NOTE ON PACKING PATTERNS IN COLORED PERMUTATIONS

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ABSTRACT. Packing patterns in permutations concerns finding the permutation with the maximum number of a prescribed pattern. In 2002, Albert, Atkinson, Handley, Holton and Stromquist showed that there always exists a layered permutation containing the maximum number of a layered pattern among all permutations of length n. Consequently the packing density for all but two (up to equivalence) patterns up to length 4 can be obtained. In this note we consider the analogous question for colored patterns and permutations. By introducing the concept of "colored blocks" we characterize the optimal permutations with the maximum number of a given colored pattern when it contains at most three colored blocks. As examples we apply this characterization to find the optimal permutations of various colored patterns and subsequently obtain their corresponding packing densities.

1. INTRODUCTION

Given a permutation π of length n a pattern σ is said to be contained in π or σ *occurs* in π if a subsequence of π is order isomorphic to σ. For instance, the permutation $\pi = 51342$ contains two $\sigma = 321$ patterns as the subsequences 532 and 542. The occurrence of a pattern in a permutation has been vigorously studied over the past decades. However, most of the existing work involves studying permutations which avoid a particular pattern; i.e., pattern avoidance.

A symmetric problem to pattern avoidance is the study of which permutation contains the most instances of a given pattern; i.e., pattern packing. The central problem of patten packing is maximizing the function $p(\pi,\sigma)$ among all permutations of given length, where $p(\pi,\sigma)$ is the number of occurrences of a pattern σ in the permutation π. This question was first considered in the unpublished work of Stromquist [4] and in the PhD thesis of Price [2] where much of the foundational work on pattern packing can be found. In what follows we call a permutation $\tilde{\pi}$ *optimal* if

$$p(\tilde{\pi},\sigma) \geq p(\pi,\sigma)$$

for any permutation π of the same length. A useful way to compare the "packability" of two different patterns is by comparing their *packing densities*, defined as

$$\delta(\sigma) = \lim_{|\pi|\to\infty} \frac{p(\tilde{\pi},\sigma)}{\binom{|\pi|}{|\sigma|}}.$$

Date: April 5, 2016.

1991 *Mathematics Subject Classification.* 05A15, 05A16.

Key words and phrases. Permutation, pattern, packing, density.

This work was partially supported by grants from the Simons Foundation (#245307).

Definition 1. *A permutation π of length n is called* layered *if it can be partitioned into segments $\pi_1, \pi_2, \ldots, \pi_r$ called* layers *such that each layer is decreasing and all elements in π_i are less than all elements in π_{i+1} for $1 \leq i \leq r-1$.*

For example, the permutation $\pi = 32154876$ contains three layers $\pi_1 = 321$, $\pi_2 = 54$, $\pi_3 = 876$. Because $\pi_1 < \pi_2 < \pi_3$ (notating all elements of one layer being less than all elements of another) we have a layered permutation. A simple representation of a layered permutation with lattice points is shown in Figure 1.

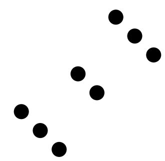

FIGURE 1. A layered permutation 32154876

Building upon previous work, Albert et al. [1] considered packing densities of layered patterns. One important result used there is from [4].

Theorem 1.1 ([4])**.** *Given a layered pattern σ there exists an optimal permutation $\tilde{\pi}$ that is also layered.*

Since all but two symmetry classes of permutations of length 4 are layered permutations (the exceptions being represented by 1342 and 2413), most of the packing densities have been found for patterns up to length four while the others have been conjectured.

We next introduce the concept of colored permutations.

Definition 2. *An m-*colored permutation *χ of* length *n is a permutation of length n in which each element is assigned one of m distinct colors.*

For example, let $\chi = 3_a 2_a 5_b 1_a 4_b$ be a two-colored permutation where 3,2 and 1 have color a while 5 and 4 have color b. Analogously to the case of non-colored patterns the colored pattern $\phi = 2_a 1_a 3_b$ occurs in χ as the subsequences $3_a 1_a 4_b$ and $3_a 2_a 4_b$.

