result, but in [8], as further developed in [19], abstracted in [6] and [16], and in the excellent combinatorial strengthening [3], this proximity takes on an arithmetic nature encoded by the Gowers uniformity norms and related objects. Although we do not take the combinatorial viewpoint here, relying instead on a restriction estimate as in [7], it seems that combinatorial methods such as those of [3] would be necessary to extend our result to the case of longer progressions.

In Section 2 we set up notation and describe the approach.

Acknowledgments

Thanks to Bochen Liu for a careful reading and helpful remarks, as well as pointing out an error in the original definition of A^*_μ. Many thanks also to the anonymous referees, whose careful reading and many remarks have been very helpful and improved this paper.

2. Notation and approach

Throughout, μ will always refer to a (possibly complex) Radon measure supported on $[0,1] \approx \mathbb{T}$. We denote the variation norm of the measure μ by $\|\mu\|$.

For a set A and measure μ, set $\mu_A := \mu|_A$ the restriction of μ to the set A, and let A^*_μ denote the set of points of positive density with respect to the measure μ; that is, for each $x \in A^*_\mu$,

$$\limsup |\mu|(A \cap B(x,r))/|\mu|(B(x,r)) > 0.$$

Note that $\mu(A/A^*_\mu) = 0$ by [13], pg. 91, Remark 1.

Throughout, let (ϕ_n) be an approximate identity on $\mathbb{T}$ (that is, a sequence of positive functions with L^1-norm 1 converging weak* to the dirac delta function), with the property that $\text{supp}(\widehat{\phi_n}) \subset [-2^n, 2^n]$ and $\widehat{\phi_n}|_{[-2^{n-1},2^{n-1}]} \equiv 1$.

The primary additional technical aspect of the argument involves the "progression counting functional" $\Lambda_3(\mu)$, defined as

$$\lim_{n\to\infty} \int \phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\, dx\, dr$$

which first appeared in the context of measures, in a slightly different form, in [12], and more generally the measure $\cap^3\mu$ defined by

$$\int f\, d\cap^3 \mu := \lim_{n\to\infty} \int f(x,r)\phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\, dx\, dr.$$

We develop their existence theory in Section 6.

Recall that the quantity α in (a) is a Hausdorff dimension estimate, while the quantity β of (b) is an estimate on the Fourier dimension of μ.

Theorem 2.1. *Suppose that μ is a measure on $\mathbb{T}$ satisfying* (a) *and* (b) *with $\beta > \frac{1}{2}$ and $\alpha > \frac{9-2\beta}{8}$.*

Then the finite measure $\cap^3\mu : C(\mathbb{T}^2) \to \mathbb{C}$ given by

$$g \mapsto \int g(x,r)\, d\cap^3 \mu(x,r) := \Lambda_g(\mu) \tag{1}$$
$$:= \lim_{n\to\infty} \int g(x,r)\phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\, dx\, dr$$

is well-defined.

Lemma 2.2. *Suppose that the probability measure μ satisfies* (a) *and (*b*) for some $\beta > \frac{1}{2}$ and $\alpha > \frac{9-2\beta}{8}$. Then the trivial progressions of step size* 0 *lie outside the support of the measure $\cap^3\mu$, or in other words*

$$\int 1_{\{0\}}(r)\, d\cap^3 \mu = 0.$$

In Section 6 we show that the measure $\cap^3\mu$ exists and that it gives no mass to the degenerate progressions of step size zero; the main tools here are a Littlewood-Paley type decomposition, the Fourier decay assumption (*b*), and the use of uniformity norm-type estimates to bound progression-counting multilinear functionals. In Section 3, we show that the measure $\cap^3\mu_A$ exists and that it is supported on the set of (x, r) for which $x, x+r, x+2r$ is an arithmetic progression contained in A^*_μ. Section 5 contains the proof of Theorem 1.1. In Section 4, we obtain the proof of Lemma 4.2, which is the main ingredient in proving the results of Section 3; the tools here are Bohr sets and the restriction estimate Theorem 1.3.

3. Existence of the restricted measure on 3APs

In this section, for measurable sets $A \subset \mathbb{T}$ we demonstrate the existence of the measure $\cap^3(\mu_A)$ on $\mathbb{T}\times\mathbb{T}$ under the assumption that μ satisfies (*a*) and (*b*) with $\beta > \frac{1}{2}$ and α sufficiently close to 1, and we discuss the support properties of the measure $\cap^3(\mu_A)$. Throughout this section, we will be assuming the results of Section 6, specifically the truth of Theorem 2.1 and Lemma 2.2.

Lemma 3.1. *Suppose that μ is a probability measure on $\mathbb{T}$ which satisfies (*b*) for a $\beta > \frac{1}{2}$ and (*a*) for $\alpha > \frac{9-2\beta}{8}$, and that the set $A \subset \mathbb{T}$ is measurable. Then the Radon measure $\cap^3\mu_A$ exists.*

Proof. Consider the more general measure

$$d\cap^3(\mu_g)(x,r) := w^* - \lim \phi_n * (g\,d\mu)(x)\phi_n * (g\,d\mu)(x-r)\phi_n * (g\,d\mu)(x-2r)\,dx\,dr$$

for a bounded measurable function g, provided the above weak* limit exists. First note that we have boundedness of $g \in L^\infty(\mu) \mapsto \cap^3(\mu_g)$ on its domain of definition, since for any $f \in C(\mathbb{T}^2)$

$$\begin{aligned}
&|\int f\, d\cap^3(\mu_g)| \\
=&\lim_{n\to\infty} |\int f(x,r)\phi_n * (g\,d\mu)(x)\phi_n * (g\,d\mu)(x-r)\phi_n * (g\,d\mu)(x-2r)\,dx\,dr| \\
\le&\lim_{n\to\infty} \|f\|_{L^\infty}\|g\|^3_{L^\infty} \int \phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\,dx\,dr \\
=&\|f\|_{L^\infty}\|g\|^3_{L^\infty}\|\cap^3\mu\|
\end{aligned}$$

and for $\beta > \frac{1}{2}$ and α close enough to 1 this last is finite by Theorem 2.1.

Thus it suffices to show that the limit defining the operator

$$g \mapsto \cap^3(\mu_g)$$

$$L^\infty(\mu) \mapsto M(\mathbb{T}\times\mathbb{T})$$

exists on a dense subset. Conditional on $\beta > \frac{1}{2}$ and α being suffciently close to 1, by taking trigonometric polynomials as this dense subset we again use Theorem 2.1 to obtain existence of $\cap^3(\mu_g)$ for such g by applying linearity and the fact that for $\xi \in \mathbb{Z}$, the measure $d\tilde{\mu}(x) := e^{2\pi i\xi x}\, d\mu(x)$ satisfies

- $|\tilde{\mu}|(B(x,r)) \lesssim r^\alpha$
- $|\widehat{\tilde{\mu}}(\xi)| \lesssim (1+|\xi|)^{-\frac{\beta}{2}}$.

Thus for any $g \in L^\infty$, $\cap^3\mu_g$ exists, and so in particular does $\cap^3\mu_A$. □

We turn now to the support properties of the measure $\cap^3(\mu_A)$.