Colored permutations are similar to permutations of a multi-set. The question of pattern avoidance on multi-sets has been studied in past years (see, for instance, [3]). In this note we focus on pattern packing in colored permutations.

We first define *colored blocks*, which are central to our study. Colored blocks are analogous to layers in non-colored permutations.

Definition 3. *In a colored permutation χ, a* colored block *is a maximal monochromatic segment $\chi_i^{(a)}$ in which each element has color a and every element not in $\chi_i^{(a)}$ is either larger than or smaller than all elements in $\chi_i^{(a)}$.*

Remark 1. *Note that every entry in a colored permutation is in exactly one of its colored blocks.*

In other words, a colored block is a monochromatic segment of elements with consecutive numerical values. For instance, the permutation $\chi = 3_a 1_a 2_a 6_a 5_b 4_b$ contains three colored blocks, $\chi_1^{(a)} = 3_a 1_a 2_a$, $\chi_2^{(a)} = 6_a$, and $\chi_3^{(b)} = 5_b 4_b$. A graphical representation of colored blocks is shown in Figure 2. An important note is that colored blocks are both numerically and chromatically disjoint.

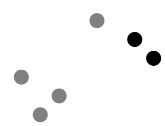

FIGURE 2. A Colored Permutation $3_a 1_a 2_a 6_a 5_b 4_b$

In the rest of this note we provide some observations on the optimal colored permutations for colored patterns which contain either two or three colored blocks. For convenience we will reuse the notation $p(\chi,\phi)$ to represent the number of occurrences of the colored pattern ϕ in the colored permutation χ. Colored blocks will often be denoted simply by their color and/or location, i.e. $\chi_1^{(a)} = A_1$, $\chi_2^{(b)} = B_2$, $\chi_3^{(a)} = A_3$, etc. Similarly for colored patterns $\phi_1^{(a)} = \alpha_1$, $\phi_2^{(b)} = \beta_2$, etc. The collection of all colored blocks in χ of color a (b) will be denoted by χ_A (χ_B).

2. PATTERN PACKING WITH TWO OR THREE COLORED BLOCKS

Note that a single-colored (or non-colored) permutation has exactly one colored block (namely the permutation itself). In what follows we assume the permutations/patterns under consideration to be at least two-colored.

2.1. **Patterns with two colored blocks.**

Since the number of colored blocks is at least the number of colors, we may assume that a colored pattern ϕ with two colored blocks $\phi_1^{(r)} = \rho$ and $\phi_2^{(b)} = \beta$ to have exactly two colors and is of the form $\rho\beta$. Furthermore, we may assume without loss of generality that all elements in ρ are less than all elements of β (similarly to layers denoted $\alpha < \beta$). For the remainder of this section color r will be referred to as "red" and b will be referred to as "blue".

Theorem 2.1. *For a pattern ϕ with two blocks of the form $\rho\beta$ with $\rho < \beta$, there is an optimal length-n permutation $\tilde{\chi}$ of the form RB with $R < B$.*

Remark 2. *The proof below follows the simple idea that sliding all of the red entries to the left and all of the blue entries to the right leaves every instance of a ϕ-pattern intact.*

Proof. Let χ be an optimal permutation of length n with colored blocks $\chi_1\chi_2\ldots\chi_k$. First we claim that $\chi_1 = R_1$ is red. If χ_1 were blue,

$$p(\chi,\phi) = p(\chi_1,\rho)\cdot p(\chi_{>1},\beta) + p(\chi_{>1},\phi)$$

where $\chi_{>i_0}$ ($\chi_{<i_0}$) is the collection of all colored blocks in χ after (before) χ_{i_0}. Clearly the first term in the sum is zero. By recoloring χ_1 red, this term is replaced with

$$p(R_1,\rho)\cdot p(\chi_{>1},\beta) \geq 0,$$

and thus $p(\chi,\phi)$ will only increase. Similarly, we may assume $\chi_k = B_k$ is blue. Along the same lines we claim there is no blue block immediately preceding a red block. Otherwise, let $\chi_j = B_j$ and $\chi_{j+1} = R_{j+1}$ in χ, we have

$$p(\chi,\phi) = P(\chi_{<j},\rho)\cdot p(B_j,\beta) + P(\chi_{>j+1},\beta)\cdot p(R_{j+1},\rho) + p(\chi_{<j}\chi_{>j+1},\phi).$$