Lemma 3.2. *Suppose that μ is a probability measure on $\mathbb{T}$ satisfying (a) and (b) with $\beta > \frac{1}{2}$ and a value of $\alpha > \frac{9-2\beta}{8}$. Suppose further that the set $A \subset \mathbb{T}$ is measurable with $\mu(A) > 0$. Then we have*

$$\cap^3\mu_A(\mathbb{T}\times\mathbb{T} \setminus \Big(\{(x,r) \in \mathbb{T}\times(\mathbb{T}\setminus\{0\}) : x, x+r, x+2r \in A^*_\mu\}\Big) = 0$$

where we recall from Section 2 that A^*_μ denotes the points of positive density of the set A with respect to the measure μ.

Before proving Lemma 3.2, we need the following fact about the projection of $\cap^3\mu$ to the x-coordinate.[5] Interestingly, it seems inaccessible without recourse to restriction esimates (a direct application of the U^2-techniques of Section 6 fails, for instance), leaving open the question of what shape, if any, an analogue would take in the case of longer progressions.

Let $\mathcal{D}^2\mu$ denote the measure $\int_T f(x)\, d\mathcal{D}^2\mu(x) := \int f(x)\, d\cap^3\mu(x,r)$ provided that $\cap^3\mu$ exists.

Proposition 3.3. *Suppose that μ is a probability measure on $\mathbb{T}$ satisfying (a) and (b) with $\beta > \frac{1}{2}$ and a value of $\alpha > 1-\beta/4$. Suppose further that the set $A \subset \mathbb{T}$ is measurable with $\mu(A) > 0$, and that the measure $\cap^3\mu_A$ exists. Then we have*

$$\mathcal{D}^2\mu_A \ll \mu_A.$$

Proof. For $f \in L^\infty$ we will show that $\left|\int f\, d\mathcal{D}^2\mu_A\right| \lesssim \left(\int |f|^2\, d\mu_A\right)^{\frac{1}{2}}$, which applied to indicator functions of sets implies absolute continuity.

By Dominated Convergence, it suffices to show the above for the dense subclass of f with $\widehat{f} \in \ell^1(\mathbb{Z})$. Fix such an f.

Taking Fourier transforms,

[5] In the discrete case, it is known that the dual function $\mathcal{D}^k f := x \mapsto \mathbb{E}_{\vec{u}} \prod_{\iota\in\{0,1\}^k\setminus\{0\}} f(x-\iota\cdot u)$ of a function f with $L^{2^k}(m)$ bounds is in $L^\infty(m)$(m-denoting Haar measure), or indeed, that the dual function is continuous (see Lemma 2.6 of [10]). What we show may be thought of as a singular U^2 version of this fact.

$$(2) \qquad \int f(x) \prod_{i=0}^{2} \phi_n * \mu_A(x - ir)\, dr\, dx = \sum_{\xi \in \mathbb{Z}} \widehat{\phi_n}(\xi)\overline{\widehat{\phi_n}(2\xi)}\widehat{\mu_A}(\xi)\overline{\widehat{\mu_A}(2\xi)}\widehat{f\phi_n * \mu_A}(\xi).$$

We have by the triangle inequality that

$$\left|\widehat{\phi_n}(\xi)\overline{\widehat{\phi_n}(2\xi)}\widehat{\mu_A}(\xi)\overline{\widehat{\mu_A}(2\xi)}\widehat{f\phi_n * \mu_A}(\xi)\right| \le \left|\widehat{\mu_A}(\xi)\overline{\widehat{\mu_A}(2\xi)}\left(|\widehat{f}| * |\widehat{\mu_A}|\right)(\xi)\right|.$$

Choosing $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ with $p_i > p_\circ$ (when $\beta > 0$, this is possible by taking $p_i = 3$ for each i provided $\alpha > 1 - \beta/4$ so that $p_\circ < 3$)

$$\begin{aligned}
&\sum_{\xi \in \mathbb{Z}} \left|\widehat{\mu_A}(\xi)\overline{\widehat{\mu_A}(2\xi)}\left(|\widehat{f}| * |\widehat{\mu_A}|\right)(\xi)\right| \\
\le& \|\widehat{\mu_A}\|_{\ell^{p_1}(\mathbb{Z})}\|\widehat{\mu_A}\|_{\ell^{p_2}(2\mathbb{Z})}\||\widehat{f}| * |\widehat{\mu_A}|\,\|_{\ell^{p_3}(\mathbb{Z})} \\
\lesssim& [\mu(A)]^{1/2}[\mu(A)]^{1/2}\|\widehat{f}\|_{\ell^1}\|\widehat{\mu_A}\|_{\ell^{p_3}} < [\mu(A)]^{3/2}\|\widehat{f}\|_{\ell^1} < \infty
\end{aligned}$$

where in the first inequality of the last line, we have used the restriction estimate Theorem 1.3 and Young's convolution inequality, and in the second inequality we have applied Theorem 1.3 again.

This shows absolute convergence (uniform in n) of the sum in (2).

But then evaluating (2) in the limit as $n \to \infty$, again applying the restriction estimate Theorem 1.3 we get

$$\begin{aligned}
&\left|\int f\, d\mathcal{D}^2\mu_A\right| = \left|\sum_{\xi \in \mathbb{Z}} \widehat{\mu_A}(\xi)\overline{\widehat{\mu_A}(2\xi)}\widehat{f \cdot \mu_A}(\xi)\right| \\
\le& \|\widehat{\mu_A}\|_{\ell^{p_1}(\mathbb{Z})}\|\widehat{\mu_A}\|_{\ell^{p_2}(2\mathbb{Z})}\|\widehat{f\mu_A}\|_{\ell^{p_3}(\mathbb{Z})} \\
\lesssim& [\mu(A)]^{1/2}[\mu(A)]^{1/2}\left(\int |f|^2\, d\mu_A\right)^{1/2} \le \|f\|_{L^2(\mu)},
\end{aligned}$$

as was to be shown.

□

Proof of Lemma 3.2. If $\beta > \frac{1}{2}$ and α is close enough to 1, then by Lemma 3.1 $\cap^3\mu_A$ exists. We show first that

$$\cap^3\mu_A\left(\{(x,r) \in \mathbb{T} \times \mathbb{T} : x, x+r, x+2r \in A^*_\mu\}^c\right) = 0.$$

Suppose that $E \subset \mathbb{T} \times \mathbb{T}$ is a collection of (x,r) which do not belong to the above set. Then for each $(x,r) \in E$, one of $\{x, x+r, x+2r\} \notin A^*_\mu$. We will decompose E into three sets corresponding to each of these three possibilities, $E = E' \cup E'' \cup E'''$.

We consider the case of E', the other cases being similar. Recall that $\mu(A \setminus A^*_\mu) = 0$.

Let $B = A^*_\mu$ be the projection of E' onto the x-axis.

Then since $\cap^3\mu_A(B^c \times \mathbb{T}) = \mathcal{D}^2\mu_A(B^c)$, $\mathcal{D}^2\mu_A \ll \mu_A$ and $\mu_A(B^c) = 0$, we have $\cap^3\mu_A(B^c \times \mathbb{T}) = 0$. (The cases of E'' and of E''' require the analogous variants of Proposition 3.3, which we leave to the reader.)

Thus $\cap^3(\mu_A)$ is in fact concentrated on the (x, r) such that $x, x+r, x+2r \in A^*_\mu$.

All that remains now is to show that $\cap^3(\mu_A)(\mathbb{T} \times \{0\}) = 0$. But this is immediate from the obvious inequality

$$\cap^3(\mu_A) \leq \cap^3(\mu)$$

and Lemma 2.2. □

4. The Main Estimate : Proof of Lemma 4.2

Recall that for a given frequency set $S \subset \mathbb{Z}$ and radius $1 \geq \epsilon > 0$, a Bohr set $B = B(S, \epsilon)$ is defined by

$$B = \{x \in \mathbb{T} : |e^{2\pi i \xi x} - 1| < \epsilon \text{ for all } \xi \in S\}$$

and (see, e.g., Section 4.4 of [18]) that

$$\epsilon^{|S|} \lesssim |B|. \tag{3}$$

We will need the following bound.

Lemma 4.1. *Suppose that the measure μ on $\mathbb{T}$ satisfies* (a) *and* (b). *Then if $S \subset \mathbb{Z}$ is a finite set of frequencies and $\boldsymbol{B} = |B|^{-1}1_B$ is the normalized indicator of the Bohr set B of radius ϵ and frequency set S, one has*

$$\|\boldsymbol{B} * \boldsymbol{B} * \mu\|_{L^\infty} \lesssim \epsilon^{-(1-\alpha)|S|\left(1-\alpha+\frac{\beta}{2}\right)^{-1}}$$

Proof. Set $\mu_1 = \boldsymbol{B} * \boldsymbol{B} * \mu$.

Let $N \in \mathbb{N}$ be a large integer to be specified in a moment.

Since $\|\widehat{\boldsymbol{B}}^2\|_{\ell^1} = \|\widehat{\boldsymbol{B}}\|^2_{\ell^2} = \|\boldsymbol{B}\|^2_{L^2} < \infty$ and $|\widehat{\mu_1}| \leq |\widehat{\boldsymbol{B}}|^2$, we may apply Fourier inversion to decompose

$$|\mu_1(x)| \leq |\boldsymbol{B} * \boldsymbol{B} * \mu(x)| \leq |\sum_{|\xi| \leq N} \widehat{\boldsymbol{B}}(\xi)^2 \widehat{\mu}(\xi) e^{2\pi i \xi x}| + |\sum_{|\xi| > N} \widehat{\boldsymbol{B}}(\xi)^2 \widehat{\mu}(\xi) e^{2\pi i \xi x}|$$
$$= I + II.$$

Taking ψ a mollifier with $\psi \approx N 1_{B(0,N^{-1})}$, $\widehat{\psi} \approx 1_{B(0,N)}$, we have

$$I \approx \left|\sum \widehat{\psi} \widehat{\boldsymbol{B} * \boldsymbol{B} *} \mu(\xi) e^{2\pi i \xi x}\right| = |\psi * \boldsymbol{B} * \boldsymbol{B} * \mu| \leq \psi * |\mu|$$
$$\lesssim N^{1-\alpha} \tag{4}$$

where in the last inequality, we have used $\|\psi * |\mu|\|_{L^\infty} \lesssim N^{1-\alpha}$ which follows from the definition of ψ and the Hausdorff dimension assumption on the positive measure $|\mu|$ (for details see, e.g., Lemma 4.1 of [4]).

Further

$$\begin{aligned} II \leq & \sum_{|\xi|>N} |\widehat{\boldsymbol{B}}|^2|\widehat{\mu}| \leq (\sum |\widehat{\boldsymbol{B}}|^2)\|\widehat{\mu}\|_{\ell^\infty(B(0,N)^c)} \lesssim N^{-\frac{\beta}{2}}\|\boldsymbol{B}\|_{L^2} \\ \leq & N^{-\frac{\beta}{2}}\epsilon^{-|S|} \end{aligned} \tag{5}$$

by Plancherel together with (3).

Combining (4) and (5) we obtain

$$\|\mu_1\|_{L^\infty} \lesssim N^{1-\alpha} + N^{-\frac{\beta}{2}}\epsilon^{-|S|}.$$

This is minimized when both terms on the right are equal. Thus we let

$$\log N = -|S|\left(1-\alpha+\frac{\beta}{2}\right)^{-1}\log\epsilon,$$

at which point we find

$$\|\mu_1\|_{L^\infty} \lesssim \epsilon^{-(1-\alpha)|S|\left(1-\alpha+\frac{\beta}{2}\right)^{-1}}. \tag{6}$$

□

Taking $\alpha \uparrow 1$ in the above estimate, we see that if $\beta > 0$, $\|\mu_1\|_{L^\infty} \lesssim 1$ for α sufficiently close to 1, and a closer inspection yields the bound $\limsup_{\alpha\to 1}\|\mu_1\|_{L^\infty} \leq 2$ in the limit as $\max(C_H, C_F) \downarrow 1$.

Lemma 4.2. *Suppose that the probability measure μ on $\mathbb{T}$ satisfies* (a) *and* (b) *and that $A \subset \mathbb{T}$ satisfies*

$$\mu(A) > \delta. \tag{7}$$

Suppose further that $\beta > \frac{1}{2}$. Then there is a lower bound α_0 on $\alpha \in (0,1)$, depending on β, δ, C_H, and C_F, and a number $c = c(\delta) > 0$ such that if

$$\alpha > \alpha_0$$

then

$$\Lambda_3(\mu_A) > c.$$

Proof. Let $\epsilon, \eta > 0$. Let

$$S = \{\xi \in \mathbb{Z} : |\widehat{1_A\mu}(\xi)| \geq \eta\}$$

be the large spectrum of μ_A and let

$$B = \{x : |e^{2\pi i\xi x} - 1| < \epsilon \forall \xi \in S\}$$

be the associated Bohr set of radius ϵ.

Set

$$\boldsymbol{B} := \frac{1}{|B|}1_B.$$

By the lower bound (3) on the size of a Bohr set

$$\|\boldsymbol{B}\|_{L^\infty} \lesssim \epsilon^{-|S|}. \tag{8}$$

We first show that $|S|$ can be bounded only in terms of η. This is a consequence of the restriction estimate Theorem 1.3. We have for p in the range guaranteed by Theorem 1.3 that

$$|S|\eta^p \leq \sum_{\xi\in S} |\widehat{\mu_A}(\xi)|^p \leq \sum |\widehat{1_A\mu}(\xi)|^p \lesssim \int |\widehat{1_A\mu}(\xi)|^p \lesssim \|1_A\|_{L^2(\mu)}^p \leq 1$$

where in the second to last inequality we applied the restriction estimate, and in the inequality before it we invoked Lemma 4.4 from the end of this section.

From the above we obtain that for appropriate p,

$$|S| \lesssim \eta^{-p}. \tag{9}$$

We will compare μ_A to $\boldsymbol{B} * \boldsymbol{B} * \mu_A =: \mu_1$ and find that they contain approximately the same number of 3-term progressions. By assuming α sufficiently close to 1 in terms of δ in order to bound $\|\mu_1\|_{L^\infty}$, a standard lower bound on $\cap^3(\mu_1)(\mathbb{T}^2)$ will then give us the result.