Let χ' be obtained from χ by switching χ_j and χ_{j+1}, we have

$$p(\chi',\phi) = p(\chi,\phi) + p(R_{j+1},\rho)\cdot p(B_j,\beta).$$

Thus $p(\chi,\phi)$ may only increase.

Consequently, we now have an optimal permutation χ of the form $\chi = \chi_R\chi_B$. Because any $\phi = \rho\beta$ pattern occurring in χ must consist of a ρ pattern from χ_R and a β pattern from χ_B, we have

$$p(\chi,\phi) \leq p(\chi_R,\rho)\cdot p(\chi_B,\beta)$$

with equality if $\chi_R < \chi_B$. Hence there is an optimal permutation $\tilde{\chi} = RB$ with $R < B$. ∎

For example, to pack the pattern $\chi = 2_r 1_r 3_b 4_b$ an optimal permutation of length n consists of a decreasing sequence of the elements $\lfloor\frac{n}{2}\rfloor\ldots 1$ colored red followed by an increasing sequence of the elements $(\lfloor\frac{n}{2}\rfloor + 1)\ldots n$ colored blue. More detailed applications will be discussed in Section 3.

2.2. **Patterns with three colored blocks.** In this subsection we consider patterns with three colored blocks through several different cases. Some arguments are similar to those in the previous subsection and we omit some details.

First consider the case when the pattern has three distinct colors. Assume without loss of generality $\phi_1^{(r)} = \rho$, $\phi_2^{(b)} = \beta$, and $\phi_3^{(g)} = \gamma$ (with colors red, blue and green) and thus the colored pattern has the form $\phi = \rho\beta\gamma$.

Theorem 2.2. *Given a pattern ϕ with three colored blocks of distinct colors of the form $\rho\beta\gamma$, there is an optimal permutation $\tilde{\chi}$ (of length n) of the form RBG with the same numerical ordering as $\rho\beta\gamma$.*

Remark 3. *For instance, given a pattern ϕ of the form $\rho\beta\gamma$ with $\rho < \gamma < \beta$, there is an optimal permutation of the form RBG such that $R < G < B$.*

Proof. Following the same arguments as Theorem 2.1, it is easy to show that the optimal permutation is of the form

$$R_1\ldots R_i B_{i+1}\ldots B_j G_{j+1}\ldots G_k = \chi_R\chi_B\chi_G.$$

Then

$$p(\chi,\phi) \leq p(\chi_R,\rho)\cdot p(\chi_B,\beta)\cdot p(\chi_G,\gamma)$$

with equality if any elements $a \in \chi_R$, $b \in \chi_B$ and $c \in \chi_G$ assume the same numerical ordering as ρ, β and γ. ∎

Suppose now that a pattern ϕ has three colored blocks with only two colors. Assume without loss of generality that there is one red block and two blue blocks. First consider the case when the two blue blocks are adjacent, i.e., $\phi = \rho\beta_1\beta_2$. This case is representative of all patterns with two adjacent blue blocks since reversing the permutation/pattern turns all $\rho\beta_1\beta_2$ patterns into $\beta_1\beta_2\rho$ patterns.

Theorem 2.3. *For a pattern ϕ with three colored blocks of the form $\rho\beta_1\beta_2$, there is an optimal permutation $\tilde{\chi}$ (of length n) that is also of the form RB_1B_2 and the numerical ordering of the colored blocks in $\tilde{\chi}$ is the same as that of the colored blocks in ϕ.*

Proof. First note that since the two blue blocks are numerically disjoint, no element from β_1 may be (numerically) adjacent to an element in β_2. That is, either $\beta_1 < \rho < \beta_2$ or $\beta_2 < \rho < \beta_1$. Without loss of generality we will assume the former.