We next obtain a lower bound on $\Lambda_3(\mu_1)$. According to Lemma 4.1,

$$\|\boldsymbol{B} * \boldsymbol{B} * \mu_A\|_{L^\infty} \leq \|\boldsymbol{B} * \boldsymbol{B} * \mu\|_{L^\infty} \lesssim \epsilon^{-(1-\alpha)|S|\left(1-\alpha+\frac{\beta}{2}\right)^{-1}}.$$

So by choosing $\alpha \geq \alpha_0' := \beta/(|S|\ln(\epsilon^{-1}) - 1)$, we find that $\|\mu_1\|_{L^\infty} \leq U$ for some constant $U \downarrow 2$ as $\alpha \uparrow 1, C_H, C_F \downarrow 1$. We suppose that we have done so.

By a version of the dense Roth's theorem on $\mathbb{T}$ due in the discrete case to Varnavides [20] (see also Proposition 2.2 of [6] for a general statement or Lemma 6.1 of [2] for a derivation of the continuous inequality from the discrete), we have

$$\Lambda_3(\mu_1) \geq c\left(\frac{\|\mu_1\|_{L^1}}{\|\mu_1\|_{L^\infty}}\right) \tag{10}$$

where $c(x)$ increases as x increases. Using the bound on $\|\mu_1\|_{L^\infty}$ from (6) and that $\|\mu_1\|_{L^1} > \delta$ since $\boldsymbol{B}$ is normalized, we see that $c \geq c(\delta/U) > 0$ remains bounded below as a function of $\delta > 0$ as $\epsilon \downarrow 0$ provided α is chosen as above.

We would then be done if we knew that $|\Lambda_3(\mu_A) - \Lambda_3(\mu_1)| < c$, so we now bound this quantity, keeping in mind that we are free to set the parameters ϵ, η to be anything we want provided α depends on them as specified.

By writing $\Lambda_3(\mu_A)$ and $\Lambda_3(\mu_1)$ on the Fourier side, we have

$$(11)\qquad |\Lambda_3(\mu_A)-\Lambda_3(\mu_1)| = \Big|\sum_{\xi\in\mathbb{Z}} \widehat{\mu_A}(\xi)^2\widehat{\mu_A}(-2\xi)(1-\widehat{\boldsymbol{B}}(\xi)^4\widehat{\boldsymbol{B}}(-2\xi)^2)\Big|$$

$$\le \sum_{\xi\in S} |\widehat{\mu_A}(\xi)|^2|\widehat{\mu_A}(-2\xi)||1-\widehat{\boldsymbol{B}}(\xi)^4\widehat{\boldsymbol{B}}(-2\xi)^2|$$

$$(12)\qquad + \sum_{\xi\notin S} |\widehat{\mu_A}(\xi)|^2|\widehat{\mu_A}(-2\xi)||1-\widehat{\boldsymbol{B}}(\xi)^4\widehat{\boldsymbol{B}}(-2\xi)^2|$$

$$\lesssim \|1-\widehat{\boldsymbol{B}}(\xi)^4\widehat{\boldsymbol{B}}(-2\xi)^2\|_{L^\infty(\xi\in S)}|S| + \|\widehat{\mu_A}\|^{2-p}_{\ell^\infty(S^c)}\|\widehat{\mu_A}\|^{p+1}_{\ell^{p+1}}$$

for any $p>0$.

In the last line, we have used Holder's inequality after pulling out the ℓ^∞ norm of $|\widehat{\mu_A}|^{2-p}$. We wish to choose p so that the restriction estimate of Theorem 1.3 may be applied to bound $\|\widehat{\mu_A}\|_{\ell^{p+1}}$ by a constant, but we also want the exponent $2-p$ to be positive. This is possible provided that

$$2 > p \ge p_\circ - 1$$

where $p_\circ$ is the critical exponent in Theorem 1.3.

Thus we want $p_\circ < 3$. Observing the formula for $p_\circ$ given in Theorem 1.3, it suffices to assume that $\alpha > 1-\beta/4$. So set $\alpha_0 = \max(\alpha_0', 1-\beta/4)$ and assume $\alpha > \alpha_0$.

Choosing such a p in (11), and applying Theorem 1.3 and the bound

$$\|1-\widehat{\boldsymbol{B}}(\xi)^4\widehat{\boldsymbol{B}}(-2\xi)^2\|_{\ell^\infty(\xi\in S)} \lesssim \epsilon^2$$

(cf. [7], Lemma 6.7) we obtain

$$(13)\qquad |\Lambda_3(\mu_A)-\Lambda_3(\mu_1)| \lesssim \epsilon^2|S| + \eta^{2-p}.$$

By choosing ϵ and η appropriately, we can finish the proof by showing that $|\Lambda_3(\mu_1) - \Lambda_3(\mu_A)|$ is smaller than the lower bound c of $\Lambda_3(\mu_1)$. To do this, we examine the estimate (13).

Combining (9) with (13), we have

$$|\Lambda_3(\mu_A)-\Lambda_3(\mu_1)| \le C\epsilon^2|S| + C\eta^{2-p} \le C\epsilon^2\eta^{-p} + C\eta^{2-p}.$$

Since we have fixed α depending on ϵ, η so that the bound in (10) is at least the constant $c = c(\delta/U)$, we may choose first η, and then ϵ small enough that the right hand side above is less than, say, $\frac{1}{2}c$, and thus

$$\Lambda_3(\mu_A) > \Lambda_3(\mu_1) - \frac{1}{2}c > \frac{1}{2}c$$

giving us the result. □

Remark 4.3. *A computation shows that in the above theorem we may take* $\eta = \epsilon = (c/2C)^{1/(2-p)}$*, so that we can set*

$$\alpha_0' = 1 - \frac{\beta\eta^p \ln(2)/2}{\ln(\epsilon^{-1}) - \eta^p \ln(2)} = 1 - \frac{\epsilon^p \ln(2)/2}{\ln(\epsilon^{-1}) - \epsilon^p \ln(2)}\beta =: 1 - \gamma'(c)\beta.$$

With C_H, C_F *close enough to* 1 *that we obtain a sufficient upper bound on* $\|\mu_1\|_{L^\infty}$ *(and thus lower bound on c) that* $c \sim c(\delta/2)$ *we then have that for any* $\rho > 0$ *we may take* $\alpha - \rho \geq \max(1-\beta/4, 1-\gamma(\delta)\beta)$ *where* $\gamma(\delta) = \gamma'(c(\delta/2))$*. Since the constant* C *can be taken to be* > 1*,* $c \leq 1$*,* $p < 2$*, and* γ *is growing as a function of* p*, we can calculate that* $\gamma(1) \leq 1/6$ *when* $\mu(A) = 1$*, in which case Lemma 4.2 remains valid when* $\alpha > 1 - \beta/6$ *(provided* C_H, C_F *sufficiently close to* 1*).*

Lemma 4.4. *Suppose that* ν *is a measure compactly supported in* T^d*. Then for any* $p \in (1, \infty]$

$$\sum_{\xi \in \mathbb{Z}^d} |\widehat{\nu}(\xi)|^p \approx \int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^p \, d\xi.$$

Proof. We first prove that for $p < \infty$

$$\sum_{\xi \in \mathbb{Z}^d} |\widehat{\nu}|^p \lesssim \int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^p \, d\xi$$

(if $p = \infty$, the corresponding inequality is immediate).

Let g be any Schwartz function equal to 1 on the support of ν and for which $\operatorname{supp} \widehat{g} \subset [-\frac{1}{2}, \frac{1}{2}]^d$. Then since $d\nu = g \, d\nu$,

$$\begin{aligned} |\widehat{\nu}(\xi)| = |\widehat{\nu} * \widehat{g}(\xi)| &= |\int_{[-\frac{1}{2},\frac{1}{2}]^d} \widehat{\nu}(\xi - \eta)\widehat{g}(\eta) \, d\eta| \\ &\leq \|\widehat{g}\|_{L^\infty} \int_{[-\frac{1}{2},\frac{1}{2}]^d} |\widehat{\nu}(\xi - \eta)| \, d\eta. \end{aligned}$$

So $|\widehat{\nu}(\xi)| \lesssim \int_{[-\frac{1}{2},\frac{1}{2}]^d} |\widehat{\nu}(\xi - \eta)| \, d\eta$. Thus by Holder's inequality

$$\begin{aligned} \sum_{\xi \in \mathbb{Z}^d} |\widehat{\nu}(\xi)|^p &\lesssim \sum_{\xi \in \mathbb{Z}^d} \left(\int_{[-\frac{1}{2},\frac{1}{2}]^d} |\widehat{\nu}(\xi - \eta)| \, d\eta \right)^p \\ &\leq \sum_{\xi \in \mathbb{Z}^d} \int_{[-\frac{1}{2},\frac{1}{2}]^d} |\widehat{\nu}(\xi - \eta)|^p \, d\eta = \int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^p \, d\xi. \end{aligned}$$

Now we prove the converse inequality. Fix a Schwartz function h equal to 1 on $\operatorname{supp}(\nu)$ such that $\operatorname{supp}(h) \subset B(0,1)$. Then for any $\xi \in \mathbb{R}^d$

$$\widehat{\nu}(\xi) = \widehat{h\nu}(\xi) = \sum_{n \in \mathbb{Z}^d} \widehat{\nu}(n)\hat{h}(\xi - n)$$

by (111) of [21]. By a variant of Young's inequality, we then have for any $1 \leq p \leq \infty$ that $\|\widehat{v}\|_{L^p} \leq \|\widehat{v}\|_{\ell^p}\|\hat{h}\|_{L^1}$. Indeed,

$$\|\widehat{v}\|_{L^1} = \int \Big| \sum_{n\in\mathbb{Z}^d} \widehat{v}(n)\hat{h}(\xi - n)\Big| \leq \|\widehat{v}\|_{\ell^1}\|\hat{h}\|_{L^1}$$

$$\|\widehat{v}\|_{L^\infty} = \sup_{\xi\in\mathbb{R}^d} \Big| \sum_{n\in\mathbb{Z}^d} \widehat{v}(n)\hat{h}(\xi - n)\Big| \lesssim \|\widehat{v}\|_{\ell^\infty}\|\hat{h}\|_{L^1}$$

where we have used the first part of the Lemma to bound $\sum_{n\in\mathbb{Z}^d} |\hat{h}(\xi - n)|$ by $\|\hat{h}\|_{L^1}$, and interpolating between the two gives for any $p \in [1, \infty]$

$$\|\widehat{v}\|_{L^p} \lesssim \|\widehat{v}\|_{\ell^p}.$$

□

5. 3APs in dense subsets : Proof of Theorem 1.1

The following is only a few lines removed from our main result, Theorem 1.1.

Proposition 5.1. *Let $\delta > 0$, and suppose that the probability measure μ on $\mathbb{T}$ satisfies (b) with $\beta > \frac{1}{2}$ and (a) with α sufficiently close to 1 depending on β, δ, and the implicit constants C_H and C_F in (a) and (b).*

Then if A is any measureable set with $\mu(A) > \delta$, A^*_μ contains 3APs; indeed, μ gives positive measure to the set of x such that $x, x+r, x+2r$ is a non-trivial 3AP contained within A^*_μ.

Proof. By Lemma 3.1, the measure $\cap^3(\mu_A)$ exists and by Lemma 3.2 it is concentrated inside a set identifiable with the non-trivial 3APs contained in A^*_μ. By Lemma 4.2, $\cap^3(\mu_A)$ is not the trivial measure and hence A^*_μ must contain 3APs. By Proposition 3.3, μ gives positive measure to the collection of base points x for such 3APs. □

Proof of Theorem 1.1. Let $L = A^*_\mu \setminus A$ denote the collection of points of density of A not contained in A. Of course, $\mu(L) = 0$. We may thus find an arbitrarily small open set around L since μ is Radon; in particular, we may find a subset $A' \subset A$ such that $A'^*_\mu \subset A$ and such that

$$\mu(A') > \delta$$

remains valid.

Applying Proposition 5.1, we find that A'^*_μ contains 3APs and that μ gives the collection of their basepoints positive measure. Consequently so too for A. □

6. Existence of the measure on 3APs

The goal of this section is to prove Theorem 2.1 and Lemma 2.2.

We will need the following bound, which follows from the Hausdorff dimension condition (a) via straightforward arguments (see Lemma 4.1 of [4]).

$$\|\phi_n * \mu\|_{\ell^\infty} \lesssim 2^{(1-\alpha)n}. \tag{14}$$

For bounded functions $f:\mathbb{T}\to\mathbb{R}$ and $g:\mathbb{T}^2\to\mathbb{R}$, define

$$\Lambda_g(f) := \int g(x,r)f(x)f(x-r)f(x-2r)\,dx\,dr$$

and for three such functions f_1, f_2, f_3

$$\Lambda_g(f_1,f_2,f_3) := \int g(x,r)f_1(x)f_2(x-r)f_3(x-2r)\,dx\,dr.$$

Following Gowers [5], set also

$$\Delta^1 f(x;r) = f(x)\overline{f(x-r)}$$
$$\Delta^2 f(x;r,s) = \Delta^1 f(x;r)\overline{\Delta^1 f(x-s;r)} = f(x)\overline{f(x-r)f(x-s)}f(x-r-s)$$
$$\Delta^1 g(x,y;r) = g(x,y)\overline{g(x,y-r)}$$
$$\Delta^2 g(x,y;r,s) = \Delta^1 g(x,y;r)\overline{\Delta^1 g(x-s,y-s;r)}$$
$$=g(x,y)\overline{g(x,y-r)g(x-s,y-s)}g(x-s,y-r-s).$$

One easily verifies the following standard identity.

Lemma 6.1. *Let f_1 and f_2 be bounded functions on $\mathbb{T}$ and g be bounded on $\mathbb{T}^2$. Then*

$$\Lambda_g(f_1)-\Lambda_g(f_2) = \overline{\underline{\sum}}_{f_1,f_2}\Lambda_g(h_1,h_2,h_3)$$

where the symbol $\overline{\underline{\sum}}_{f_1,f_2}$ refers to a sum taken over the set

$$\{(f_1-f_2,f_1,f_1),(f_2,f_1-f_2,f_1),(f_2,f_2,f_1-f_2)\}.$$

We postpone the proofs of the following lemmas to the end of this section.

Lemma 6.2. *Let f be a function on $\mathbb{T}$ which satisfies*

$$|\hat{f}(\xi)| \le C_F(1+|\xi|)^{-\frac{\beta}{2}} \text{ for all } \xi\in\mathbb{Z} \tag{15}$$

$$supp(\hat{f}) \subset B(0,2^{n+1})\setminus B(0,2^n) \tag{16}$$

for some $\beta>0$.