Once again arguments from Theorem 2.1 yield that all red blocks in an optimal permutation χ can be placed before any blue blocks. That is, an optimal permutation is of the form

$$\chi = R_1 \dots R_i B_{i+1} \dots B_k = \chi_R \chi_B.$$

Note that any $\rho\beta_1\beta_2$ pattern is a result of a ρ pattern in χ_R and a $\beta_1\beta_2$ pattern in χ_B. For any particular pattern ρ in χ_R, let $\chi_{B_{<\rho}}$ be the set of all blue blocks less than ρ and $\chi_{B_{>\rho}}$ be the set of all blue blocks greater than ρ. Since $\rho\beta_1\beta_2$ patterns are only formed using β_1 patterns from $\chi_{B_{<\rho}}$ and β_2 patterns from $\chi_{B_{>\rho}}$, the contribution from this ρ pattern to $p(\chi,\phi)$ is at most

$$p(\chi_{B_{<\rho}},\beta_1)\cdot p(\chi_{B_{>\rho}},\beta_2).$$

This can be achieved (regardless of the choice of ρ) by putting the blue blocks in increasing order. Under this assumption, let $\chi_{B_{<j}}$ ($\chi_{B_{>j}}$) denote the collection of blue blocks before (after) B_j in χ_B and j_0 be such that

$$p(\chi_{B_{<j_0+1}},\beta_1)\cdot p(\chi_{B_{>j_0}},\beta_2) \geq p(\chi_{B_{<j+1}},\beta_1)\cdot p(\chi_{B_{>j}},\beta_2)$$

for any $i+1 \leq j \leq k-1$, we now have

$$p(\chi,\phi) \leq p(\chi_R,\rho)\cdot p(\chi_{B_{<j_0+1}},\beta_1)\cdot p(\chi_{B_{>j_0}},\beta_2).$$

Equality holds if

$$\chi_{B_{<j_0+1}} < \chi_R < \chi_{B_{>j_0}}.$$

Consequently each of $R = \chi_R$, $B_1 = \chi_{B_{<j_0+1}}$ and $B_2 = \chi_{B_{>j_0}}$ is a single block and the optimal permutation is of the form RB_1B_2 with $B_1 < R < B_2$. ■

Lastly we consider the case when the pattern is of the form $\phi = \beta_1\rho\beta_2$.

Theorem 2.4. *For a pattern ϕ with three colored blocks of the form $\beta_1\rho\beta_2$, there is an optimal permutation $\tilde{\chi}$ that is of the form B_1RB_2 with same numerical ordering as those in ϕ.*

Proof. First we may assume (following the same argument as before), that in an optimal permutation χ, the first and last blocks are blue, i.e.,

$$\chi = \chi_1^{(b)} \cdots \chi_i^{(c_i)} \cdots \chi_k^{(b)}$$

where $c_i \in \{r, b\}$. Consider any ρ pattern formed in the sequence of the s red blocks $\chi_R := \chi^{(r)}_{j_1}\chi^{(r)}_{j_2}\cdots\chi^{(r)}_{j_s}$. A $\beta_1\rho\beta_2$ pattern can only be formed by a β_1 pattern in the the sequence of blue blocks before $\chi^{(r)}_{j_1}$ and a β_2 pattern in the sequence of blue blocks after $\chi^{(r)}_{j_s}$. Thus the number of ϕ patterns formed from this particular ρ pattern is at most

$$p(\chi_{B_{<j_1}}, \beta_1) \cdot p(\chi_{B_{>j_s}}, \beta_2)$$

where $\chi_{B_{<j_1}}$ denotes the sequence of blue blocks before block j_1 and $\chi_{B_{>j_s}}$ denotes the sequence of blue blocks after block j_s.