Then for all $\eta\in\mathbb{Z}$

$$|\widehat{\Delta^2 f}(\xi;\eta)| \lesssim 2^{-(2\beta-1)n}$$

and

$$supp\,\widehat{\Delta^2 f}(0;\cdot) \subset B(0,2^{n+4}).$$

Lemma 6.3. *Let $f,h_1,h_2:\mathbb{T}\to\mathbb{R}$ and $g:\mathbb{T}^2\to\mathbb{R}$ be bounded functions. Then*

$$\left|\int g(x,r)h_1(x)h_2(x-r)f(x-2r)\,dx\,dr\right|$$
$$\le \|h_1\|_{L^\infty}\|h_2\|_{L^\infty}\left(\int\left|\int \Delta^1 g(2x-r,x-r;-s)\,\Delta^1 f(r;2s)\,dr\right|^2 dx\,ds\right)^{1/4}.$$

Lemma 6.4. *Let f be a function on $\mathbb{T}$ which satisfies*

$$|\hat{f}(\xi)| \leq C_F(1+|\xi|)^{-\frac{\beta}{2}} \text{ for all } \xi \in \mathbb{Z}. \tag{17}$$

$$supp(\hat{f}) \subset B(0,2^{n+1}) \setminus B(0,2^n) \tag{18}$$

for some $\beta > 0$. Then for any $h_1, h_2 \in L^\infty(\mathbb{T})$ and $g : \mathbb{T}^2 \to \mathbb{R}$, for all $p \in [1,\infty]$

$$|\Lambda_g(h_1,h_2,f)| \lesssim 2^{-\frac{1}{4}(2\beta-1-\frac{2}{p'})n} \|h_1\|_{L^\infty} \|h_2\|_{L^\infty} \|\Delta^2 \widehat{g(0,\xi;\eta)}\|_{\ell^1_\xi \ell^p_\eta}^{\frac{1}{4}}.$$

Lemma 6.5. *Suppose that μ is a measure on $\mathbb{T}$ satisfying* (b) *with $\beta > 0$. Then for any $g : \mathbb{T}^2 \to \mathbb{R}$ and $p \in [1,\infty]$, for all $N > m \in \mathbb{N}$,*

$$\left|\Lambda_g(\phi_N * \mu) - \Lambda_g(\phi_m * \mu)\right| \lesssim \left(\sum_{n=m}^{N-1} 2^{-\frac{1}{4}(2\beta-1-\frac{2}{p'})n} \max_{i=0,1} \|\phi_{n+i} * \mu\|_{L^\infty}^2\right) \|\widehat{\Delta^2 g}(0,\xi;\eta)\|_{\ell^1_\xi \ell^p_\eta}^{\frac{1}{4}}$$

where the mixed norm $\|G(\xi,\eta)\|_{\ell^1_\xi \ell^p_\eta} := \sum_\xi \left(\sum_\eta |G(\xi,\eta)|^p\right)^{\frac{1}{p}}$.

Using the above lemmas we can complete the proof of Theorem 2.1 and Lemma 2.2.

Proof of Theorem 2.1. It suffices to show that the functional defined by the limit in (1) exists for g in a dense subclass of $C(\mathbb{T}^2)$ and is bounded on that subclass. We take as our subclass the collection of all trigonometric polynomials, and by linearity it suffices to check existence and boundedness for a monomial of the form $g(x,r) = e^{2\pi i(x\xi'+r\eta')}$.

To show existence of the limit, it suffices to show that given $\epsilon > 0$ there is some m such that for all large N,

$$\left|\Lambda_g(\phi_N * \mu) - \Lambda_g(\phi_m * \mu)\right| < \epsilon. \tag{19}$$

To this end, we apply Lemma 6.5 with $p = 1$. By our assumption on g, it is easy to see that

$$\|\widehat{\Delta^2 g}\|_{\ell^1} = 1.$$

Inserting this and the bound (14) into Lemma 6.5 gives

$$\left|\Lambda_g(\phi_N * \mu) - \Lambda_g(\phi_m * \mu)\right| \lesssim \sum_{n=m}^{N-1} 2^{2(1-\alpha)(n+1)} 2^{-(2\beta-1)n/4} \approx \sum_{n=m}^{N-1} 2^{-n(\frac{2\beta-1}{4}-2(1-\alpha))} \tag{20}$$

and the exponent $-(\frac{2\beta-1}{4} - 2(1-\alpha))$ will be negative when $\beta > \frac{1}{2}$ and $\alpha > \frac{9-2\beta}{8}$.

Since the sum (20) is a geometric series and remains finite in the limit $N \uparrow \infty$, we can take m large enough to guarantee that (19) holds, as desired.

Thus

$$\lim_{n\to\infty} \int g(x,r)\phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\, dx\, dr$$

exists for g a trigonometric polynomial. In particular, the limit exists for for $g \equiv 1$.

Finally, we complete the proof by showing that the above limit is bounded where defined. Indeed, setting

$$M := \Lambda_1(|\mu|)$$

we have

$$\begin{aligned}&|\lim_{n\to\infty}\int g(x,r)\phi_n * \mu(x)\phi_n * \mu(x-r)\phi_n * \mu(x-2r)\,dx\,dr|\\ \leq&\|g\|_{L^\infty}\lim_{n\to\infty}\int \phi_n * |\mu|(x)\phi_n * |\mu|(x-r)\phi_n * |\mu|(x-2r)\,dx\,dr\\ \leq&M\|g\|_{L^\infty}\end{aligned}$$

using that the limit defining $\Lambda_g(|\mu|)$ exists for the trigonometric polynomial $g \equiv 1$. □

Proof of Lemma 2.2. Choose a compactly supported Schwartz function g with $[-\frac{1}{2}, \frac{1}{2}] \subset \operatorname{supp} g \subset [-1, 1]$. Let $\epsilon > 0$ and set $g_\epsilon(r) = g((4\epsilon)^{-1} r)$.

It is not hard to see that $\Delta^k g$ is rapidly decaying since g is Schwartz.

We have

$$\cap^3\mu(\{|r| \leq \epsilon\}) \leq |\Lambda_{g_\epsilon}(\mu)|. \tag{21}$$

We will show that

$$\Lambda_{g_\epsilon}(\mu) \lesssim \epsilon^{2\frac{p-1}{p}}$$

for some $p > 1$ to be specified later.

Since $\Lambda_g(\phi_n * \mu) \to \int g\, d\cap^3 \mu$ by Theorem 2.1, we can send $N \to \infty$ and set $m = 0$ in the bound from Lemma 6.5, which gives

$$|\int g_\epsilon\, d\cap^3 \mu - \Lambda_{g_\epsilon}(\phi_0 * \mu)| \lesssim \left(\sum_{n=0}^{\infty} 2^{-\left(\frac{2\beta-1-\frac{2}{p'}}{4}-2(1-\alpha)\right)n}\right) \|\widehat{\Delta^2 g_\epsilon}(0;\cdot)\|_{\ell^p}^{\frac{1}{4}}$$

for some $p > 1$. Since $\beta > \frac{1}{2}$, provided that $\alpha > \frac{9-2\beta}{8}$ we may find $p > 1$ such that the sum in the right hand side above is finite. Fix such a p. We then have

$$\int g_\epsilon\, d\cap^3 \mu \lesssim \Lambda_{g_\epsilon}(\phi_0 * \mu) + \|\widehat{\Delta^2 g_\epsilon}(0;\cdot)\|_{\ell^p}^{\frac{1}{4}} = C\epsilon + \|\widehat{\Delta^2 g_\epsilon}(0;\cdot)\|_{\ell^p}^{\frac{1}{4}}$$

We are now done, since for any function G on $[0,1]^2$ with $\hat{G} \in L^1(\mathbb{R}^2)$, for $p > 1$ if $G_\epsilon(\vec{x}) = G((4\epsilon)^{-1}\vec{x})$ then applying Lemma 4.4 and a change of variables,

$$\|\hat{G}_\epsilon\|_{\ell^p} \lesssim \|\hat{G}_\epsilon\|_{L^p} = \epsilon^{2\frac{p-1}{p}} \|\hat{G}\|_{L^p}$$

and $G = \Delta^2 g(0;\cdot)$ is such a function. □

All that remains is to prove Lemmas 6.2, 6.3, 6.4, and 6.5.

Proof of Lemma 6.2. The (trivial) identity

$$\widehat{\Delta^1 f}(\eta_1;\tau) = \overline{\hat{f}(\tau)}\hat{f}(\eta_1+\tau)$$

together with the Fourier decay condition (15) and the support condition (16) give

$$|\widehat{\Delta^1 f}(\eta_1;\tau)| \lesssim 1_{\{2^n\le|\tau|,|\tau+\eta_1|\le 2^{n+1}\}}\,(1+|\tau|)^{-\beta/2}\,(1+|\eta_1+\tau|)^{-\beta/2} \tag{22}$$

One readily verifies that

$$\widehat{\Delta^2 f}(\xi;\eta) = \sum_{\tau\in\mathbb{Z}} \widehat{\Delta^1 f}(-\eta_1;\tau)\overline{\widehat{\Delta^1 f}(-\xi-\eta_1;\tau-\eta_2)}$$

which shows with (22) that $\eta \in B(0,2^{n+4})$, and it follows by Cauchy-Schwarz that

$$|\widehat{\Delta^2 f}(\xi;\eta)| \le \sqrt{\sum_{\tau\in\mathbb{Z}}|\widehat{\Delta^1 f}(\eta_1;\tau)|^2}\sqrt{\sum_{\tau\in\mathbb{Z}}|\widehat{\Delta^1 f}(\xi+\eta_1;\tau)|^2}.$$

Applying (22) to the above gives

$$\begin{aligned}|\widehat{\Delta^2 f}(\xi;\eta)| &\lesssim \left(\sum_{\tau\in\mathbb{Z}} 1_{\{2^n\le|\tau|,|\tau+\eta_1|\le 2^{n+1}\}}\,(1+|\tau|)^{-\beta}\,(1+|\eta_1+\tau|)^{-\beta}\right)^{\frac{1}{2}}\\ &+\left(\sum_{\tau\in\mathbb{Z}} 1_{\{2^n\le|\tau|,|\tau+\xi+\eta_1|\le 2^{n+1}\}}\,(1+|\tau|)^{-\beta}\,(1+|\xi+\eta_1+\tau|)^{-\beta}\right)^{\frac{1}{2}}\\ &\lesssim 2^{-(2\beta-1)n}.\end{aligned}$$

□

Proof of Lemma 6.3. We have

$$\begin{aligned}&\left|\int g(x,r)h_1(x)h_2(x-r)f(x-2r)\,dx\,dr\right|\\ \le&\|h_1\|_{L^\infty}\left(\int\left|\int g(x,r)h_2(x-r)f(x-2r)\,dr\right|^2dx\right)^{\frac{1}{2}}\\ =&\|h_1\|_{L^\infty}\left(\int g(x,r)\overline{g(x,s+r)}h_2(x-r)\overline{h_2(x-r-s)}f(x-2r)\overline{f(x-2r-2s)}\,dr\,dx\,ds\right)^{\frac{1}{2}}\\ =&\|h_1\|_{L^\infty}\left(\int \triangle^1 g(x,r;-s)\triangle^1 h_2(x-r;s)\triangle^1 f(x-2r;2s)\,dr\,dx\,ds\right)^{\frac{1}{2}}\\ \le&\|h_1\|_{L^\infty}\|h_2\|_{L^\infty}\left(\int\left|\int \triangle^1 g(x+r,r;-s)\triangle^1 f(x-r;2s)\,dr\right|^2dx\,ds\right)^{1/4}\end{aligned}$$

$$=\|h_1\|_{L^\infty}\|h_2\|_{L^\infty}\left(\int\left|\int \Delta^1 g(2x-r,x-r;-s)\,\Delta^1 f(r;2s)\,dr\right|^2 dx\,ds\right)^{1/4}$$

where in the last line we have applied the change of variables $r\mapsto -r+x$.

□

Proof of Lemma 6.4. By Lemma 6.3,

$$\begin{aligned}
&|\int g(x,r)h_1(x)h_2(x-r)f(x-2r)\,dx\,dr|\\
\leq&\|h_1\|_{L^\infty}\|h_2\|_{L^\infty}\left(\int\left|\int \Delta^1 g(2x-r,x-r;-s)\,\Delta^1 f(r;2s)\,dr\right|^2 dx\,ds\right)^{1/4}\\
=&\|h_1\|_{L^\infty}\|h_2\|_{L^\infty}\left(\int \Delta^2 g(2x-r,x-r;-s)\,\Delta^2 f(r;2s,t)\,dx\,dr\,ds\,dt\right)^{1/4}.
\end{aligned}$$

Applying Plancherel gives

$$\begin{aligned}
&|\int\int \Delta^2 g(x+r,\frac{1}{2}x;\frac{1}{2}s,t)\,\Delta^2 f(r;s,t)\,dr\,dt\,ds\,dx|\\
=&2|\sum_{\xi\in\mathbb{Z},\eta\in\mathbb{Z}^2}\widehat{\Delta^2 g}(0,2\xi;\eta_1/2,\eta_2)\overline{\widehat{\Delta^2 f}(\xi;\eta)}|
\end{aligned}$$

$$\lesssim\|\widehat{\Delta^2 g}(0,\xi;\eta_1/2,\eta_2)\|_{\ell^1_\xi\ell^p_\eta}\left(\sup_{\xi\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}^2}|\widehat{\Delta^2 f}(\xi;\eta)|^{p'}\right)^{\frac{1}{p'}} \tag{23}$$

by Holder's inequality.