Let j_0 (not necessarily unique) be a value such that $c_{j_0} = r$ and

$$p(\chi_{B_{<j_0}}, \beta_1) \cdot p(\chi_{B_{>j_0}}, \beta_2) \geq p(\chi_{B_{<j}}, \beta_1) \cdot p(\chi_{B_{>j}}, \beta_2)$$

for all j, then

$$p(\chi, \beta_1\rho\beta_2) \leq p(\chi_{B_{<j_0}}, \beta_1) \cdot p(\chi_R, \rho) \cdot p(\chi_{B_{>j_0}}, \beta_2)$$

with equality if all red blocks are located in between the first blue block immediately preceding and following $\chi^{(r)}_{j_0}$. Consequently χ is an optimal permutation of the form

$$\chi^{(b)}_1 \cdots \chi^{(b)}_\ell \chi^{(r)}_{\ell+1} \cdots \chi^{(r)}_m \chi^{(b)}_{m+1} \cdots \chi^{(b)}_k = \chi_{B_1}\chi_R\chi_{B_2}$$

with colored blocks $\chi_{B_1} := \chi^{(b)}_1 \ldots \chi^{(b)}_\ell$, $\chi_R := \chi^{(r)}_{\ell+1} \cdots \chi^{(r)}_m$, and $\chi_{B_2} := \chi^{(b)}_{m+1} \cdots \chi^{(b)}_k$.

Through arguments similar to those of Theorem 2.3 one can see that the numerical ordering of $\chi_{B_1}\chi_R\chi_{B_2}$ is the same as $\beta_1\rho\beta_2$. ■

3. APPLICATIONS TO SPECIFIC PATTERNS

In this section, we apply our findings to some specific colored patterns and obtain their corresponding packing densities. For convenience we let $p(n, \phi)$ denote the specific number of occurrences of ϕ in an optimal permutation of length n.

3.1. Patterns of Length 2.

For non-colored patterns of length 2, the packing density is trivially equal to one.

In the colored case, Theorem 2.1 implies that the optimal permutation of length n of the colored pattern 1_r2_b (or equivalently 2_r1_b) is of the form RB with $R < B$. Then

$$p(n, 1_r2_b) = p(R, 1) \cdot p(B, 1) = |R| \cdot |B|.$$

Given $|R| + |B| = n$, it is easy to see that

$$p(n, 1_r2_b) = \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{2n^2 - 1 + (-1)^n}{8}.$$

Therefore the packing density of all length two colored patterns (in which two distinct colors occur) is given by

$$\delta(1_r2_b) = \lim_{n\to\infty} \frac{p(n, 1_r2_b)}{\binom{n}{2}} = \frac{1}{2}.$$

3.2. **Patterns of Length 3.** For non-colored patterns of length 3, the packing densities for the decreasing and increasing patterns are trivial. The layered pattern 132 has packing density $2\sqrt{3}-3$ as established in [4].

3.2.1. *The pattern* $2_r1_b3_b$ *(and equivalents).* Theorem 2.3 implies that the optimal permutation of length n of the colored pattern $2_r1_b3_b$ is of the form RB_1B_2 with $B_1 < R < B_2$, then

$$p(n,2_r1_b3_b) = p(R,1)\cdot p(B_1,1)\cdot p(B_2,1) = |R|\cdot|B_1|\cdot|B_2|.$$

With $|R|+|B_1|+|B_1| = n$, it is easily shown that

$$p(n,2_r1_b3_b) = \begin{cases} \frac{n^3}{27}, & \text{if } n\equiv 0 \mod 3\\ \frac{(n-1)^2(n+2)}{27}, & \text{if } n\equiv 1 \mod 3\\ \frac{(n+1)^2(n-2)}{27}, & \text{if } n\equiv 2 \mod 3\end{cases}$$

Consequently the packing density of $2_r1_b3_b$ (and equivalent patterns) is

$$\delta(2_r1_b3_b) = \lim_{n\to\infty}\frac{p(n,2_r1_b3_b)}{\binom{n}{3}} = \frac{2}{9}.$$

3.2.2. *The pattern* $1_r3_b2_b$ *(and equivalents).* Theorem 2.1 implies that the optimal permutation of length n of the colored pattern $1_r3_b2_b$ is of the form RB in which $R<B$, then

$$p(n,1_r3_b2_b) = p(R,1)\cdot p(B,21) = |R|\binom{|B|}{2}.$$

Let $|B| = k$, then

$$p(n,1_r3_b2_b) = \max_{1\le k\le n}\left\{(n-k)\binom{k}{2}\right\},$$

achieved when $k\sim\frac{2n}{3}$. Consequently $\delta(1_r3_b2_b) = \frac{4}{9}$.