By Lemma 6.2, this is bounded by

$$\|\widehat{\Delta^2 g}(0,\xi;\eta_1/2,\eta_2)\|_{\ell^1_\xi\ell^p_\eta}2^{-(2\beta-1-\frac{2}{p'})n}$$

and two applications of Lemma 4.4 in order to move to the continuous context, change variables, and back, give the result. □

Proof of Lemma 6.5. We decompose into telescoping series and invoke Lemma 6.1

$$\begin{aligned}
&\left|\Lambda_g(\phi_N*\mu)-\Lambda_g(\phi_m*\mu)\right| = |\sum_{n=m}^{N-1}\Lambda_g(\phi_{n+1}*\mu)-\Lambda_g(\phi_n*\mu)|\\
=&|\sum_{n=m}^{N-1}\overline{\underline{\sum}}_{\phi_{n+1}*\mu,\phi_n*\mu}\Lambda_g(h_1,h_2,h_3)|
\end{aligned}$$

$$\leq\sum_{n=m}^{N-1}\overline{\underline{\sum}}_{\phi_{n+1}*\mu,\phi_n*\mu}\left|\Lambda_g(h_1,h_2,h_3)\right|. \tag{24}$$

We next calculate a bound on each individual term in the above sum. By a linear change of variables, it suffices to suppose that $h_3 = \phi_{n+1} * \mu - \phi_n * \mu := \mu_n$.[6]

We can apply Lemma 6.4, which gives

$$|\Lambda_g(h_1, h_2, \mu_n)| \lesssim 2^{-\frac{2\beta-1-\frac{2}{p'}}{4}n} \|h_1\|_{L^\infty} \|h_2\|_{L^\infty} \|\widehat{\Delta^2 g}(0,\xi;\eta)\|^{\frac{1}{4}}_{\ell^1_\xi \ell^p_\eta}.$$

Inserting this into the right hand side of (24) gives the result.

□

7. Concluding Remarks

It is interesting to note that if the dependence of the Hausdorff dimension α in Theorem 1.1 on the measure δ of the set A were dropped, we would in fact have that μ-almost every point x was the starter of a three-term arithmetic progression contained in the support of μ. It is not a priori clear whether this dependence is necessary, or whether for some α, β strictly less than 1, and some C_H, C_F, it might be true that whenever μ satisfied (*a*) and (*b*) that every A for which $\mu(A) > 0$ must contain 3APs. In the discrete context, Green's result applies to any dense subset of the primes, regardless of density, but the primes have "dimension" 1 within the natural numbers. On the other hand Conlon, Fox, and Zhao [3] have developed a combinatorial approach strong enough to obtain a relative Szemerédi theorem (kAPs for $k \geq 3$) for subsets of pseudorandom sets of integers having "dimension" less than 1 within the integers, and their result applies for any dense subset of a sufficiently pseudorandom set, which gives reasonably strong evidence that such a result may be possible in our context.

There is some further weak evidence the dependence of α on δ may be unnecessary, in the form of a result of Conlon and Gowers [6] (see also [11] for the $k = 3$ case) that as $N \uparrow \infty$, with probability converging to 1, every random subset E of $[N]$ large enough for the statement to not trivially fail has the property that all size $\delta|E|$ subsets of E contain kAPs. The natural cutoff for 3APs in this result of Conlon and Gowers is that the set $E \subset [N]$ satisfy $|E| \approx N^{\frac{1}{2}}$. Thus it is natural to conjecture that the appropriate bound on the Hausdorff dimension α of a measure μ on $\mathbb{T}$ in order that it contain 3APs be $\alpha > \frac{1}{2}$. Though we have achieved results for any $\beta > \frac{1}{2}$, we require a far larger α.

In a different direction, it should be true that for a large set of x, the collection of r for which $x, x+r, x+2r$ lies in $\mathrm{supp}(\mu)$, or indeed in A for a set A of positive μ-measure, be large in Hausdorff dimension.

In [2], we applied a notion of "higher order" Fourier dimension to extend the results of [12] to longer progressions. It would be natural to extend the results of the present paper to this setting, namely, to demonstrate that given a measure of $(k-1)$st order Fourier dimension sufficiently close to 1, that any subset of sufficiently large measure

[6]The change of variables may result in a different g. However, since the change of variable is linear, it will affect the bounds we obtain by at most a constant which doesn't affect the conclusion of the theorem.

must contain kAPs. The obvious approach to such a result would be to adapt the methods of [3].

References

[1] Jean Bourgain. "Pointwise ergodic theorems for arithmetic sets". *Inst. Hautes Études Sci. Publ. Math.* (1989). With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein, 5–45.

[2] M. Carnovale. "Long progressions in sets of fractional dimension" (2013). arXiv: 1308.2919.

[3] Y. Zhao D. Conlon J. Fox. "A relative Szemerédi theorem" ().

[4] M. Burak Erdoğan. "On Falconer's distance set conjecture". *Rev. Mat. Iberoam.* 22. (2006), 649–662.

[5] W. T. Gowers. "A new proof of Szemerédi's theorem". *Geom. Funct. Anal.* 11. (2001), 465–588.

[6] W. T. Gowers. "Decompositions, approximate structure, transference, and the Hahn-Banach theorem". *Bull. Lond. Math. Soc.* 42. (2010), 573–606.

[7] Ben Green. "Roth's theorem in the primes". *Ann. of Math. (2)* 161. (2005), 1609–1636.

[8] Ben Green and Terence Tao. "The primes contain arbitrarily long arithmetic progressions". *Ann. of Math. (2)* 167. (2008), 481–547.

[9] Kyle Hambrook and Izabella Łaba. "On the sharpness of Mockenhaupt's restriction theorem". *Geom. Funct. Anal.* 23. (2013), 1262–1277.

[10] Bernard Host and Bryna Kra. "A point of view on Gowers uniformity norms". *New York J. Math.* 18 (2012), 213–248.

[11] Yoshiharu Kohayakawa, Tomasz Łuczak, and Vojtěch Rödl. "Arithmetic progressions of length three in subsets of a random set". *Acta Arith.* 75. (1996), 133–163.

[12] Izabella Łaba and Malabika Pramanik. "Arithmetic progressions in sets of fractional dimension". *Geom. Funct. Anal.* 19. (2009), 429–456.

[13] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*. Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge: Cambridge University Press, 1995, pp. xii+343.

[14] Themis Mitsis. "A Stein-Tomas restriction theorem for general measures". *Publ. Math. Debrecen* 60. (2002), 89–99.

[15] Gerd Mockenhaupt. "Salem sets and restriction properties of Fourier transforms". *Geom. Funct. Anal.* (1996), 1579–1587.

[16] O. Reingold et al. "Dense Subsets of Pseudorandom Sets". *Electronic Colloquium on Computational Complexity* (2008).

[17] Elias M. Stein. "Maximal functions. I. Spherical means". *Proc. Nat. Acad. Sci. U.S.A.* 73. (1976), 2174–2175.

[18] Terence Tao and Van H. Vu. *Additive combinatorics*. Vol. 105. Cambridge Studies in Advanced Mathematics. Paperback edition [of MR2289012]. Cambridge University Press, Cambridge, 2010, pp. xviii+512.

[19] Terence Tao and Tamar Ziegler. "The primes contain arbitrarily long polynomial progressions". *Acta Math.* 201. (2008), 213–305.

[20] P. Varnavides. "On certain sets of positive density". *J. London Math. Soc.* 34 (1959), 358–360.

[21] Thomas H. Wolff. *Lectures on harmonic analysis*. Vol. 29. University Lecture Series. With a foreword by Charles Fefferman and preface by Izabella Łaba, Edited by Łaba and Carol Shubin. American Mathematical Society, Providence, RI, 2003, pp. x+137.

Department of Mathematics, The Ohio State University, 231 W 18th Ave, Columbus, Ohio, USA

E-mail address: carnovale.2@osu.edu