3.3. **Longer colored patterns.** Patterns of length over three may also be studied so long as they contain no more than three colored blocks and each colored block is equivalent to a non-colored pattern with known packing density.

3.3.1. *Pattern* $1_b3_r4_r2_b$. Theorem 2.4 implies that the optimal colored permutation $\tilde{\chi}$ is of the form B_1RB_2 with $B_1<B_2<R$, then

$$p(n,1_b3_r4_r2_b) = p(B_1,1)\cdot p(R,12)\cdot p(B_2,1).$$

For convenience, let $|B_1| = x$, $|R| = y$, and $|B_2| = z$. Then $x+y+z = n$ and thus for any fixed y

$$p(B_1,1)\cdot p(B_2,1) = x\cdot z \le \frac{2(n-y)^2-1+(-1)^{n-y}}{8}$$

with equality when $|x-z|\le 1$. Consequently

$$p(n,1_b3_r4_r2_b) = \max_{2\le y\le n-2}\left\{\frac{2(n-y)^2-1+(-1)^{n-y}}{8}\cdot\binom{y}{2}\right\},$$

achieved when $y\sim\frac{n}{2}$. Hence $\delta(1_b3_r4_r2_b) = \frac{3}{16}$.

3.3.2. *Pattern* $3_b2_b4_r6_r5_r1_b$. Theorem 2.3 implies that the optimal permutation $\tilde{\chi}$ is of the form B_1RB_2 with $B_2 < B_1 < R$. Hence

$$p(n, 3_b2_b4_r6_r5_r1_b) = p(B_1, 21) \cdot p(R, 132) \cdot p(B_2, 1).$$

Letting $|B_1| = x$, $|R| = y$, $|B_2| = z$ and fixing y again, we have $x + z = n - y$ and

$$p(B_1, 21) \cdot p(B_2, 1) \leq \binom{x}{2} \cdot z.$$

This expression is maximized when $x \sim \frac{2(n-y)}{3}$ and $z \sim \frac{n-y}{3}$. From [4] we have $p(y, 132) \sim (2\sqrt{3} - 3)\frac{y^3}{6}$, hence

$$\begin{aligned} &p(n, 3_b2_b4_r6_r5_r1_b) \\ &\sim \max_{3 \leq y \leq n-3} \left\{ \frac{2(n-y)}{6} \left(\frac{2(n-y) - 3}{6} \right) \cdot \left(\frac{2(n-y)}{3} \right) \cdot (2\sqrt{3} - 3)\frac{y^3}{6} \right\}, \end{aligned}$$

achieved when $y \sim \frac{n}{2}$. Thus $\delta(3_b2_b4_r6_r5_r1_b) = \frac{5}{9}(2\sqrt{3} - 3)$.

4. Concluding Remarks

In this note we considered the question of packing colored patterns into colored permutations. It is worth noting that, with our characterizations of the optimal permutations, the colored version of the pattern packing question is in some sense easier than the non-colored version and encompasses a wider range of patterns. For instance, the optimal permutation for the colored pattern $6_r1_r3_r2_r5_b4_b$ can indeed be characterized since it contains only three colored blocks and each block is a layered pattern. However, the non-colored pattern 613254 is not layered and its optimal permutation is much more difficult to characterize.

It is natural to conjecture that the following holds in general, which we post as a question.

Question 4.1. *Is it true that the optimal colored permutation with respect to a given colored pattern always shares the same number and arrangement of the colored blocks as those of the pattern?*

Acknowlegements

We are in debt to the anonymous referee whose valuable suggestions improved the quality of this paper.

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